Exponential decay of the solutions of Klein-Gordon equation in unbounded domains

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Damped Klein-Gordon equation

- $\mathbb{R}^d$ endowed with $K(x) \in \mathcal{C}_b^2(\mathbb{R}^d)$ with bounded geometry:

$$\forall (x,\xi) \in \mathbb{R}^{2d} \ , \quad K_{sup}|\xi|^2 \geq \xi^\intercal . K(x).\xi \geq K_{inf}|\xi|^2 \ .$$

- $\Delta_K = \mathrm{div}(K(x)\nabla\cdot)$ Laplacian operator associated to the metric
- $H^1(\mathbb{R}^d)$ endowed with the scalar product

$$\langle u|v\rangle_{H^1} = \int_{\mathbb{R}^d} (\nabla u(x))^\intercal . K(x).(\overline{\nabla v(x)}) + u(x)\overline{v(x)}\, dx \ .$$

- Damping $\gamma \geq 0$ in $\mathbb{L}^\infty(\mathbb{R}^d)$ with support ω.

We consider the **damped Klein-Gordon equation** :

$$\begin{cases} \partial_{tt}^2 u + \gamma(x)\partial_t u = \Delta_K u - u & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+^* \\ (u(\cdot,0), \partial_t u(\cdot,0)) = U_0 = (u_0, u_1) & \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \end{cases}$$

Damped Klein-Gordon equation

We set

$$U = (u, \partial_t u) \quad X = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \quad A = \begin{pmatrix} 0 & Id \\ \Delta_K - Id & -\gamma \end{pmatrix} .$$

The solutions of the damped Klein-Gordon equation are the trajectories of the semigroup $U(t) = e^{At}U_0$.

The energy $E(U(t)) = \frac{1}{2}\|e^{At}U_0\|_X^2$ is a Lyapunov function

$$\partial_t E(U(t)) = -\int_{\mathbb{R}^d} \gamma(x)|u_t(x)|^2 \, dx \ .$$

If γ does not vanish in large areas, $E(U(t))$ decays to zero for any solution $U(t)$.

Uniform stabilisation?

$$\exists M, \lambda > 0 \ , \quad \|e^{At}\|_{\mathcal{L}(X)} \ \leq \ Me^{-\lambda t} \ ?$$

Stabilisation and Hamiltonian flow

With the Laplacian operator $\Delta_K = \mathrm{div}(K(x)\nabla\cdot)$, we associate the symbol $g(x,\xi) = \xi^\intercal.K(x).\xi$ and the Hamiltonian flow $\varphi_t(x_0,\xi_0) = (x(t),\xi(t))$ defined on $\mathbb{R}^{2d}$ by

$$\begin{aligned} \varphi_0(x_0,\xi_0) &= (x_0,\xi_0) \\ \partial_t\varphi_t(x,\xi) &= (\partial_\xi g(x(t),\xi(t)), -\partial_x g(x(t),\xi(t)) \ . \end{aligned}$$

We introduce the mean value of the damping on geodesics

$$\langle\gamma\rangle_T(x,\xi) = \frac{1}{T}\int_0^T \gamma(\varphi_t(x,\xi))dt$$

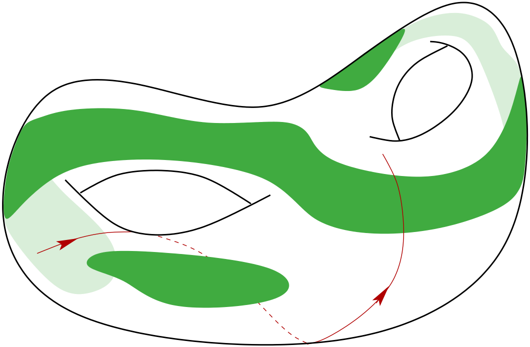
where $\gamma(x,\xi) := \gamma(x)$. We also consider the sphere

$$\Sigma = \{(x,\xi)\in\mathbb{R}^{2d} \ , \ \xi^\intercal K(x)\xi = 1\} \ .$$

On Ω compact manifold (possibly with boundary), C. Bardos, G. Lebeau, J. Rauch and M. Taylor have shown that the stabilisation $\|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$ is (almost) equivalent to the existence of T such that

