A duality principle for non convex variational problems, Numerical examples

PHAN Tran Duc Minh
IMATH, University of Toulon (FRANCE)

joint work with:

Guy BOUCHITTÉ (IMATH, University of Toulon)
Ilaria FRAGALÀ (Politecnico di Milano)

Benasque
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A non convex problem

Let Ω be a bounded domain of $\mathbb{R}^N$. We consider the problem

$$\mathcal{I}(\Omega) := \inf_{u \in H^1(\Omega)} \left\{ \int_\Omega (\varphi(\nabla u) + g(u))\, dx : u = u_0 \text{ on } \partial\Omega \right\}, \quad (1)$$

where $\varphi : \mathbb{R}^N \to [0, +\infty)$, $g : \mathbb{R} \to [0, +\infty)$ are functions such that

- $\varphi(z)$ is convex continuous, $\varphi(0) = 0$;
- $g(t)$ is lower semicontinuous and $\exists M$ countable such that for $t \in \mathbb{R} \setminus M$, $\limsup_n g(t_n) \leq g(t)$ whenever $t_n \to t$;
- there exist $\alpha, \beta > 0 : \alpha |z|^2 \leq \varphi(z) + g(t) \leq \beta(1 + |z|^2)$.

For simplicity, we assume that $u_0 = 0$.
We emphasize that g is not assumed to be convex.

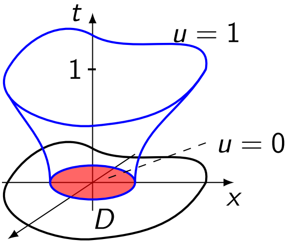
Example

$$\varphi(z) = \frac{|z|^2}{2}, \quad g(t) = \begin{cases} \lambda & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases},$$

$$\mathcal{I}(\Omega) = \inf \left\{ \int_\Omega \left[\frac{|\nabla u|^2}{2} + g(u) \right] dx : u = 1 \text{ on } \partial\Omega \right\}.$$

$\rightsquigarrow$ Free boundary Pb in term of $D = \{u > 0\}$,
u solves

$$\begin{cases} -\Delta u_D = 0 \text{ in } D \\ u_D = 1 \text{ on } \partial\Omega \\ u_D = 0 \text{ in } \Omega \backslash D. \end{cases}$$



$\rightsquigarrow$ Shape functional

$$J : D \to \lambda|D| + \frac{1}{2}\int_\Omega |\nabla u_D|^2.$$

Pb in 1D case

Let $N = 1$, $\Omega = (0, h)$, the Pb is stated as

$$\mathcal{I}(\lambda, h) = \inf \left\{ \int_0^h \frac{u'^2}{2} dx + \lambda |\{u > 0\}| \;\middle|\; \begin{array}{l} u(0) = 1 \\ u(h) = 1 \end{array} \right\}. \qquad (2)$$

Taking first integral of Euler's equation, $-u'' + g'(u) = 0$, we have

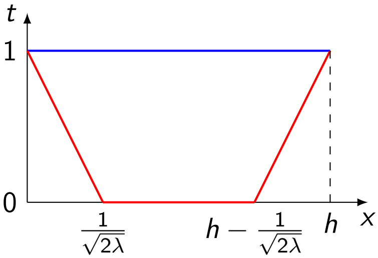
$$\begin{aligned} u'\varphi'(u') - [\varphi(u') + g(u)] &= \mu \\ \Leftrightarrow u'.u' - \left(\frac{u'^2}{2} + \lambda 1_{\{u>0\}} \right) &= \mu \\ \Leftrightarrow u' = \pm\sqrt{2\left(\mu + \lambda 1_{\{u>0\}}\right)}. \end{aligned}$$

Solutions are piecewise affine functions of slopes in $\left\{0, \pm\sqrt{2}\sqrt{\lambda}\right\}$.

