Smoothness results for the minimization of the first eigenvalue of a two-phase material and applications to non existence

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Eigenvalue problem

We want to mix two materials (electric, thermal,…) given by their diffusion constants α, β with $0 < \alpha < \beta$ in order to minimize the first eigenvalue of the operator

$$u \in H_0^1(\Omega) \longmapsto -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))\nabla u\big)$$

with $\Omega \subset \mathbb{R}^N$, $N \geq 2$, under the restriction $|\omega| \leq \kappa$, with $0 < \kappa < |\Omega|$, i.e., we consider

$$(\Lambda_m) \qquad \min_{|\omega|\leq\kappa} \min_{u\in H_0^1(\Omega)} \frac{\int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

Remark: If $\kappa = |\Omega|$, the solution is the trivial one $\omega = \Omega$.

In order to give some physical interpretation of the problem, we recall:

If u is the solution of

$$\begin{cases} \partial_t u - \operatorname{div}\Big(\big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)\nabla u\Big) = 0 \ \text{ in } \mathbb{R}^+ \times \Omega \\ u = 0 \ \text{ on } \mathrm{R}^+ \times \partial\Omega \\ u_{|t=0} = u_0 \end{cases}$$

Then

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} e^{-\lambda_1 t}, \quad \forall t \geq 0.$$

Thus our problem can be used to obtain the optimal distribution of two materials in heat conduction in order to obtain the most insulated one

Compliance problem

$$\Omega \subset \mathbb{R}^N, N \geq 2, \text{ bounded, open,}$$
$$\beta > \alpha > 0,\ 0 < \kappa < |\Omega|,\ \tilde{f} \in H^{-1}(\Omega)$$

$$\max_{|\omega| \leq \kappa} \int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^2 dx$$

$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))\nabla u_\omega\big) = \tilde{f} \text{ in } \Omega \\ u_\omega = 0 \text{ on } \partial\Omega. \end{cases}$$

Remark: If $\kappa \geq |\Omega|$, the solution is the trivial one $\omega = \Omega$.

Remark: This problem has been specially studied if $N = 2, \tilde{f} = 1$

(F. Murat - L. Tartar 1985, J. Goodman - R.V. Kohn - L. Reyna 1986)

Assuming Ω simply connected, it consists in mixing two isotropic elastic materials in the cross-section of a beam in order to minimize the torsion.

It also applies to the optimal arrangement of two viscous fluids moving parallel to the axis of a pipe (Poiseuille flow) in order to maximize the flux

Using

$$\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u_\omega|^2 dx$$

$$= -\left(\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u_\omega|^2 dx - 2 \prec \tilde{f}, u_\omega \succ\right)$$

$$= -\min_{u\in H_0^1(\Omega)} \left(\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u|^2 dx - 2 \prec \tilde{f}, u \succ\right).$$

The problem can be stated as

$$\min_{\substack{u\in H_0^1(\Omega)\\ |\omega|\le\kappa}} \left(\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u|^2 dx - 2 \prec \tilde{f}, u \succ\right)$$

F. Murat (1972): This type of problems has not solution in general. Thus, it it is usual to work with a relaxation.

F. Murat, L. Tartar (1985). A relaxation is given by replacing $\alpha\chi_\omega + \beta(1-\chi_\omega)$ by the armonic mean value of α and β with proportions θ and $1-\theta$, with $\theta \in L^\infty(\Omega;[0,1])$, i.e.

$$\min_{\substack{u\in H_0^1(\Omega) \\ \theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa}} \left(\int_\Omega \frac{\alpha\beta|\nabla u|^2}{\beta\theta+\alpha(1-\theta)}dx - 2 \prec \tilde{f}, u \succ\right)$$

$$= \beta \min_{\substack{u\in H_0^1(\Omega) \\ \theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa}} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta}dx - 2 \prec f, u \succ\right)$$

or

$$\begin{cases} \displaystyle\max_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa} \int_\Omega \frac{|\nabla u_\theta|^2}{1+c\theta}dx \\ -\mathrm{div}\left(\dfrac{\nabla u_\theta}{1+c\theta}\right) = f \text{ in } \Omega, \quad u_\theta = 0 \text{ on } \partial\Omega \end{cases}$$

$$c = \frac{\beta-\alpha}{\alpha},\ f = \frac{1}{\beta}\tilde{f}$$

Another formulation (F. Murat, L. Tartar (1985)).

Recall: If u_θ is the solution of

$$-\mathrm{div}\frac{\nabla u_\theta}{1+c\theta} = f \text{ in } \Omega, \qquad u_\theta = 0 \text{ on } \partial\Omega.$$

Then, $\sigma_\theta = \frac{\nabla u_\theta}{1+c\theta}$ is the solution of $\min\limits_{\substack{\sigma\in L^2(\Omega)^N \\ -\mathrm{div}\sigma = f \text{ in } \Omega}} \int_\Omega (1+c\theta)|\sigma|^2 dx.$

$$\text{Thus} \quad \min_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\, dx\le\kappa} \; \min_{u\in H_0^1(\Omega)} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta}dx - 2 \prec f, u \succ\right)$$

$$= - \max_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\, dx\le\kappa} \; \min_{\substack{\sigma\in L^2(\Omega)^N \\ -\mathrm{div}\sigma = f \text{ in } \Omega}} \int_\Omega (1+c\theta)|\sigma|^2 dx$$

$$= - \min_{\substack{\sigma\in L^2(\Omega)^N \\ -\mathrm{div}\sigma = f \text{ in } \Omega}} \; \max_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\, dx\le\kappa} \int_\Omega (1+c\theta)|\sigma|^2 dx$$

Remark:

The functional $\sigma \mapsto \displaystyle\max_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa} \int_\Omega (1+c\theta)|\sigma|^2 dx$

is strictly convex. So the problem

$$\min_{\substack{\sigma\in L^2(\Omega)^N \\ -\mathrm{div}\sigma=f \text{ in } \Omega}} \ \max_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa} \int_\Omega (1+c\theta)|\sigma|^2 dx$$

has a unique solution $\hat{\sigma}$, i.e. although the solution $\left(\hat{\theta},\hat{u}\right)$ of

$$\min_{\substack{u\in H_0^1(\Omega) \\ \theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa}} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta} dx - 2 \prec f, u \succ\right)$$

can be not unique, $\hat{\sigma} = \frac{\nabla\hat{u}}{1+c\hat{\theta}}$ is unique.

Taking the minimum in θ in

$$\min_{u\in H_0^1(\Omega)}\ \min_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa}\left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta}dx-2\prec f,u\succ\right),$$

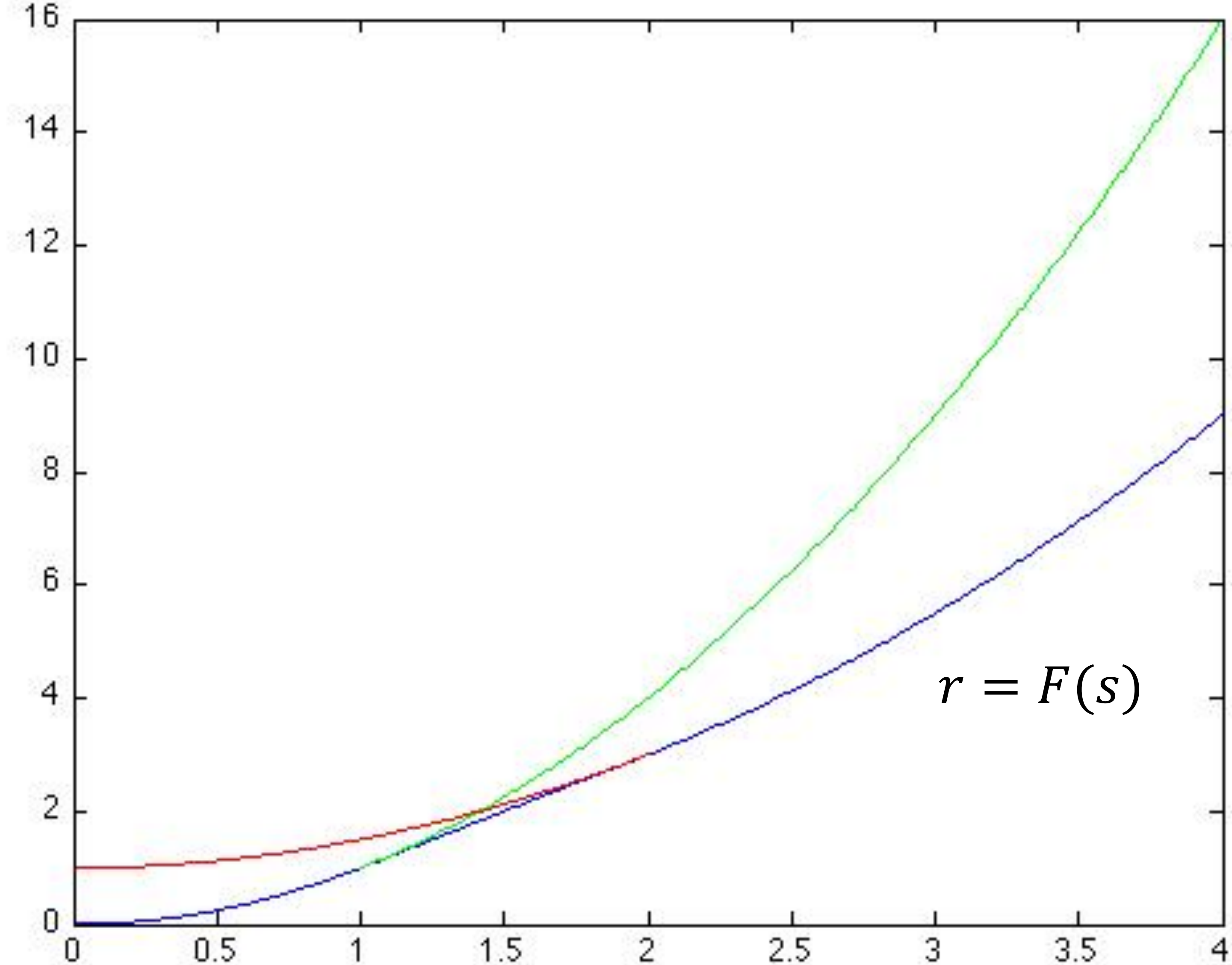
we deduce the existence of $\mu>0$ such that u is a solution of

$$\min_{u\in H_0^1(\Omega)}\left(\int_\Omega F(|\nabla u|)dx-2\prec f,u\succ\right)$$

with $F\in W^{2,\infty}(0,\infty)$ given by

$$2F(s)=\begin{cases} s^2 & \text{if } 0\le s\le\mu\\ 2\mu s-\mu^2 & \text{if } \mu\le s\le(1+c)\mu\\ \dfrac{s^2}{1+c}+\mu^2 & \text{if } (1+c)\mu\le s.\end{cases}$$

Besides $\theta=1$ if $|\sigma|<\mu$, $\theta=0$ if $|\sigma|>\mu$. Thus (θ,u) is unique in $\{|\sigma|\neq\mu\}$



16
14
12
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
$r = F(s)$

We have then proved that u is a solution of the nonlinear problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{F'(|\nabla u|)}{|\nabla u|}\nabla u\right) = 2f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

The main difficulty is that the problem has not good ellipticity properties to get u twice derivable.