$$\min_{(x,\xi)\in\Sigma} \langle\gamma\rangle_T(x,\xi) > 0 .$$

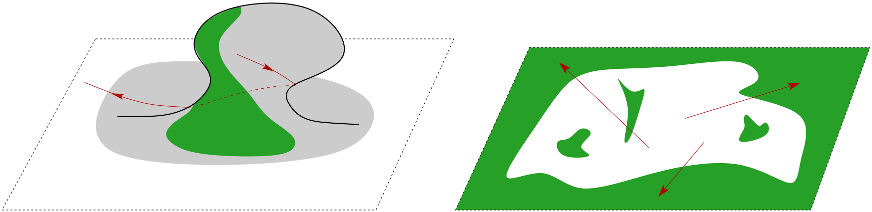
This is the famous **geometric control condition**.



Stabilisation and Hamiltonian flow

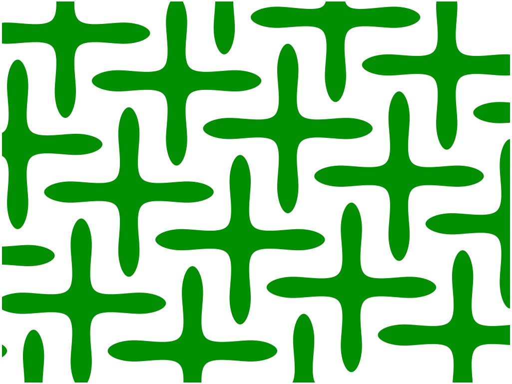
If Ω is not bounded:

- decay of the local energy [Lax, Morawetz, Phillips, 1963], [Morawetz, Ralston, Strauss, 1978], [N. Burq, 1998], [Aloui, Khenissi, 2002]...
- decay of the global energy (including semilinear cases) if $\gamma(x) \geq \alpha > 0$ outside a compact set [Zuazua, 1990-1992], [Feireisl, 1995], [Dehman, Lebeau, Zuazua, 2003], [R.J., C. Laurent, 2013].



Main result

Stabilisation if the damping may vanish?



Main result

Theorem – N. Burq & R.J. (2014-2015)

Assume that there exists $\tilde{\gamma} \in \mathcal{C}^0(\mathbb{R}^d, \mathbb{R})$ *uniformly continuous such that* $\gamma \geq \tilde{\gamma} \geq 0$ *and that there exist* $T > 0$ *and* $\alpha > 0$ *satisfying* $\langle \tilde{\gamma} \rangle_T(x, \xi) \geq \alpha > 0$ *for all* $(x, \xi) \in \Sigma$*.*

Then the semigroup generated by the damped Klein-Gordon equation satisfies

$$\exists M, \lambda > 0 \; , \quad \forall t \geq 0 \; , \quad |||e^{At}|||_{\mathcal{L}(X)} \leq M e^{-\lambda t} \; .$$

The damped wave equation

Theorem – R. Chill & A. Haraux (2003)

Let u be solution of

$$\begin{cases} \partial_{tt}^2 u + \partial_t u = \Delta_K u & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+^* \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0, u_1) & \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \end{cases}$$

and v be solution of

$$\begin{cases} \partial_t v = \Delta_K v & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+^* \\ v(\cdot,0) = u_0 + u_1 & \end{cases}$$

Then, $\|u(t) - v(t)\|_{H^1} \leq C \frac{\|u_0\|_{H^1} + \|u_1\|_{L^2}}{t}$.

In particular, there are solutions of the damped wave equation decaying with polynomial rate $1/t^{d/4}$ for $d \leq 3$.

The damped wave equation

Work in progress: R.J. & J. Royer

Let γ be as in the main theorem. Assume that γ is $\mathbb{Z}^d-$periodic and denote by $\langle\gamma\rangle$ is mean value on $[0,1]^d$.

Let u be solution of

$$\begin{cases} \partial_{tt}^2 u + \gamma(x)\partial_t u = \Delta_K u & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+^* \\ (u(\cdot,0), \partial_t u(\cdot,0)) = (u_0, u_1) & \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \end{cases}$$

and v be solution of

$$\begin{cases} \langle\gamma\rangle\partial_t v = \Delta_K v & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+^* \\ v(\cdot,0) = u_0 + u_1 \end{cases}$$

Then, $\|u(t) - v(t)\|_{H^1} \leq C\frac{\|u_0\|_{H^1} + \|u_1\|_{L^2}}{t}$.