Thus, solutions are $\bar{u}_0 \equiv 1$ or of the form:

$$\bar{u}_1(x) = \begin{cases} -\sqrt{2}\sqrt{\lambda}x + 1 & \text{if } 0 \le x \le \frac{1}{\sqrt{2}\sqrt{\lambda}} \\ 0 & \text{if } \frac{1}{\sqrt{2}\sqrt{\lambda}} < x < h - \frac{1}{\sqrt{2}\sqrt{\lambda}} \\ \sqrt{2}\sqrt{\lambda}x - (\sqrt{2}\sqrt{\lambda}h - 1) & \text{if } h - \frac{1}{\sqrt{2}\sqrt{\lambda}} \le x \le h. \end{cases}$$

The problem reaches its minimum: $\mathcal{I}(\lambda, h) = \min\{\lambda h, 2\sqrt{2}\sqrt{\lambda}\}$. As $h = \frac{2\sqrt{2}}{\sqrt{\lambda}}$, the problem has at least two solutions, $\bar{u}_0$ and $\bar{u}_1$.



Duality framework(Constrained flow optimization)

We set $\mathcal{B}$ the class of fields $\sigma = (\sigma^x, \sigma^t) \in (L^\infty(\Omega \times \mathbb{R}))^{N+1}$ satisfying the following conditions:

(s1) $\operatorname{div} \sigma = 0$ in $\Omega \times \mathbb{R}$;

(s2) $\sigma(x,t) \in C(t)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$;

(s3) $\forall t \in M, 0 \le \sigma^t(x,t) + g(t)$ a.e. $x \in \Omega$. (∗)

Here $C(t) = \left\{(q^x, q^t) \in \mathbb{R}^N \times \mathbb{R} \middle| \varphi^*(q^x) - g(t) \le q^t\right\}$.

Lemma 1 [Bouchitté, Fragalà]
For every $u \in H_0^1(\Omega)$ and for every $\sigma \in \mathcal{B}$, one has

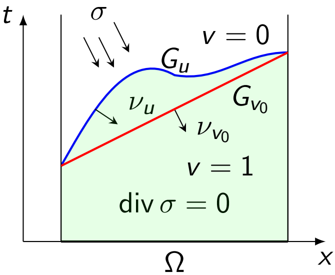
$$-\int_\Omega \sigma^t(x,0)dx \le \int_\Omega [\varphi(\nabla u) + g(u)]dx$$

(∗) $(s3)$ can be dropped if g is continuous, $\sigma^t(\cdot, t)$ coincides with the normal flow across the hyperplane $\{x_{N+1}\}$.

Geometrical interpretation

$\varphi^*(\sigma^x) - g(t) \leq \sigma^t$ a.e. in $\Omega \times \mathbb{R}$.

$$v = 1_u(x,t) = \begin{cases} 1 & \text{if } t \leq u(x) \\ 0 & \text{if } t > u(x) \end{cases}$$



$\nu_u = \frac{1}{\sqrt{1+|\nabla u|^2}}(\nabla u, -1)$ is the unit normal to the graph G_u.

$$-\int_{G_{v_0}} \sigma.\nu_{v_0} dH^N = \int_{\Omega\times\mathbb{R}} \sigma.D1_u$$

$$= \int_\Omega \left[\sigma^x(x,u(x)).\nabla u(x) - \sigma^t(x,u(x))\right] dx$$

$$\leq \int_\Omega \left[\varphi^*(\sigma^x) + \varphi(\nabla u) - \sigma^t\right] dx$$

$$\leq \int_\Omega [\varphi(\nabla u) + g(u)] dx$$

If $u_0 = 0$ then $\int_{G_{v_0}} \sigma.\nu_{v_0} dH^N = \int_\Omega \sigma^t(x,0)dx.$

Dual Pb holds in dimension $N+1$

Let us define

$$\mathcal{S}(\Omega) := \sup_{\sigma \in \mathcal{B}} \left\{ - \int_\Omega \sigma^t(x,0)dx \right\}. \qquad (3)$$

Then $\mathcal{I}(\Omega) \geq \mathcal{S}(\Omega)$ (Lemma 1).