We will prove that

$$\sigma = \frac{\nabla u}{1 + c\theta} = \frac{F'(|\nabla u|)}{2|\nabla u|}\nabla u$$

is derivable.

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Theorem (JCD): Assume $\Omega \in C^{1,1}$

$$f \in W^{-1,p}(\Omega),\ \ 2 \leq p < \infty \implies \hat{\sigma} \in L^p(\Omega)^N$$

$$f \in L^p(\Omega),\ \ N < p \implies \hat{\sigma} \in L^\infty(\Omega)^N$$

$$f \in W^{1,1}(\Omega) \cap L^2(\Omega) \implies \begin{cases} \hat{\sigma} \in H^1(\Omega)^N,\ \ P(\hat{\sigma}) = 0 \text{ on } \partial\Omega \\ \partial_i \theta \hat{\sigma}_j - \partial_j \theta \hat{\sigma}_i \in L^2(\Omega),\ \ \ 1 \leq i,j \leq N \end{cases}$$ The main

P denotes the orthogonal projection on the tangent space.

Kowhl, Stara, Wittum, 1991: Local estimates for u in $W^{1,\infty}_{loc}(\Omega)$

Proposition:
If $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$, and there exists an unrelaxed solution ($\theta = \chi_\omega$), then $\mathrm{curl}(\hat{\sigma}) = 0$.

If Ω is simply connected, $\Omega \in C^{1,1}$, then $\hat{\sigma} = \nabla w$, with w the unique solution of

$$\begin{cases} -\Delta w = f \text{ in } \Omega \\ w = 0 \text{ on } \partial\Omega \end{cases}$$

Moreover, if ω is smooth, then $\partial\omega$ must be composed by surface levels of the corresponding function u and $\frac{\partial u}{\partial \nu} = \text{constant on these surface levels.}$

Proof. It is essentially a consequence of

$$\partial_i\theta\hat{\sigma}_j - \partial_j\theta\hat{\sigma}_i \in L^2(\Omega), \quad 1 \leq i,j \leq N$$

Remark: The above conclusions appear in F. Murat, L. Tartar 1985, but assuming the solution smooth.

Theorem (F. Murat, L. Tartar): $\Omega \subset \mathbb{R}^2$ smooth, simply connected. We assume that the problem

$$\max_{|\omega| \leq \kappa} \int_{\Omega} (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^2 dx$$

$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))\nabla u_\omega\big) = 1 \text{ in } \Omega \\ u_\omega = 0 \text{ on } \partial\Omega \end{cases}$$

has a solution and that the interfaces are smooth. Then Ω is a circle.

Theorem (JCD): The previous results allow us to eliminate the assumption that the interfaces are smooth. The result holds in $\mathbb{R}^N, N \geq 2$, assuming Ω smooth, simply connected, with connected boundary.

Sketch of the proof: If there exists a solution (ω, u), then

$$(\alpha\chi_\omega + \beta(1-\chi_\omega))\nabla u = \nabla w$$

with w solution of

$$\begin{cases} -\Delta w = 1 \text{ in } \Omega \\ w = 0 \text{ on } \partial\Omega. \end{cases}$$

Moreover, $\exists \mu > 0$ such that

$$\{|\nabla w| < \mu\} \subset \omega \subset \{|\nabla w| \leq \mu\}.$$

Using

$$-\Delta|\nabla w|^2 = -2|D^2 w|^2 \text{ in } \Omega$$

we can use Hopf's Lemma to find $x_0 \in \Omega$, with

$$|\nabla w|(x_0) = \mu, \qquad \nabla|\nabla w|(x_0) \neq 0.$$

Then $\{|\nabla w| = \mu\}$ is an analytic manifold in a neighborhood of x_0.

By the optimality conditions, it agrees with $\{w = q\}$, for some $q > 0$.

Thanks to the analyticity, $\exists$ a connected component E of $\{w = q\}$, where $\{|\nabla w| = \mu\}$. We prove it is an analytic variety manifold without boundary.

By Jordan-Brower's Theorem $E = \partial C, \bar{C} \subset \Omega$, C an open set with analytic boundary.

Then, w satisfies

$$\begin{cases} -\Delta w = 1 \text{ in C} \\ w = q, \quad \dfrac{\partial w}{\partial n} = \pm\mu \text{ on } \partial \text{C}. \end{cases}$$

Here we follow Murat-Tartar's proof:
By Serrin's Theorem C is a ball and w is radial in C. By analyticity, w is in fact radial in Ω.

Then, Ω is a ball.

Return to the eigenvalue problem

$$(\Lambda_m) \qquad \min_{|\omega|\leq\kappa} \min_{u\in H_0^1(\Omega)} \frac{\int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

Remark: For $A \in L^\infty(\Omega)^N$, elliptic,

$$\lambda_1(A) = \min_{u\in H_0^1(\Omega)} \frac{\int_\Omega A\nabla u\cdot\nabla u\, dx}{\int_\Omega |u|^2 dx}$$

can be characterized as $$\frac{1}{\lambda_1(A)} = \max_{\substack{-\mathrm{div}(A\nabla u)=f\\ u\in H_0^1(\Omega)\\ \|f\|_{L^2(\Omega)}\leq 1}} \int_\Omega A\nabla u\cdot\nabla u\, dx$$

$$= -\min_{\substack{u\in H_0^1(\Omega)\\ \|f\|_{L^2(\Omega)}\leq 1}} \left(\int_\Omega A\nabla u\cdot\nabla u\, dx - 2\int_\Omega fu\, dx\right).$$

Thus, we have the relaxed formulation

$$(\Lambda_m) \quad \min_{\|f\|_{L^2(\Omega)}\leq 1} \min_{\substack{u\in H_0^1(\Omega) \\ \int_\Omega \theta dx \leq \kappa}} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta}\,dx - 2\int_\Omega fu\,dx\right) \qquad c = \frac{\beta-\alpha}{\alpha}$$

The regularity results for the compliance problem can then be applied.

Theorem: Assume $\Omega \in C^{1,1}$, then

$$\sigma = \frac{\nabla u}{1+c\theta} \in H^1(\Omega)^N \cap L^\infty(\Omega)^N, \quad \partial_i\theta\sigma_j - \partial_j\theta\sigma_i \in L^2(\Omega), \quad 1 \leq i,j \leq N.$$

Theorem: Assume there exists an unrelaxed solution χ_ω for (Λ_m). Then,

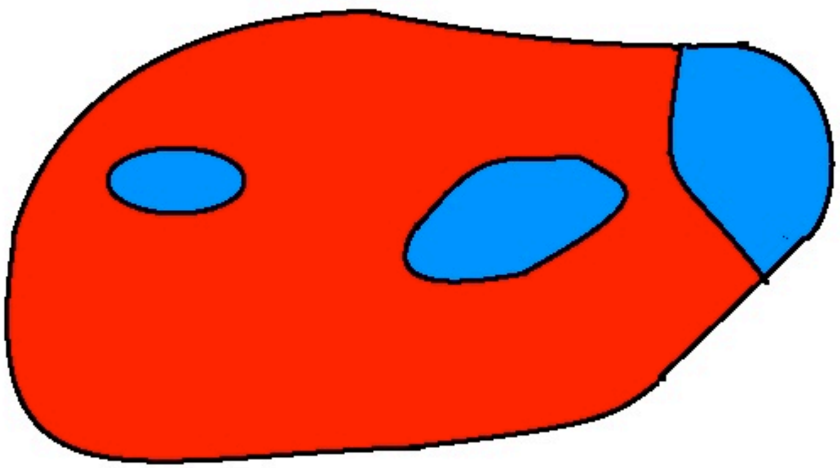
$$\sigma = \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)\nabla u \in W^{2,p}(\Omega), \quad \forall p \in [1,\infty), \quad \mathrm{curl}\sigma = 0$$

Moreover, if there exist two open sets $O \Subset U \subset \Omega$, $O \in C^2$, such that $\chi_\omega = r$ in $O, \chi_\omega = 1 - r$ in $U \backslash O$. Then, O is a sphere.

Proof.

It is a consequence of $\begin{cases} -\Delta u = \lambda_1 u \quad \text{in } O \\ u = \text{constant on } \partial O, \ \ \dfrac{\partial u}{\partial \nu} = \text{constant on } \partial O. \end{cases}$

and Serrin's theorem.



It would be only possible if the interior blue zones were circles

Counterexample: $\Omega=\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)^{N-1}$, $\alpha=1,\beta=2.$ For $\varepsilon>0$ small enough the solutions θ of

$$\min\left\{\frac{\int_\Omega \frac{|\nabla u|^2}{1+\theta}dx}{\int_\Omega |u|^2dx} : u\in H_0^1(\Omega), \theta\in L^\infty(\Omega,[0,1]), \int_\Omega \theta dx\leq|\Omega|-\varepsilon\right\}$$

is not a characteristic

Proof. If $(\chi_{\omega_\varepsilon}, u_\varepsilon)$ were a solution then $u_\varepsilon\approx\cos(2x_1)\prod_{j=2}^N\cos(x_j)$.

$\exists$ a smooth connected component O_ε of $\Omega\backslash\omega_\varepsilon$,

$$O_\varepsilon\approx\left\{\frac{x_1^2}{8}+\sum_{i=2}^N\frac{x_i^2}{2}=1-c_\varepsilon\right\},\qquad c_\varepsilon\searrow 0$$

Remark: The properties of Ω we use are that Ω is simply connected and that the positive eigenfunction corresponding to the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u \ \text{ in } \ \Omega \\ \ u = 0 \text{ on } \partial\Omega \end{cases}$$

attains his maximum in a unique point x_0 and $D^2u(x_0)$ is regular and non-scalar.
Using symmetry arguments this can be proved for example if Ω is an ellipsis, which is not a circle.

Theorem: (A. Alvino, P.L. Lions, G. Trombetti, 1987). If Ω is a ball in $\mathbb{R}^N$ there exists a solution for the (unrelaxed) eigenvalue problem. Moreover it is radial.

The exact form of the solution for a ball is a problem which has been considered by several authors

C. Conca, A. Laurain, R. Mahadevan, A. Mohammadi, L. Sanz, M. Yousefnezhad.,…

It seems to be an open problem.

Theorem (JCD): Assume $\Omega \in C^{1,1}$, simply connected, with connected boundary. If the eigenvalue problem has an optimal solution then Ω is a ball.