(works also for $\frac{1}{g(x)}\Delta_K$ by diffeomorphism)

Main tools

Theorem – Gearhart-Prüss-Huang (1985)

Let e^{At} be a $\mathcal{C}^0-$semigroup on a Hilbert space X. Assume that there exists $M > 0$ such that $|||e^{At}||| \leq M$ for all $t \geq 0$.
Then e^{At} is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and

$$\sup_{\mu\in\mathbb{R}} |||(A - i\mu Id)^{-1}|||_{\mathcal{L}(X)} \ < \ +\infty \ . \tag{1}$$

We argue by contradiction: assume that there exists (μ_n) such that $|||(A - i\mu_n Id)^{-1}||| \to +\infty$. Two cases :

- **High frequencies** $(\mu_n) \to +\infty$
- **Low frequencies** $(\mu_n) \to \mu \in \mathbb{R}$

N.B.: we may assume that $\gamma \in \mathcal{C}_b^\infty$.

Main tools

We use the usual arguments with pseudo-differential operators:

- High frequencies: micro-local analysis,
- Low frequencies: Carleman-like estimates.

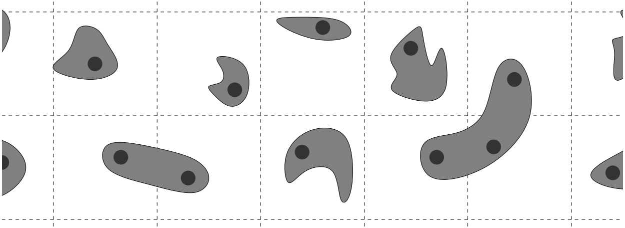
One has to be careful and to check that all arguments can be extended in the unbounded case. Examples:

- default measure is useless, since it yields a convergence only on compact sets. We have to go back to the arguments beyond.
- the construction of the Carleman weight must satisfy uniform estimates: positivity on a compact set is no more sufficient.

Main tools

To deal with low frequencies, that is to construct a suitable Carleman weight, we only need the following assumption.

Network Control Condition:
There exist L, r, $a > 0$ and a sequence $(x_n) \subset \mathbb{R}^d$ such that $\gamma(x) > a$ on $\cup_n B(x_n, r)$ and $\forall x \in \mathbb{R}^d$, $d(x, \cup\{x_n\}) \leq L$.



Theorem – N. Burq & R.J. (2014-2015)

Assume that γ belongs to $L^\infty(\mathbb{R}^d)$ and satisfies (NCC) only. Then the semigroup generated by the damped Klein-Gordon equation satisfies:
for any $k > 0$ and $U_0 \in H^{k+1} \times H^k$

$$\forall t \geq 0 \ , \quad \|e^{At}U_0\|_{H^1\times L^2} \leq \frac{C_k}{\ln(2+t)^k}\|U_0\|_{H^{k+1}\times H^k} \ .$$

Proof: use the same Carleman-type estimate as the one proved for low frequencies, but keep track of the exponential dependence of the constants with respect to the frequency.

Examples of applications

Consider now solutions with **unbounded energy** but with **bounded uniformly local energy**: fix some $\rho > 0$ and consider

$$\|u\|_{L^2_{ul}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \left(\int_{B(x,\rho)} |u|^2 \right)^{1/2}$$

and the same for higher Sobolev norms.

Corollary

Assume that there exists $\tilde{\gamma} \in \mathcal{C}^0(\mathbb{R}^d, \mathbb{R})$ uniformly continuous such that $\gamma \geq \tilde{\gamma} \geq 0$ and that there exist $T > 0$ and $\alpha > 0$ satisfying $\langle\tilde{\gamma}\rangle_T(x,\xi) \geq \alpha > 0$ for all $(x,\xi) \in \Sigma$.