Theorem

It holds $\mathcal{I}(\Omega) = \mathcal{S}(\Omega)$.

Sketch of proof.
$u \rightsquigarrow v = 1_u(x,t) \in \mathcal{A}_0$ where

$$\mathcal{A}_0 = \left\{ v(x,t) : \Omega \times \mathbb{R} \to [0,1] \,\middle|\, \begin{array}{c} v(x,\cdot) \text{ is decreasing }, \\ v(x,+\infty) = 0, v(x,-\infty) = 1, \\ Dv \text{ is a bounded measure }. \end{array} \right\}$$

• $\mathcal{I}(\Omega)$ can be reformulated as: $\inf\{F(v), v \in \mathcal{A}_0\}$ where

$$F(v) = \int_{\Omega\times\mathbb{R}} h(t, Dv), \qquad h(t,p) := -p^t(\varphi(-\frac{p^x}{p^t}) + g(t)).$$

Let $u_s(x) := \inf\{\tau \in \mathbb{R} : v(x,\tau) \leq s\}$ for $s \in [0,1]$.

Lemma 2

If $F(v) < +\infty$, for $v \in \mathcal{A}_0$, then for a.e. $s \in [0,1]$, one has $u_s \in H_0^1(\Omega)$ and $F(v) = \int_0^1 \left(\int_\Omega [\varphi(\nabla u_s) + g(u_s)]dx\right) ds$.

Remark. If $v = 1_u$ then $u_s = u$ for a.e. $s \in [0,1]$.

Consequence.

If v is solution of $\inf\{F(v), v \in \mathcal{A}_0\}$ then $\forall s \in [0,1]$, u_s is solution of $\mathcal{I}(\Omega)$.

• $F(v)$ can be rewritten as $F(v) = \sup\left\{\int_{\Omega\times\mathbb{R}} \sigma.Dv : \sigma \in \mathcal{B}\right\}$.

$$\mathcal{I}(\Omega) = \inf_{v\in\mathcal{A}_0} \sup_{\sigma\in\mathcal{B}} \left\{\int_{\Omega\times\mathbb{R}} \sigma.Dv\right\} = \sup_{\sigma\in\mathcal{B}} \inf_{v\in\mathcal{A}_0} \left\{\int_{\Omega\times\mathbb{R}} \sigma.Dv\right\} = \mathcal{S}(\Omega)$$

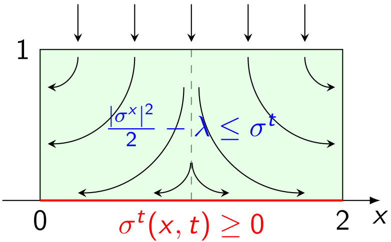
Numerical computation of optimal flow

We treat the case

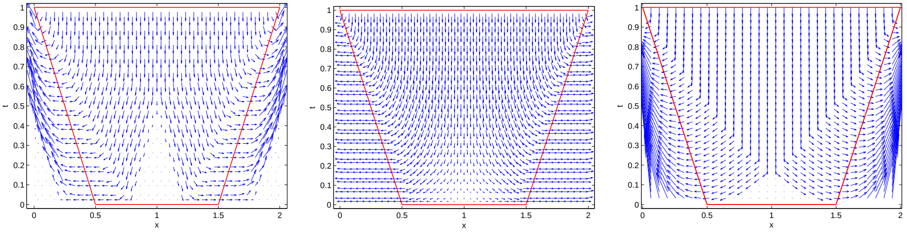
$$\Omega = [0,2], \qquad g(t) = \begin{cases} \lambda & \text{if } t > 0 \\ 0 & \text{if } t \le 0, \end{cases} \qquad \lambda = 2,$$

$$\mathcal{S}_\epsilon(\Omega) := \sup_{\sigma \in \mathcal{B}} \left\{ -\int_\Omega \sigma^t(x,1)dx - \epsilon \int_{\Omega\times[0,1]} |\sigma|^2 : \epsilon \ge 0 \right\}.$$