Sketch of the proof: Assume (ω, u) a solution. We know

$$(\alpha\chi_\omega + \beta(1-\chi_\omega))\nabla u = \nabla w$$

with w solution of

$$\begin{cases} -\Delta w = \lambda u & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\exists \mu > 0$ such that

$$\{|\nabla w| < \mu\} \subset \omega \subset \{|\nabla w| \leq \mu\}.$$

Using that locally $u = \psi(w)$, we show that $\forall x_0 \in \Omega,$ such that

$$|B(x_0, r) \cap \omega|, |B(x_0, r) \cap (\Omega \backslash \omega)| > 0, \quad \forall r > 0$$

$\exists$ C open of class $W^{3,p}$, $\forall p > 1$ with $\bar{C} \subset \Omega$, such that on $\partial C, w$ is constant, $x_0 \in \Omega$, $|\nabla w| = \mu$, and $\exists \psi$ Lipschitz with $u = \psi(w)$ in a neighborhood of ∂C.

The arguments are something different of the previous ones. We do not have analyticity and $\Delta|\nabla w|^2$ has not a determined sign.

We take C minimal in the sense $\nexists$ another set contained in C in these conditions. Then, w satisfies

$$\begin{cases} -\Delta w = \lambda\psi(w) \text{ in C} \\ w = q, \quad \dfrac{\partial w}{\partial n} = \pm\mu \text{ on } \partial\text{C}. \end{cases}$$

for some ψ Lipschitz.. Moreover, $u = \psi(w)$ in a neighborhood of $\bar{C}$.
By Serrin's Theorem C $= B(x_0, r)$ and w is radial in C.

Now, we define R by

$$R = \sup\{r > 0 \colon w \text{ is radial in } B(x_0, r),\ \overline{B(x_0, r)} \subset \Omega,$$

$$u = \psi(w) \text{ neighborhood of } \overline{B(x_0, r)}\}.$$

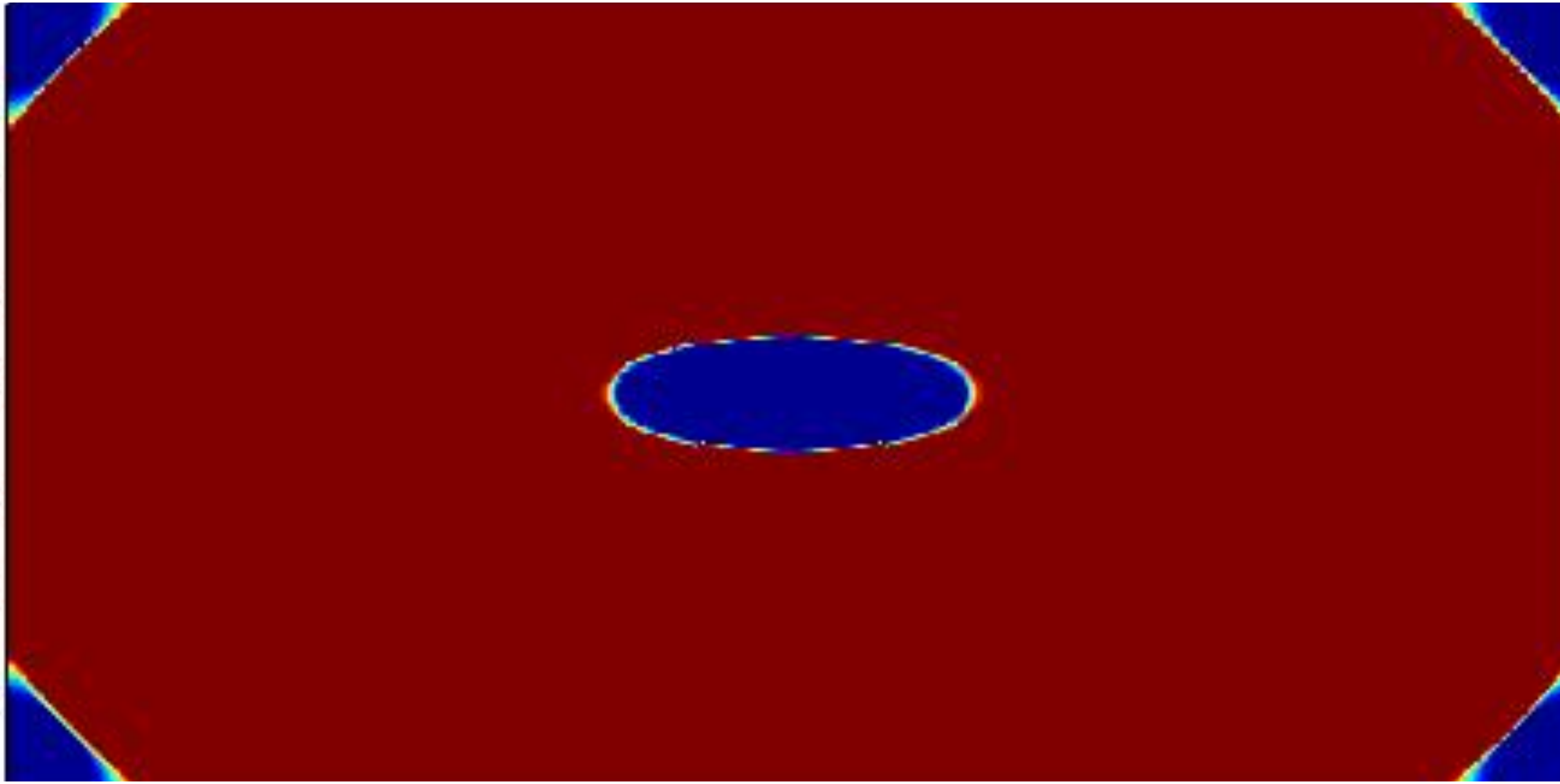
By a unique continuation argument, we show $\Omega = \overline{B(x_0, R)}$.

Numerical experiments.

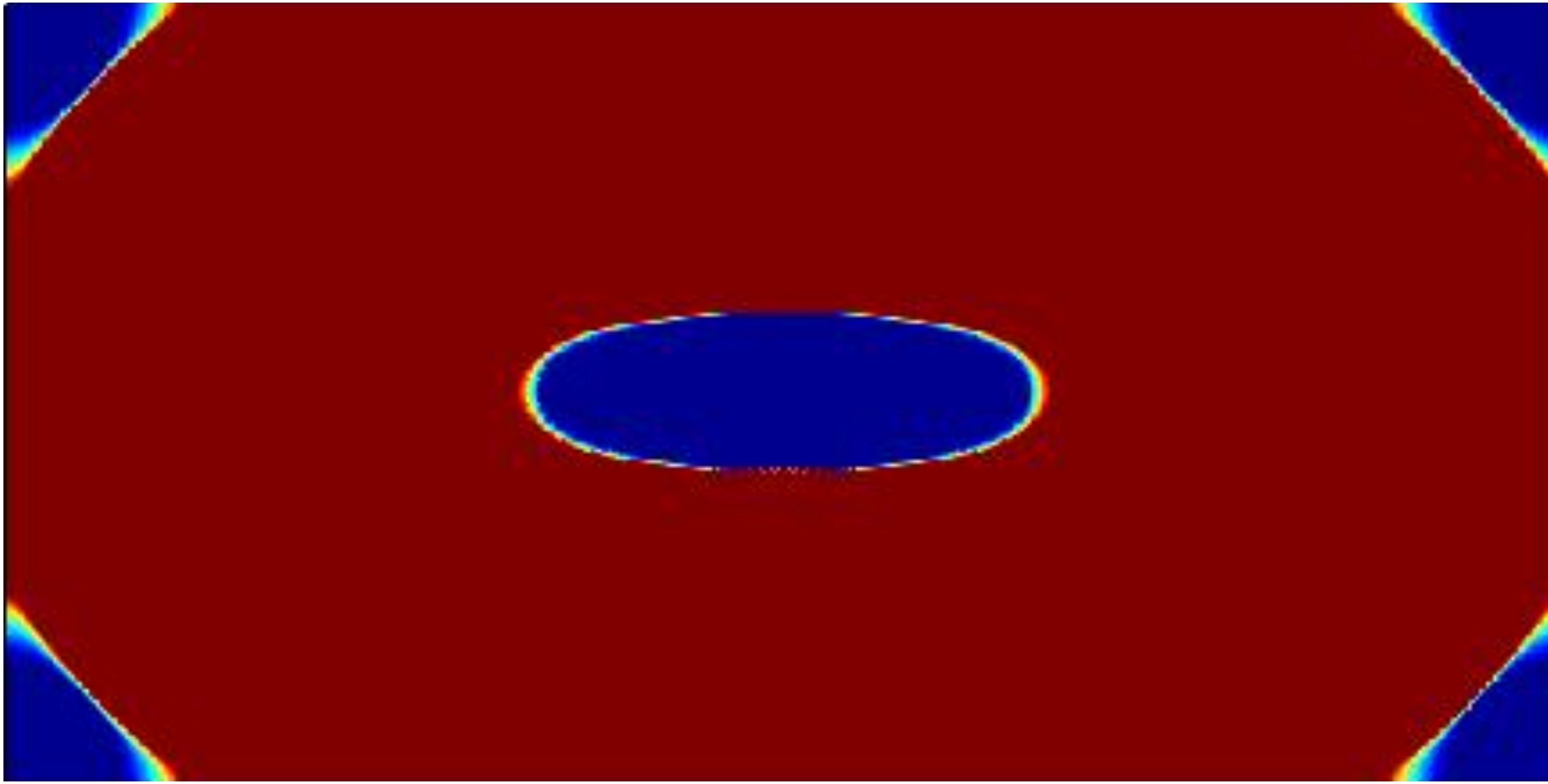
Problem $\Omega = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $|\Omega| \approx 4{,}935$, $\alpha = 1, \beta = 2$

$$\min \frac{\int_\Omega \frac{|\nabla u|^2}{1+\theta}\,dx}{\int_\Omega |u|^2\,dx}$$