Then the semigroup generated by the damped Klein-Gordon equation on the uniformly local Sobolev spaces satisfies

$$\exists M, \lambda > 0 \ , \quad \forall t \geq 0 \ , \quad |||e^{At}|||_{\mathcal{L}(H^1_{ul}\times L^2_{ul})} \leq Me^{-\lambda t} \ .$$

Examples of applications

Using H.U.M. method of Lions

Corollary

Let ω be a non-empty subset of $\mathbb{R}^d$. Assume that all the assumptions of our main result hold for $\gamma = \mathbb{1}_\omega$. Then, there exists $T > 0$ such that, for all $(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and all $(\tilde{u}_0, \tilde{u}_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, there exists a control $v \in L^1((0, T), L^2(\omega))$ such that the solution u of

$$\begin{cases} \partial_{tt}^2 u - \operatorname{div}(K(x)\nabla u) + u = \mathbb{1}_\omega v(x,t) & (t,x) \in (0,T) \times \mathbb{R}^d \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) & \end{cases},$$

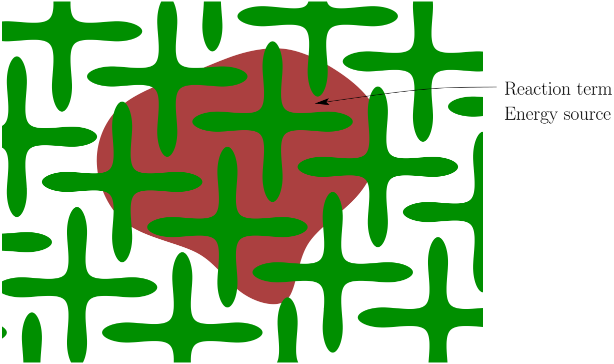
satisfies $(u, \partial u)(\cdot, T) = (\tilde{u}_0, \tilde{u}_1)$.

Examples of applications

Non-linear equation

$$\begin{cases} \partial_{tt}^2 u + \gamma(x)\partial_t u = \mathrm{div}(K(x)\nabla u) - u - f(x,u) & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) & \end{cases},$$

where f has compact support with respect to x.



Examples of applications

Assume there exists $1 \leq p < (d+2)/(d-2)$ such that

$$|f(x,u)| \leq C(1+|u|)^p \ , \quad |f'(x,u)| \leq C(1+|u|)^{p-1}$$

$$\text{and} \quad \liminf_{|u|\to+\infty} \ \max_{x\in\text{supp}(f)} \ f(x,u)u \ \geq \ 0 \ .$$

Corollary

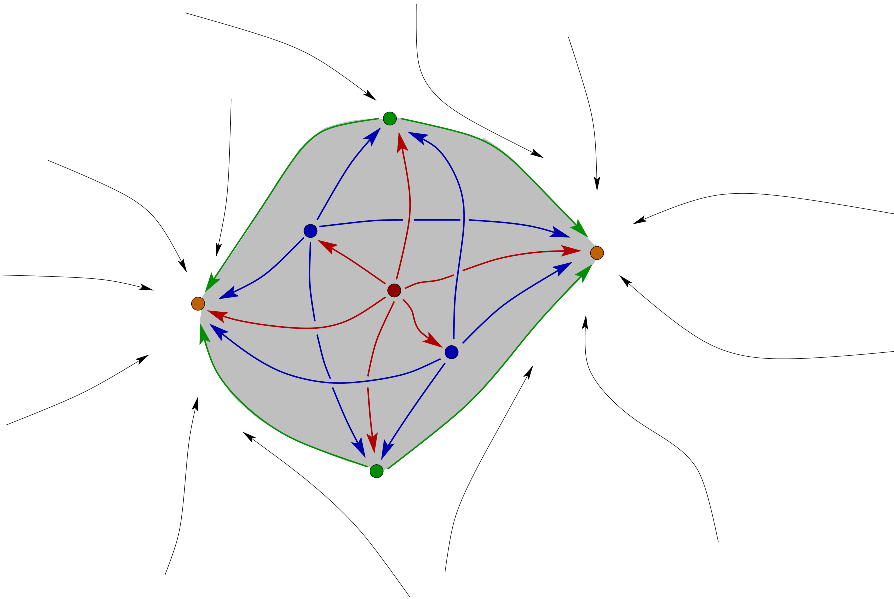
Assume the hypothesis of our main result. If $f \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is as above, with compact support in x and analytic with respect to u, then the dynamical system generated by the semilinear damped wave equation admits a compact global attractor $\mathcal{A}$ and the energy

$$E(u) := E(u,\partial_t u) = \frac{1}{2}\int_{\mathbb{R}^d} (|\partial_t u|^2 + |\nabla u^\intercal . K(x) . \nabla u| + |u|^2) + \int_{\mathbb{R}^d} V(x,u)$$

is a Lyapunov function.