- $\epsilon = 0$ the critical dual Pb
- $\epsilon > 0$ viscosity term ($\rightsquigarrow$ uniqueness of solution)



Numerics (Matlab toolbox + 2D Finite element)



(a) $\epsilon = 0$ (b) $\epsilon > 0$ (c) Singular solution

- Singular solution (c) is constructed by symmetrization of gradient rotated, $\sigma = (\partial_t V, -\partial_x V)$, of value function:

$$V(x,t) := \inf \left\{ \int_0^x \frac{u'^2}{2} dx + \lambda |\{u > 0\}| \,\middle|\, u(0) = 1, u(x) = t \right\}.$$

- Time of computation is very high. Matlab toolbox is not good for non linear constrained optimization Pb.

Min-max Formulation

Let $\quad L(v,\sigma) := \int_{\Omega\times\mathbb{R}} \sigma.Dv.$
As we have known

$$\mathcal{I}(\Omega) = \inf_{v\in\mathcal{A}_0} \sup_{\sigma\in\mathcal{B}} L(v,\sigma) = \sup_{\sigma\in\mathcal{B}} \inf_{v\in\mathcal{A}_0} L(v,\sigma) = \mathcal{S}(\Omega)$$

We now seek the saddle point of min-max problem

$$\inf_{v\in\mathcal{A}_0} \sup_{\sigma\in\mathcal{B}} L(v,\sigma)$$

Recall that $(\bar{v},\bar{\sigma})$ is solution of the problem min-max if

$$L(\bar{v},\sigma) \leq L(\bar{v},\bar{\sigma}) \leq L(v,\bar{\sigma}), \forall v \in \mathcal{A}_0, \sigma \in \mathcal{B}$$

Remark. Once when $\bar{v}$ is determined, we will obtain u_s as optimal solution of $\mathcal{I}(\Omega)$ (Lemma 2).

Discretization settings

Back to the previous Free boundary Pb.
Let $\Omega = (0,2)$, $\Sigma = \Omega \times (0,1)$. Note that

$$\int_\Sigma \sigma.Dv = \int_\Sigma \sigma.D(v-1) = \int_\Sigma -(v-1)\,\mathrm{div}\,\sigma + \int_{\partial\Sigma} (v-1)(\sigma.n)ds.$$

$$A = \left\{ v(x,t) \in BV(\Sigma) \middle| \begin{array}{c} v(\cdot,0) = 1, v(\cdot,1) = 0, \\ v(0,\cdot) = v(h,\cdot) = 1 \end{array} \right\}.$$

The min-max problem reads

$$\sup_\sigma \inf_{v \in A} \left\{ \int_\Sigma -(v-1)\,\mathrm{div}\,\sigma - \int_\Omega \sigma^t(x,1)dx \right\}$$
$$= \sup_\sigma \left\{ -\int_0^h \sigma^t(x,1)dx : \mathrm{div}\,\sigma = 0 \right\}.$$

Discretization settings:

We consider a two-dimensional Cartesian grid G^h of size $n_x \times n_t$. Let h_x, h_t are steps and (i,j) is location on the grid.

$$G^h = \{(ih_x, jh_t) : 0 \leq i < n_x, 0 \leq j < n_t\}$$

$$A^h = \left\{ v^h \in \mathbb{R}^{n_x n_t} : v^h_{i,0} = 1, v^h_{i,n_t-1} = 0, v^h_{0,j} = v^h_{n_x-1,j} = 1 \right\}$$

$$B^h = \left\{ \sigma^h \in (\mathbb{R}^2)^{n_x n_t} : (\sigma^h)_{i,j} \in C(jh_t) \right\}$$

The discrete minimax Pb

$$\min_{v^h \in A^h} \max_{\sigma^h \in B^h} \left\langle \nabla^h v^h, \sigma^h \right\rangle$$