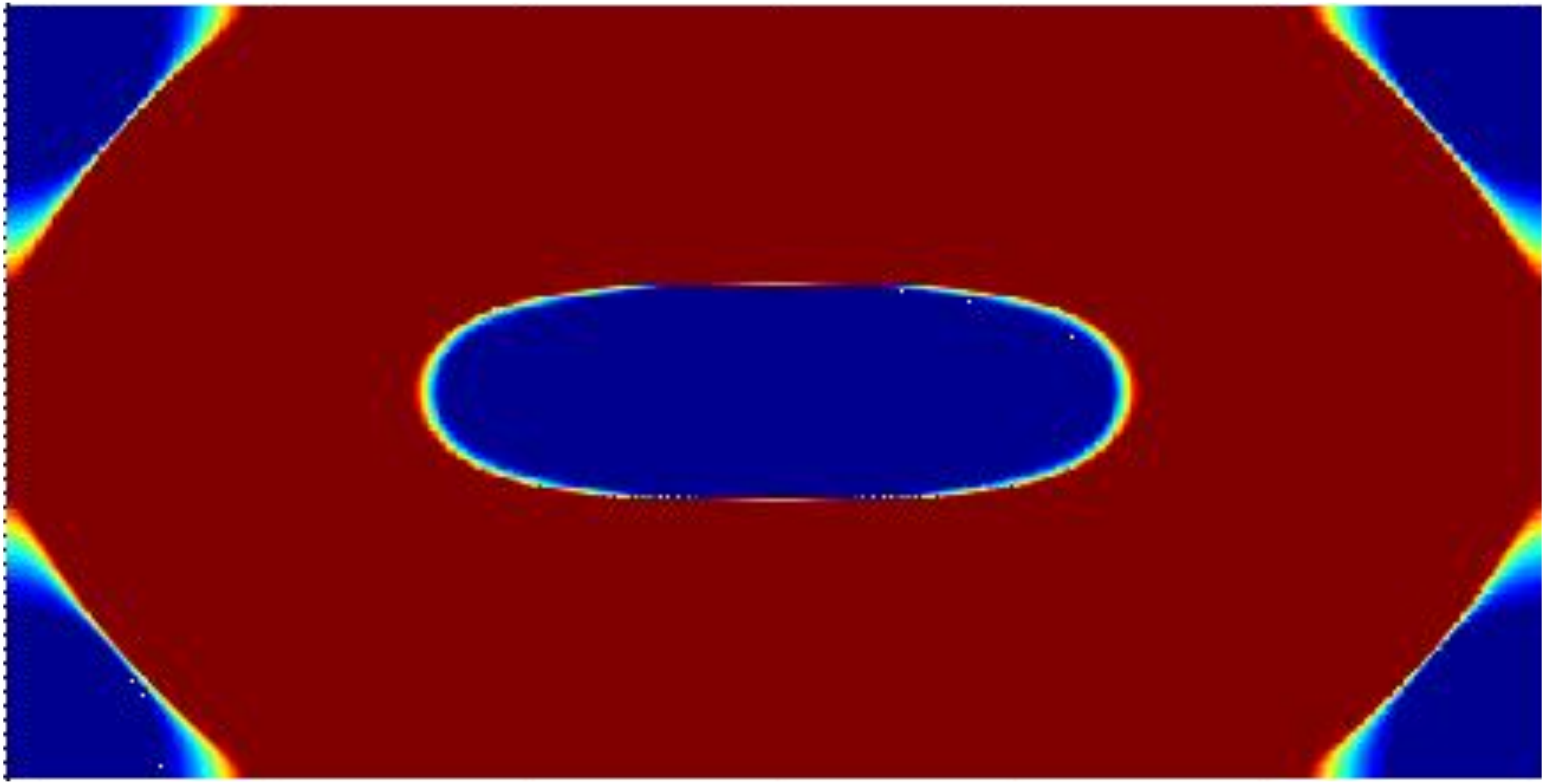
$$u \in H_0^1(\Omega), \int_\Omega \theta\,dx \le \kappa$$