If moreover $f(x,u) \geq 0$, then stabilisation holds.

Examples of applications



Examples of applications

One of key arguments:

$$\left\{\begin{array}{l}\partial^2_{tt}v = \Delta_K v - v - f'_u(x,u)v \\ v \equiv 0 \text{ on } \mathrm{supp}(\gamma)\end{array}\right. \quad \Longrightarrow \quad v \equiv 0\ .$$

- [Zuazua, 1992], [Feireisl, 1995] and [Dehman, Lebeau, Zuazua, 2003] : flat geometry, $\gamma(x) \geq \alpha > 0$ and $f(x,u)u \geq 0$ outside a compact set (unique continuation property of Ruiz).
- [R.J., Laurent, 2012] : curved geometry, f analytic and $\gamma(x) \geq \alpha > 0$ and $f(x,u)u \geq 0$ outside a compact set (regularisation result of [Hale, Raugel, 2003] and unique continuation property with semi-analytic coefficients of [Robbiano, Zuily, 1998])
- [N. Burq, R.J., 2014] : natural geometry for γ but f analytic and compactly supported in x.

Corollary – R.J. & C. Laurent (2013)

Let ω a non-empty open subset of $\mathbb{R}$. We assume that there exist $L > 0$ and $\varepsilon > 0$ such that ω contains a interval of size ε in each interval $[x, x+L]$, $x \in \mathbb{R}$. Let $f \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ compactly supported in x and satisfying

$$\liminf_{|s|\to+\infty} \max_{x\in supp(f)} f(x,s)s \geq 0 .$$

Then, for all $E_0 \geq 0$, there exists $T > 0$ such that, for all (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with energy $E \leq E_0$, there exists a control $v \in L^1((0,T), L^2(\omega))$ such that the solution of

$$\begin{cases} \partial_{tt}^2 u - \operatorname{div}(K(x)\nabla u) + u + f(x,u) = \mathbb{1}_\omega v(x,t) \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \end{cases}$$

satisfies $(u, \partial u)(\cdot, T) = (\tilde{u}_0, \tilde{u}_1)$.

Open problems

Open problems

Linear stabilisation:

- Regularity assumption "$\tilde{\gamma}$ uniformly continuous" for the damping really necessary?

Non-linear stabilisation:

- Non-linear case with f not compactly supported (but still satisfying $f(x,u)u \geq 0$ outside a compact set).
- Non-linear case with f non-analytic.

Other asymptotic behaviors:

- Asymptotic behavior of the solutions of the non-linear damped wave equation compared to the ones of the parabolic PDE?
- Non-linear behavior when the linear semigroup follows other types of decay (polynomial decay etc.)

Thank you for your attention

Aneto, Benasque 2013

Sketch of proofs

Theorem – Gearhart-Prüss-Huang (1985)

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Then e^{At} is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and

$$\sup_{\mu \in \mathbb{R}} |||(A - i\mu Id)^{-1}|||_{\mathcal{L}(X)} \ < \ +\infty \ . \tag{2}$$

We argue by contradiction: assume that there exists (μ_n) such that $|||(A - i\mu_n Id)^{-1}||| \to +\infty$. Two cases :

- **High frequencies** $(\mu_n) \to +\infty$
- **Low frequencies** $(\mu_n) \to \mu \in \mathbb{R}$

N.B.: we may assume that $\gamma \in \mathcal{C}_b^\infty$.

Sketch of proofs

We use the usual arguments with pseudo-differential operators, micro-local analysis (high frequencies) and Carleman-like estimates (low frequencies). One has to be careful and to check that all arguments can be extended in the unbounded case. Examples :

- default measure is of no use, since it yields a convergence only on compact sets. We have to go back to the arguments beyond.
- the construction of the Carleman weight must satisfy uniform estimates: positivity on a compact set is no more sufficient.