Orthogonal projections

Consider the projection

$$\begin{cases} \sigma_{n+1}^h = \mathrm{Proj}_{B^h}(\sigma_n^h + \alpha \nabla^h \overline{v}_n^h) \\ v_{n+1}^h = v_n^h - \beta(\mathrm{div}^h \, \sigma_{n+1}^h) \\ \overline{v}_{n+1}^h = 2v_{n+1}^h - v_n^h \end{cases}$$

where $\alpha\beta L^2 < 1$, div^h is adjoint to ∇^h, and L is given by

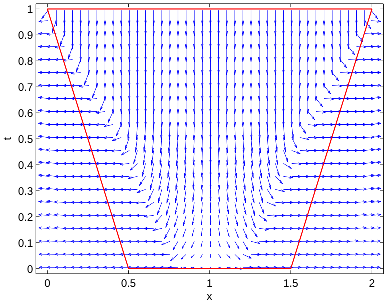
$$L = \|\nabla^h\| = \sup_{\|v^h\| \neq 0} \frac{\|\nabla^h v^h\|}{\|v^h\|} = \sqrt{\frac{4}{h_x^2} + \frac{4}{h_t^2}}$$

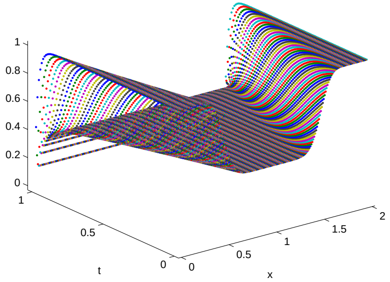
The projection $\overline{\sigma}^h = (\overline{\sigma}^x, \overline{\sigma}^t)$ of $\sigma^h \notin B^h$ is given by

$$\begin{cases} \overline{\sigma}^x & = \frac{1}{1+\theta}\sigma^x \\ \overline{\sigma}^t & = \sigma^t + \theta \\ q^x & = \sigma^x \\ qt^t & = \sigma^t + \lambda \\ 0 & = \theta^3 + (2 + q^t)\theta^2 + (1 + 2q^t)\theta + q^t - \frac{1}{2}|q^x|^2 \end{cases}$$

Scheme MAC + Orthogonal projections

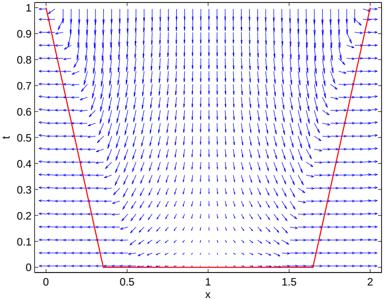
Scheme MAC is adaptive to this method. Here are some results.

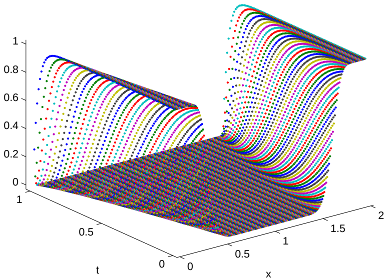




- In case of $\lambda = 2$.

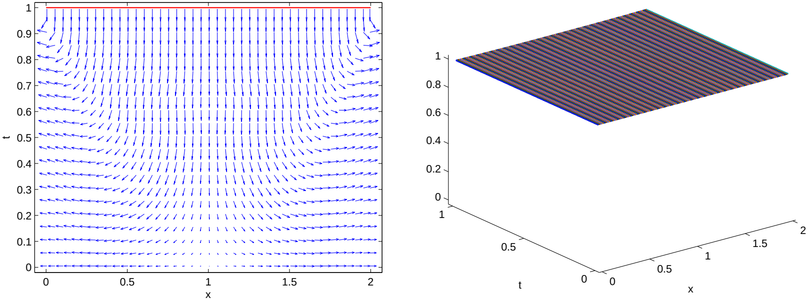
Optimal v exhibits two plateaus corresponding to solution u_0 and u_1





- In the case of $\lambda = 4$.

Optimal v exhibits two plateaus corresponding to solution u_1.



- In the case of $\lambda = 1$.

Optimal v has only one plateau corresponding to solution u_0.

Thank You!