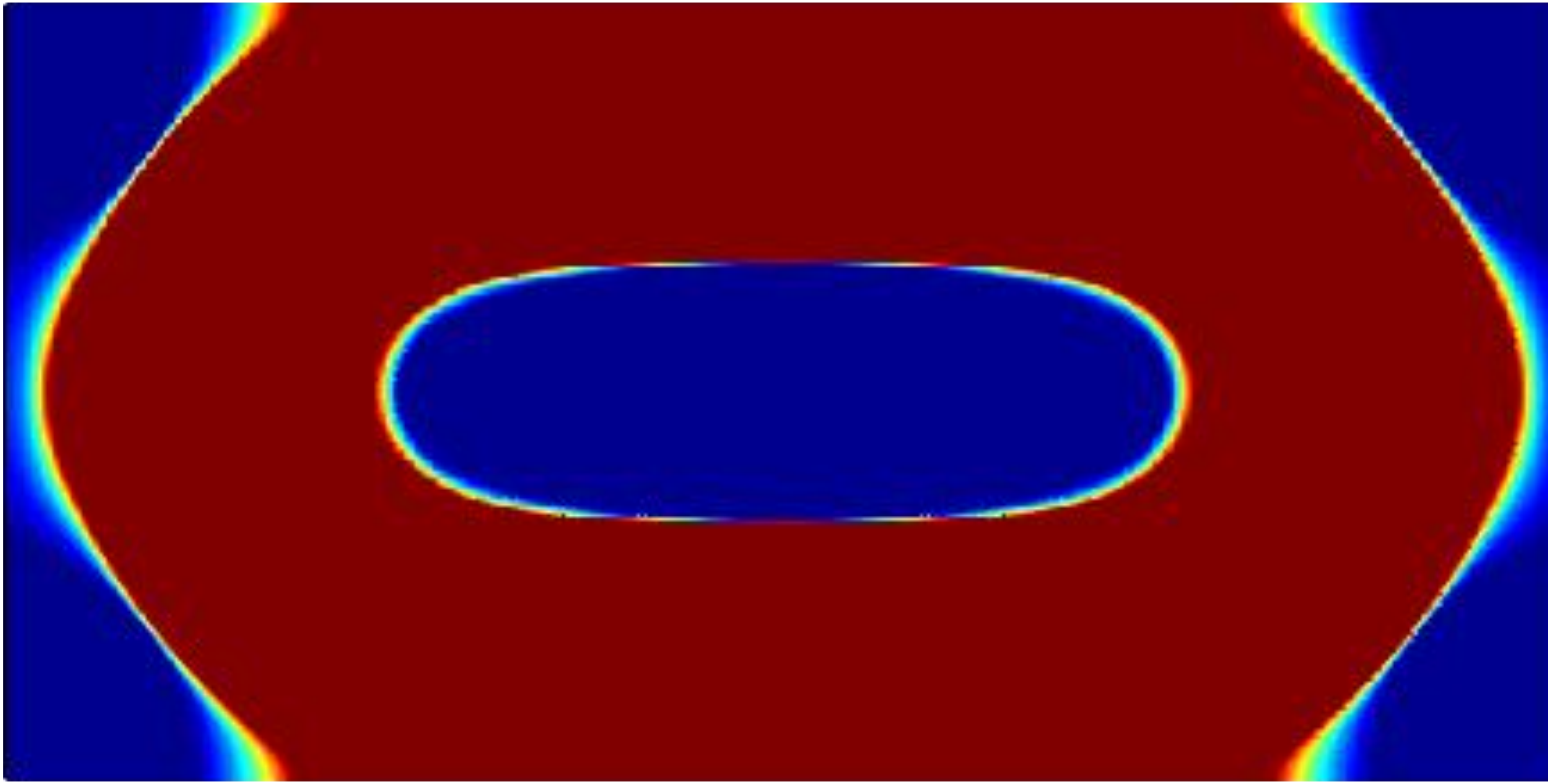
$\alpha = 1, \beta = 2, \kappa = 4.685$



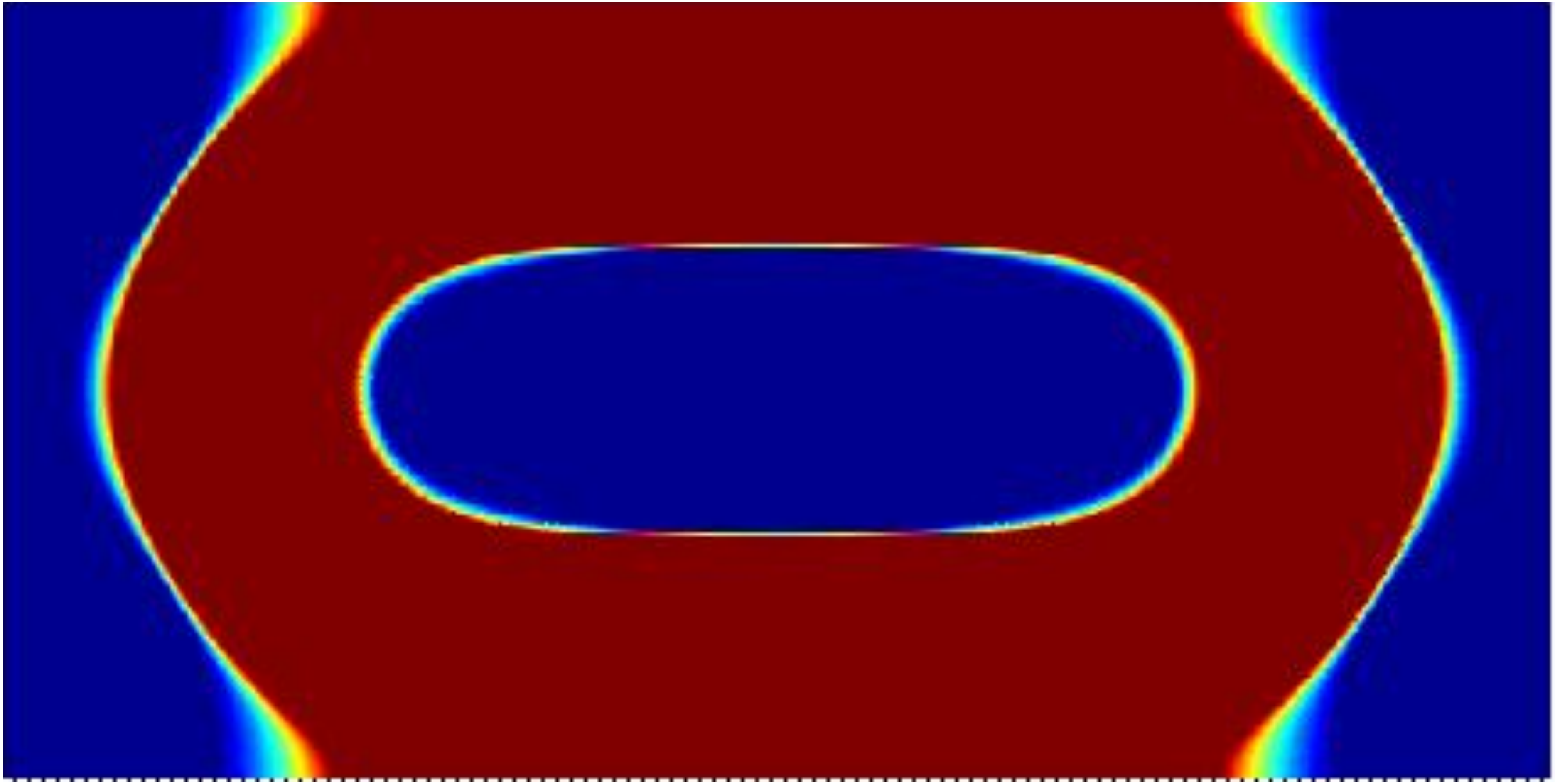
$\alpha = 1, \beta = 2, \kappa = 4.435$



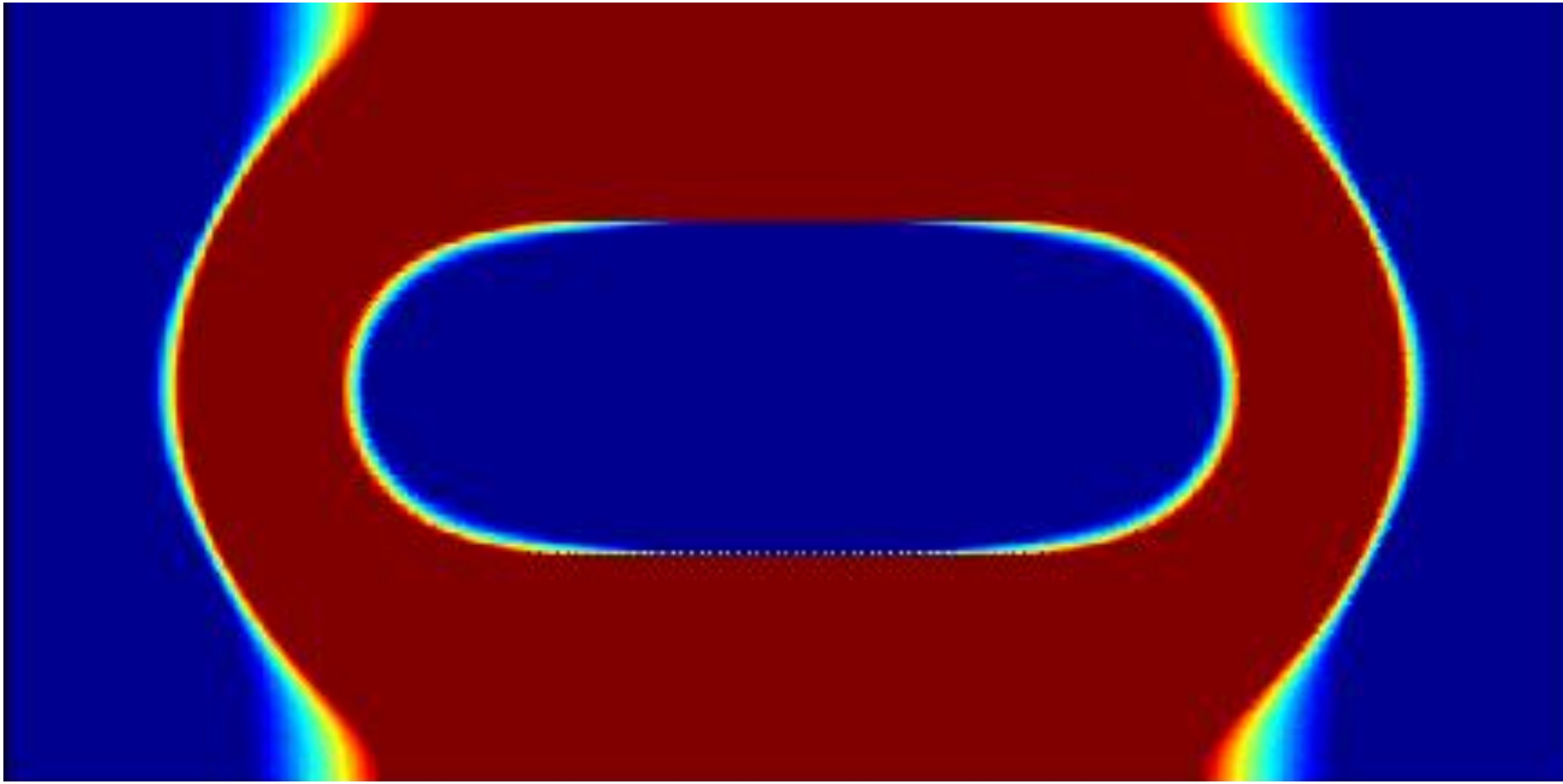
$\alpha = 1, \beta = 2, \kappa = 3.935$



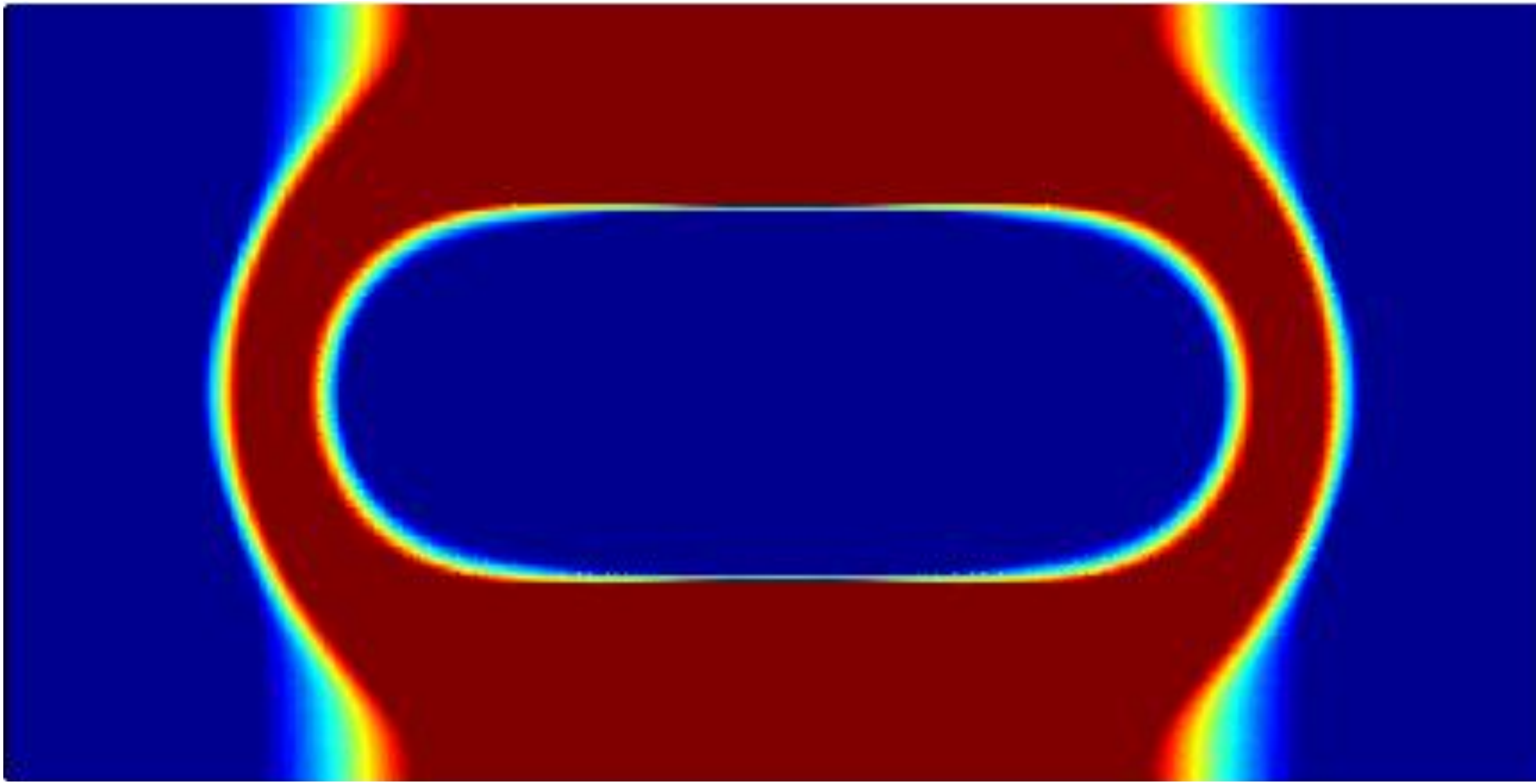
$\alpha = 1, \beta = 2, \kappa = 3.435$



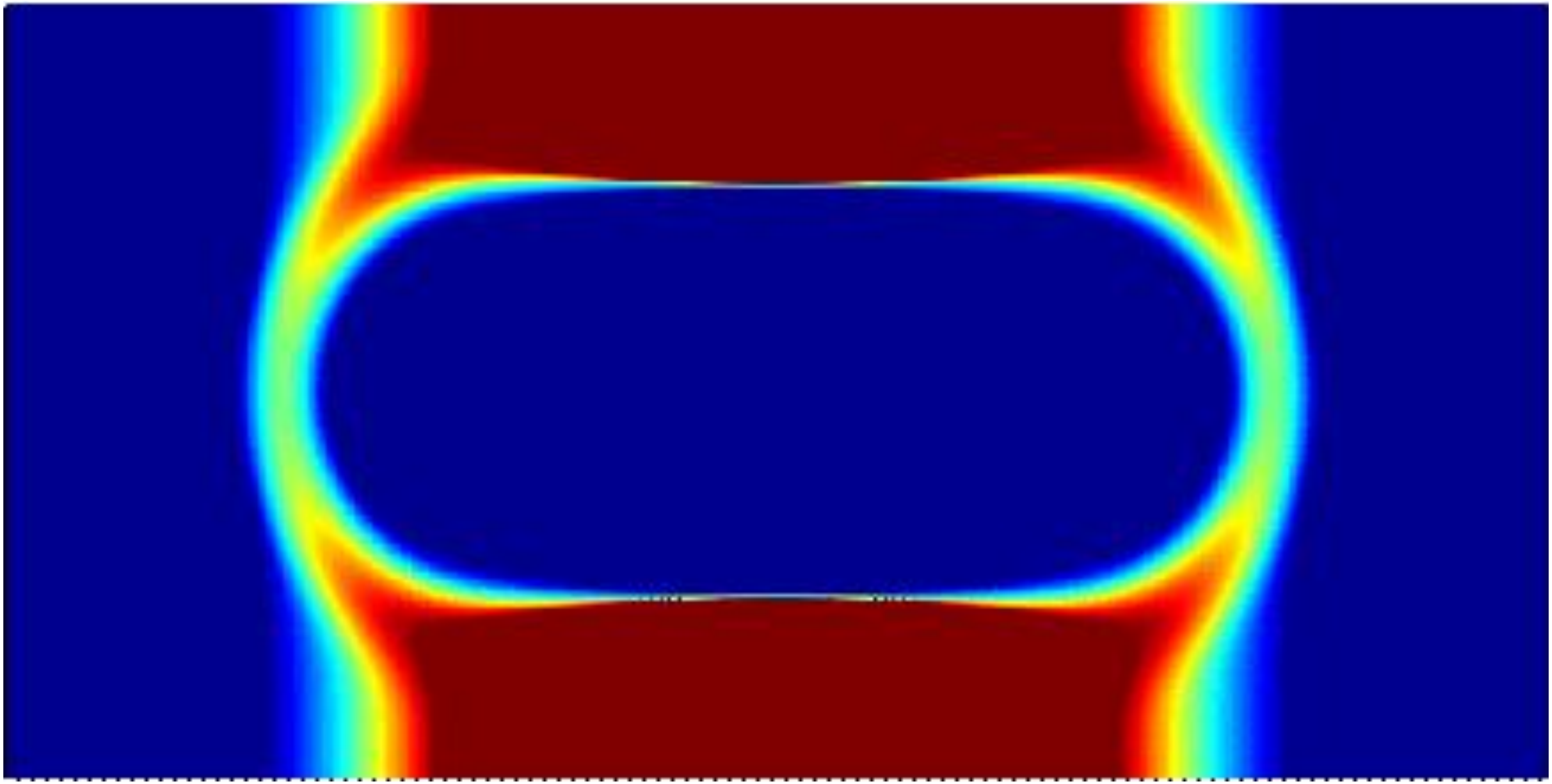
$\alpha = 1, \beta = 2, \kappa = 2.935$