Notations:

We use pseudo-differential operators.
For a symbol $a(x,\xi)$, we use Weyl's quantification

$$\mathrm{Op}_h(a)u = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2d}} e^{i(x-y).\xi}\, a\left(\frac{x+y}{2}, h\xi\right) u(y)\, dy\, d\xi \ .$$

Examples:

1. $u \mapsto f(x)u$ has for symbol $f(x)$
2. $h\nabla$ has for symbol $i\xi$
3. $h^2\Delta_K = \mathrm{Op}_h(-\xi^\intercal.K(x).\xi) + \mathcal{O}_{L^2\to L^2}(h^2)$

Proof: high frequencies

Proof: high frequencies

We use microlocal analysis and the assumption $\langle\tilde{\gamma}\rangle_T(x,\xi) \geq \alpha > 0$ for all $(x,\xi) \in \Sigma$.

We set $h = 1/\mu$, it is sufficient to show

Proposition

The operator $P_h = h^2(\Delta_K - Id) - ih\gamma(x) + Id$ has a L^2-resolvant satisfying $\|(P_h)^{-1}f\|_{L^2} \leq \dfrac{C}{h}\|f\|_{L^2}$.

We argue by contradiction: let (u_h) be a sequence such that $\|u_h\|_{L^2} = 1$ and $P_h u_h = o_{L^2}(h)$.

Proof: high frequencies

(u_h) a sequence with $\|u_h\|_{L^2} = 1$ and
$P_h u_h = h^2(\Delta_K - Id)u_h - ih\gamma(x)u_h + u_h = o_{L^2}(h)$

1. u_h is concentrating on $\{(x,\xi)\ ,\ \xi^\intercal K(x)\xi = 1/h^2\}$.
   Indeed, at the first order $\mathrm{Op}_h(-\xi^\intercal K(x)\xi + 1)u_h = o(1)$.
2. If $a(x,\xi)$ is a symbol of order 0 and if $g(x,\xi) = \xi^\intercal.K(x).\xi$, then

$$[\mathrm{Op}_h(a), P_h] = -ih\mathrm{Op}_h\big(\{\xi^\intercal K(x)\xi, a(x,\xi)\}\big) + \mathcal{O}_{L^2\to L^2}(h^2)$$

$$\langle[\mathrm{Op}_h(a), P_h]u_h|u_h\rangle_{L^2} = 2ih\langle\mathrm{Op}_h(a\gamma)u_h|u_h\rangle_{L^2} + o(h)$$

and we obtain

$$\langle\mathrm{Op}_h(2a\gamma + \{g,a\})u_h|u_h\rangle_{L^2} \xrightarrow[h\longrightarrow 0]{} 0\ .$$

Proof: high frequencies

Goal: find $a(x,\xi)$ a symbol of order 0 such that
$2a\gamma + \{g,a\} \geq \alpha > 0$ on $\Sigma = \{(x,\xi)\ ,\ \xi^\intercal K(x)\xi = 1\}$.
Then $\langle \mathrm{Op}_h(2a\gamma + \{g,a\})u_h | u_h\rangle \to 0$ would be in contradiction with $\|u_h\| = 1$.

We set $a(x,\xi) = e^{c(x,\xi)}$ with

$$c(x,\xi) = \frac{2}{T}\int_0^T (T-t)\gamma(\varphi_t(x,\xi))\,dt = \frac{2}{T}\int_0^T\int_0^t \gamma(\varphi_s(x,\xi))\,ds\,dt\,.$$

The Hamiltonian flow satisfies $\{g,a\}(x,\xi) = \partial_\tau a(\varphi_\tau(x,\xi))_{|\tau=0}$ and since

$$\begin{aligned} c(\varphi_\tau(x,\xi)) &= \frac{2}{T}\int_0^T (T-t)\gamma(\varphi_{t+\tau}(x,\xi))\,dt \\ &= \frac{2}{T}\int_\tau^{T+\tau} (T-t+\tau)\gamma(\varphi_t(x,\xi))\,dt \end{aligned}$$

we get $2a\gamma + \{g,a\} = 2e^{c(x,\xi)}\langle\gamma\rangle_T(x,\xi)$.