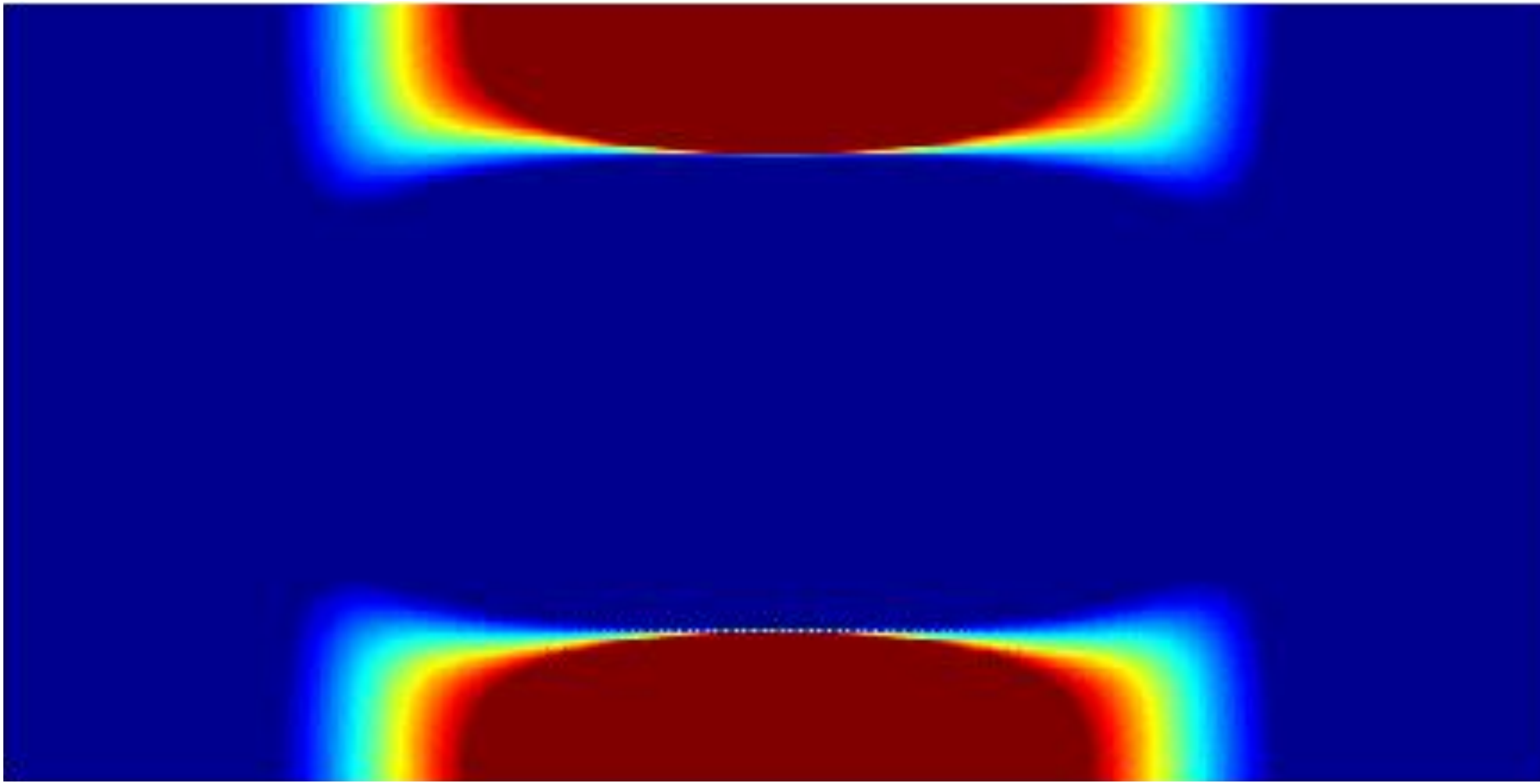
$\alpha = 1, \beta = 2, \kappa = 2.435$



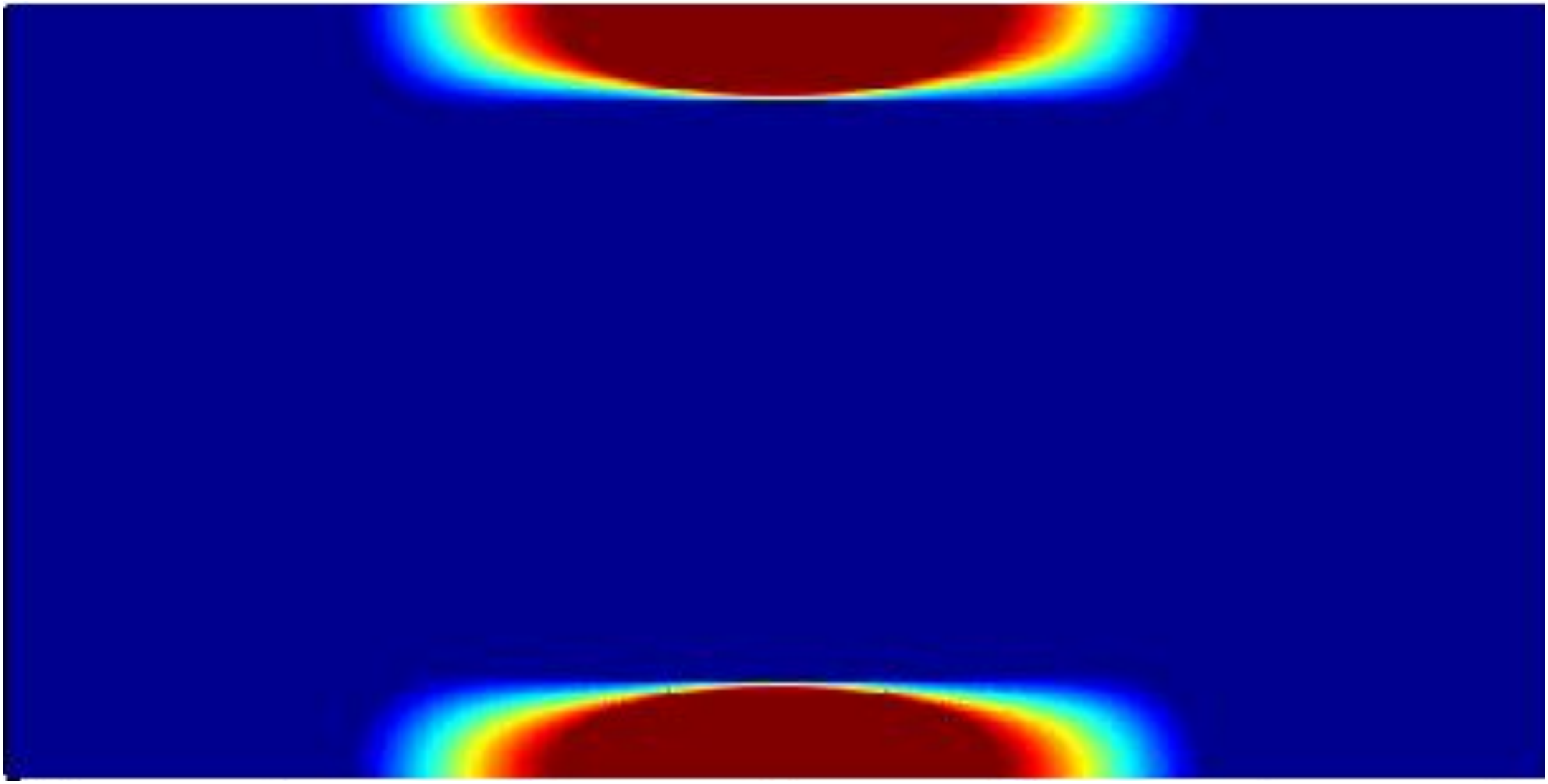
$\alpha = 1, \beta = 2, \kappa = 1.935$



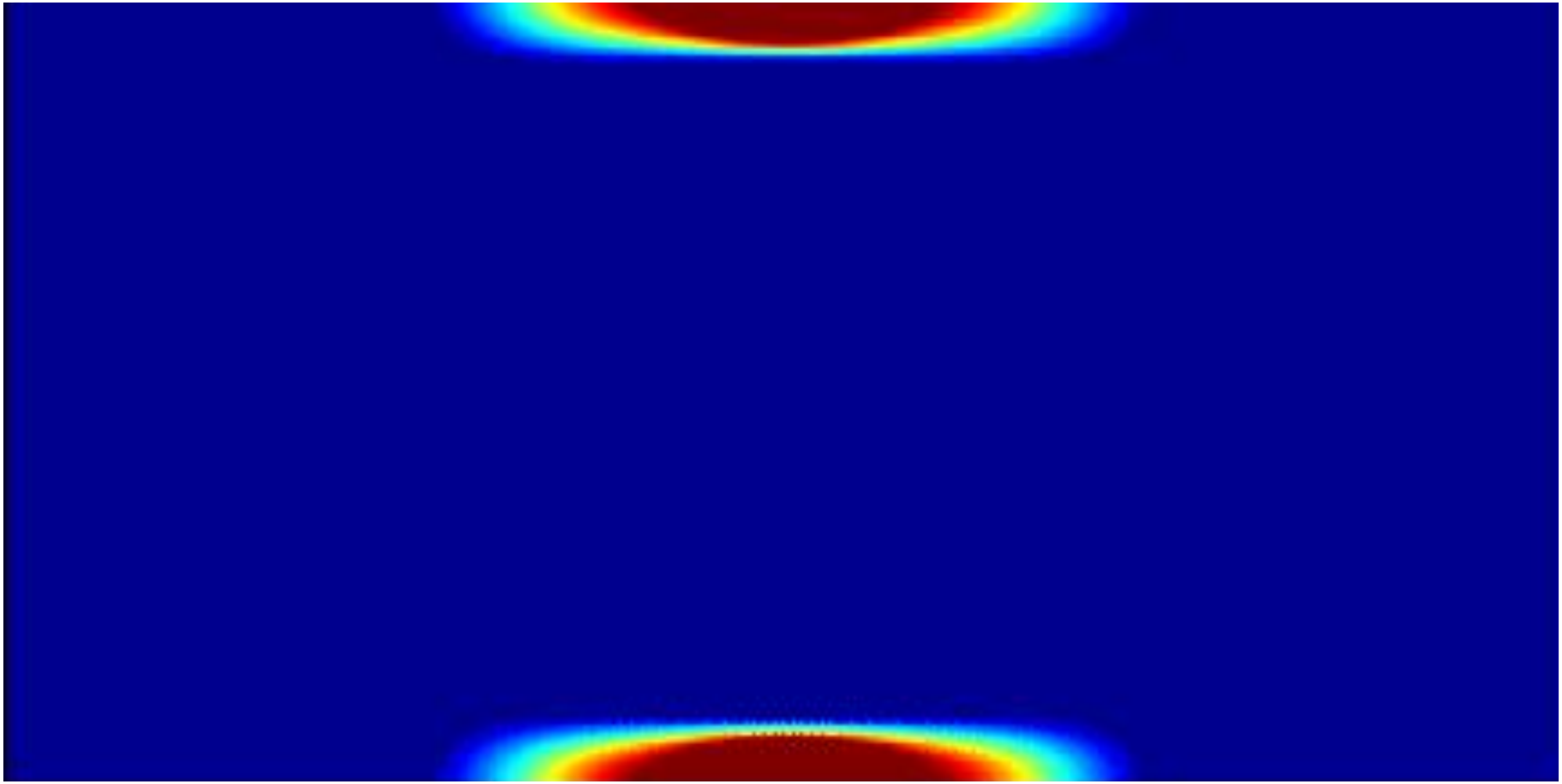
$\alpha = 1, \beta = 2, \kappa = 1.435$



$\alpha = 1, \beta = 2, \kappa = 0.935$



$\alpha = 1, \beta = 2, \kappa = 0.435$

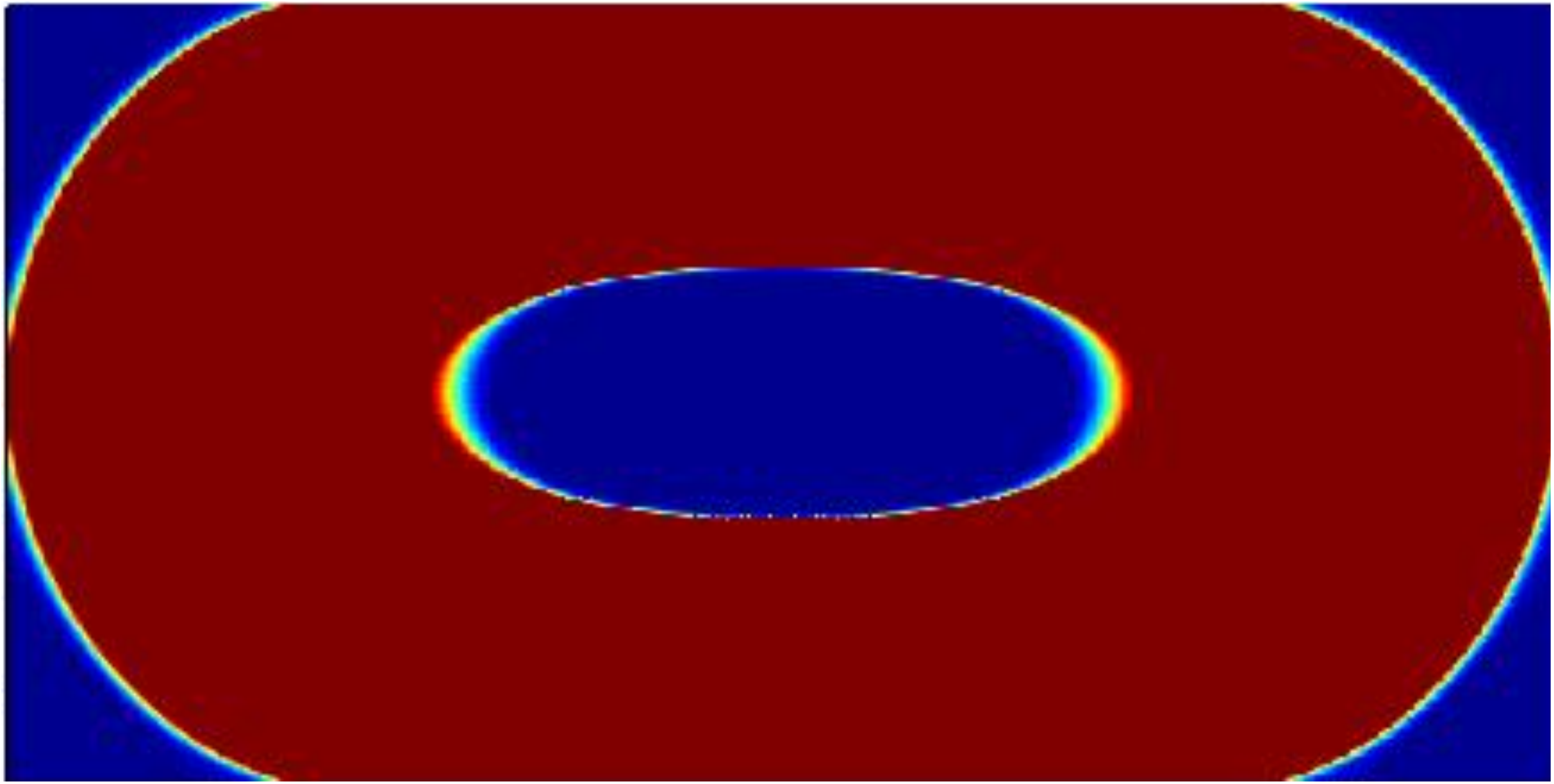


$\alpha = 1, \beta = 2, \kappa = 0.46$

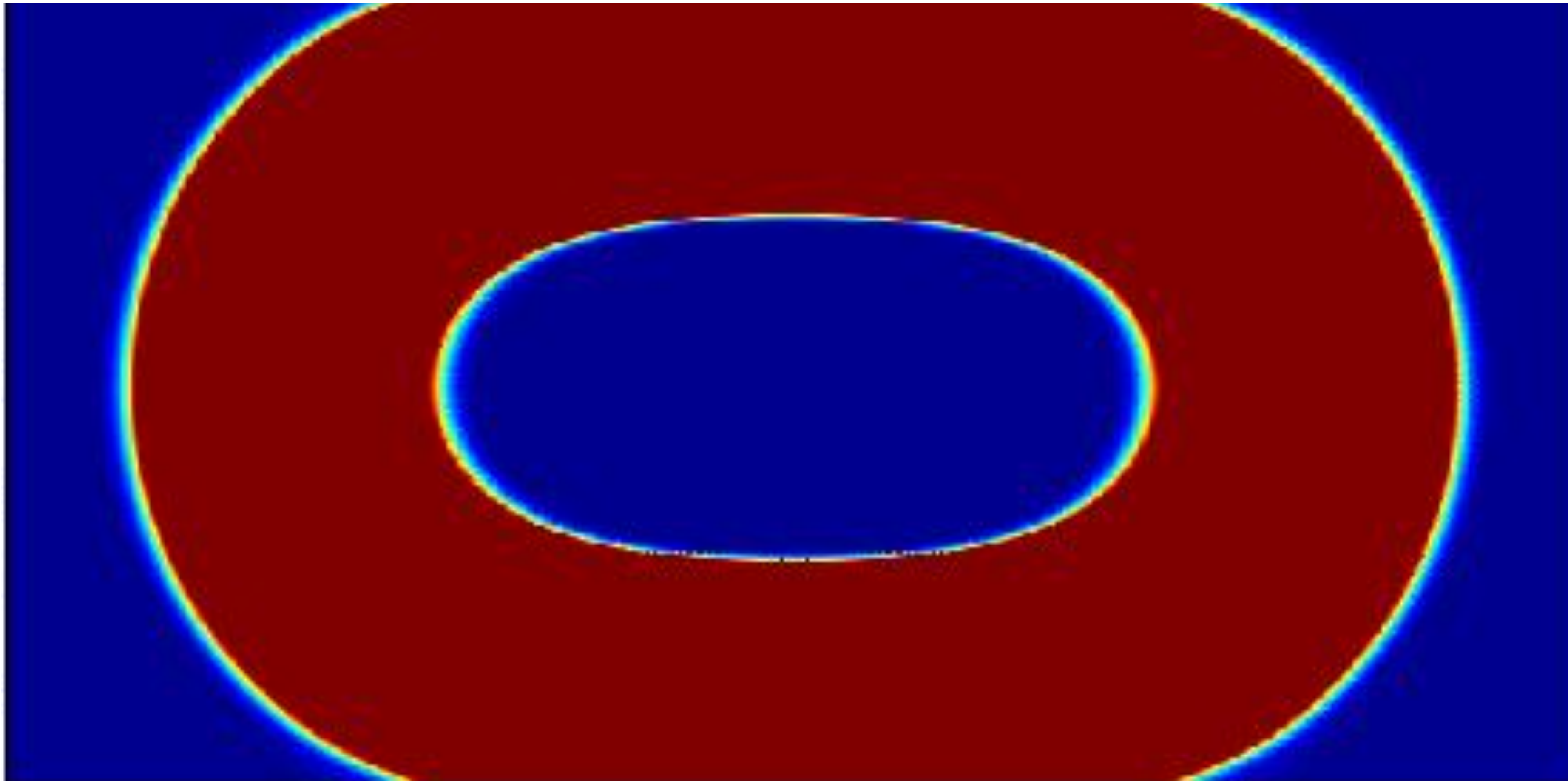
Problem $\Omega = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right),\ |\Omega| \approx 4{,}935,\ \alpha = 1, \beta = 20$

$$\min \frac{\int_\Omega \frac{|\nabla u|^2}{1 + 19\theta} dx}{\int_\Omega |u|^2 dx}$$

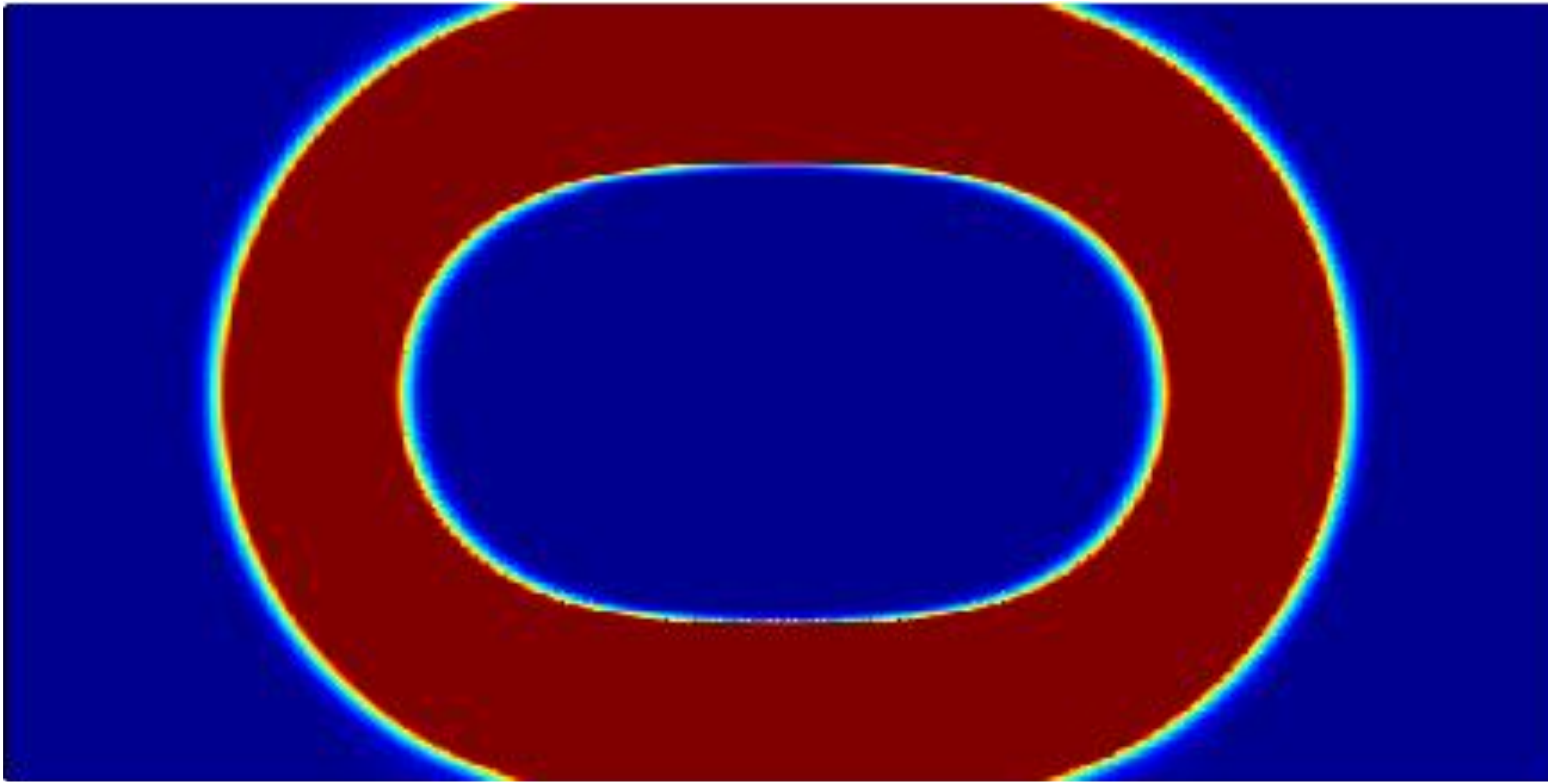
$$u \in H_0^1(\Omega), \int_\Omega \theta dx \leq \kappa$$