Proof: low frequencies

We set $\mu \in \mathbb{R}$. We have to show that $A - i\mu Id$ is invertible, which is equivalent to

$$P = (\Delta_K - Id) - i\mu\gamma(x) + \mu^2 Id$$

invertible in $\mathcal{L}(L^2)$. We again argue by contradiction and assume the existence of (u_n) such that $\|u_n\|_{L^2} = 1$ and

$$(\Delta_K - Id)u_n - i\mu\gamma(x)u_n + \mu^2 u_n = o_{L^2}(1) \; .$$

If $\mu = 0$, then ok (since we consider Klein-Gordon equation and not the wave one). If $\mu \neq 0$, mutiplying by $\overline{u}_n$ and integrating, we find

$$\int_{\mathbb{R}^d} \gamma(x)|u_n(x)|^2 \, dx \;\longrightarrow\; 0 \; .$$

Hörmander-Carleman strategy:
For φ to be fixed later, we set

$$Q_h = -h^2 e^{\varphi/h} \Delta_K e^{-\varphi/h} = \mathrm{Op}_h(q_R) + \mathrm{Op}_h(q_I)$$

with

$$\begin{aligned} q_R(x,\xi) &= \xi^\intercal K(x)\xi - \nabla\varphi^\intercal K(x)\nabla\varphi \\ q_I(x,\xi) &= 2\nabla\varphi^\intercal K(x)\xi \end{aligned}$$

A direct computation shows that

$$\|Q_h u\|^2 \geq h\langle \mathrm{Op}_h(\mu(q_R^2 + q_I^2) + \{q_R, q_I\})u|u\rangle + \mathcal{O}(h^2\|u\|_{L^2}^2)$$

with μ such that $h\mu \leq 1$. On the other hand, by conjugating P by $e^{\varphi/h}$, we find that $\|Q_h u_n\|$ goes to 0 when n goes to infinity.

Proof: low frequencies

$$h\langle \mathrm{Op}_h(\mu(q_R^2+q_I^2)+\{q_R,q_I\})u_n|u_n\rangle \xrightarrow[n \longrightarrow 0]{} 0\ .$$

To obtain a contradiction for small fixed h, it is sufficient to show that

$$\mu(q_R^2+q_I^2)+\{q_R,q_I\} \geq \alpha > 0 \quad \text{on } \mathbb{R}^d \times \mathbb{R}^d\ .$$

As $q_R = \xi^\intercal K(x)\xi - \nabla\varphi^\intercal K(x)\nabla\varphi$, it holds for $\xi \not\approx \nabla\varphi(x)$. To deal with the case $\xi = \mathcal{O}(\varphi(x))$, we set $\varphi = e^{\lambda\psi}$ with λ very large and ψ bounded such that $|\nabla\psi(x)| \geq \tilde{\alpha} > 0$ and we show $\{q_R, q_I\} \geq \alpha > 0$ (Hörmander sub-ellipticity). In other words, we choose a weight with very steep gradient so that the terms $\nabla\varphi^\intercal K(x)\nabla\varphi$ will be predominant on the other terms.

Proof: low frequencies

In short:

We conjugate the equation

$$(\Delta_K - Id)u_n - i\mu\gamma(x)u_n + \mu^2 u_n = o_{L^2}(1) \ .$$

by a weight $e^{e^{\lambda\psi(x)}/h}$ with very steep gradient. In the first order, we get an elliptic operator, which concludes.

Problem:

Find ψ bounded with $|\nabla\psi(x)| \geq \tilde{\alpha} > 0$!

Proof: low frequencies

ψ bounde and $|\nabla\psi(x)| \geq \tilde{\alpha} > 0 \Rightarrow$ absurd!

Fortunately, we know that $\int \gamma(x)|u_n|^2\,dx \to 0$ and we can choose ψ with no constraints on the support of γ (in fact, rather where $\gamma \geq \beta$ for some positive β).

The assumptions yield the existence of a value $\beta > 0$, a radius $\rho > 0$ and a length L such that, for all x in $\mathbb{R}^d$, there exists at distance at most L, a ball a radius ρ where $\gamma(x) \geq \beta > 0$.
$\Rightarrow$ **there exists a network of balls where the damping is effective**.

Up to a diffeomorphism, we assume that the ball are centered on $k \in \mathbb{Z}^d$ and we set

$$\psi(x) = \sum_{i=1}^{d} \cos(\pi x_i)\ .$$

Examples of applications

Corollary

If in addition $f(x,u)u \geq 0$ for all $(x,u) \in \mathbb{R}^{d+1}$, then the semilinear damped wave equations is stabilised in the sense where, for all $E_0 \geq 0$, there exist $K > 0$ and $\lambda > 0$ such that, for all solutions u with $E(u(0)) \leq E_0$, $E(u(t)) \leq Me^{-\lambda t}E(u(0))$ for any $t \geq 0$.