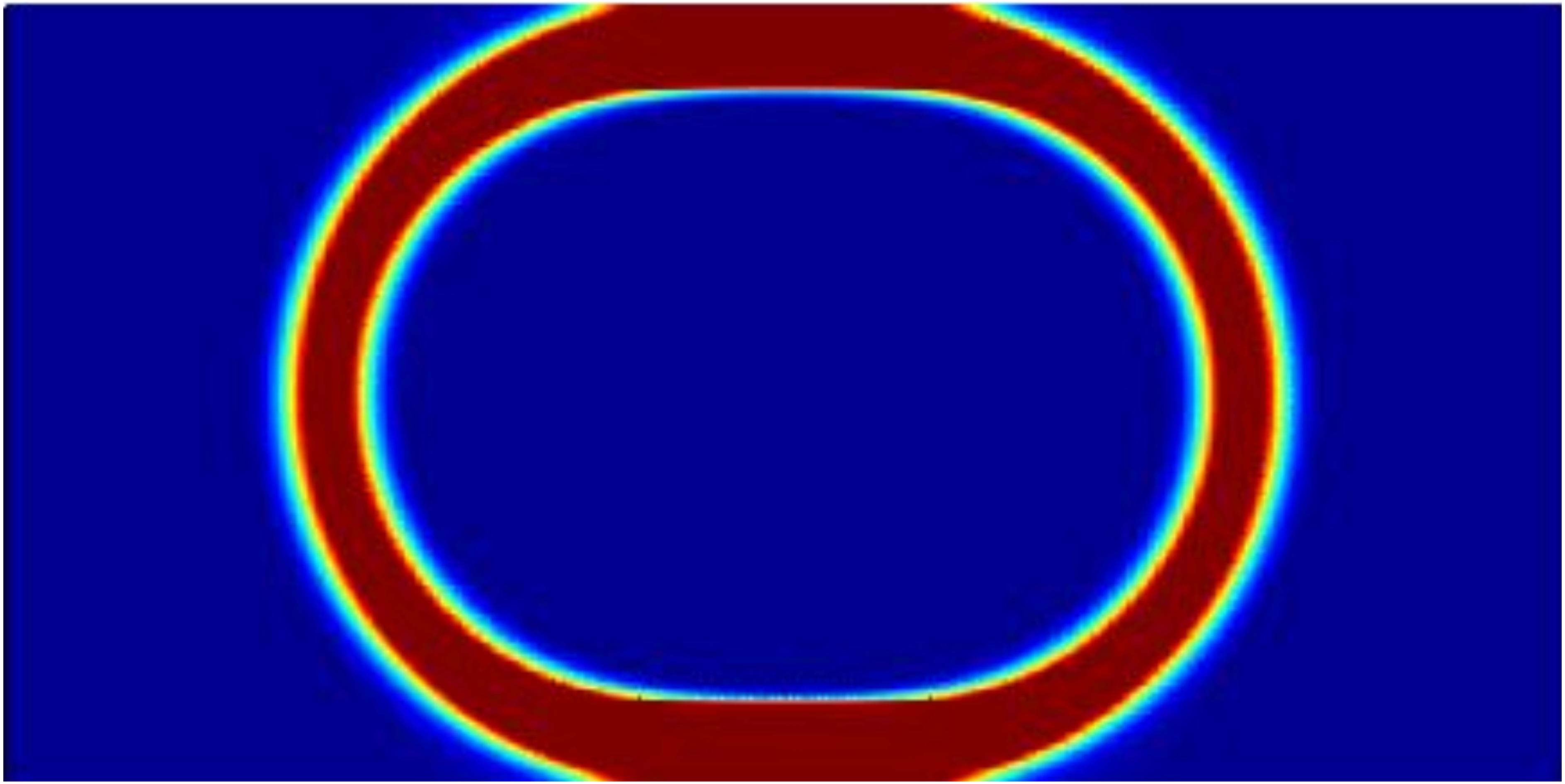
$\alpha = 1, \beta = 20, \kappa = 3.935$



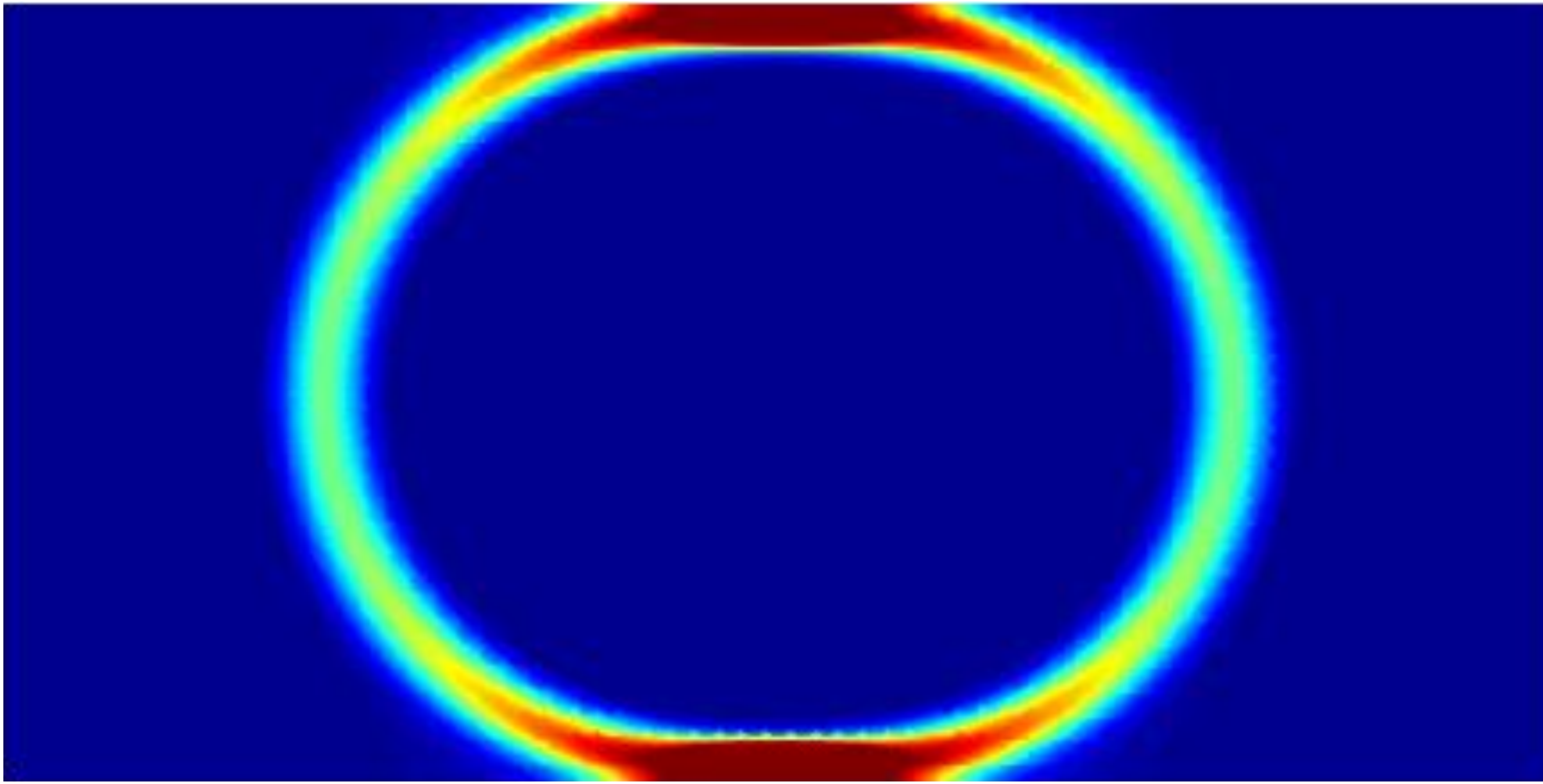
$\alpha = 1, \beta = 20, \kappa = 2.935$



$\alpha = 1, \beta = 20, \kappa = 1.935$



$\alpha = 1, \beta = 20, \kappa = 0.935$



$\alpha = 1, \beta = 20, \kappa = 0.435$

Remark. Similar results can be obtained for the probems ($p > 1$)

$$\max_{|\omega|\leq\kappa} \int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^p dx$$

$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^{p-2}\nabla u_\omega\big) = \tilde{f} \text{ in } \Omega \\ u_\omega = 0 \text{ on } \partial\Omega \end{cases}$$

and

$$\min_{|\omega|\geq\kappa} \int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^p dx$$

$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^{p-2}\nabla u_\omega\big) = \tilde{f} \text{ in } \Omega \\ u_\omega = 0 \text{ on } \partial\Omega, \end{cases}$$

which admit the relaxed formulations

$$\min_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta dx\le\kappa} \ \min_{u\in H_0^1(\Omega)} \left(\int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} dx - p \prec f,u \succ\right)$$

with $c = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}} - 1, \quad f = \frac{1}{\beta}\tilde{f}$

and

$$\max_{\substack{\theta\in L^\infty(\Omega;[0,1]) \\ \int_\Omega \theta dx\ge\kappa}} \ \min_{u\in H_0^1(\Omega)} \left(\int_\Omega (1-c\theta)|\nabla u|^p dx - p \prec f,u \succ\right)$$

with $c = \frac{\beta-\alpha}{\beta}, \quad f = \frac{1}{\beta}\tilde{f}$