The Obstacle Problem for the Total Variation Flow

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The total variation flow

$\Omega \subset \mathbb{R}^n$ bounded domain, $T > 0$, $\Omega_T := \Omega \times (0, T)$

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 \quad \text{in } \Omega_T \tag{1}$$

Boundary condition on the parabolic boundary:

$$u = u_o \quad \text{on } \partial_{\mathrm{par}}\Omega_T$$

where

$$\partial_{\mathrm{par}}\Omega_T = \big(\Omega \times \{0\}\big) \cap \big(\partial\Omega \times (0, T)\big)$$

and

$$u_o \colon \Omega \to \mathbb{R}$$

Existence results (non-complete list)

Since 2000 available for different notions of solutions (strong, weak, entropy, ...)

Mazon, Andreau, Casselles, Ballester, Diaz, Moll, Bellettini & Novaga, Bonforte & Figalli

Dealing with the initial value problem, i.e.

$$u(x,0) = u_o(x) \text{ on } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega \times (0,T).$$

Different notions in order to prove existence for initial data

$$u_o \in L^2(\Omega) \text{ or } L^1(\Omega)$$

In any case all notions of weak solutions rely on the Anzellotti pairing (a somewhat heavy tool from the theory of functions of bounded variation)

Recently: Bögelein & D. & Marcellini: Flows related to functionals from image reconstruction (cf. $\mathbf{TV} - L^2$)

Heuristics: An idea of Lichnevsky & Teman

Multiply by $v - u$ where $v: \Omega_T \to \mathbb{R}$ coincides with u on the lateral boundary $\partial\Omega \times (0, T)$ and integrate over Ω_T.

$$0 = \iint_{\Omega_T} u_t(v-u)\,dxdt + \iint_{\Omega_T} \frac{Du}{|Du|} \cdot (Dv - Du)\,dxdt =: \mathbf{I} + \mathbf{II}$$

We have:

$$\mathbf{I} = \iint_{\Omega_T} v_t(v-u)\,dxdt + \tfrac{1}{2}\|v(0) - u_o\|_{L^2}^2 - \tfrac{1}{2}\|(v-u)(T)\|_{L^2}^2$$

By convexity of $\xi \mapsto |\xi|$:

$$\mathbf{II} \geq \iint_{\Omega_T} \big(|Dv| - |Du|\big)\,dxdt$$

Variational formulation

Definition: A map $u\colon \Omega_T \to \mathbb{R}$ is called variational solution to the total variation flow if

$$\iint_{\Omega_T} |Du|\,dxdt \le \iint_{\Omega_T} |Dv|\,dxdt + \iint_{\Omega_T} v_t(v-u)\,dxdt \tag{2}$$

$$+ \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2} - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2}$$

holds true for any $v\colon \Omega_T \to \mathbb{R}$ with $v = u$ on $\partial\Omega \times (0,T)$. Formally, (2) and (1) are equivalent. □

Note: (2) can easily be formulated in the context of functions of bounded variation.

Function spaces

The natural space: functions with bounded variation

$$\mathrm{BV}(\Omega) := \left\{ u \in L^1(\Omega) : \|Du\|(\Omega) < \infty \right\},$$

where the total variation is defined by

$$\|Du\|(\Omega) := \sup \left\{ \int_\Omega u \operatorname{div} G \, dx : G \in C_o^1(\Omega, \mathbb{R}^n),\ \|G\|_{L^\infty} \le 1 \right\}.$$

Maps with values in $\mathrm{BV}(\Omega)$

$$v : (0, T) \to \mathrm{BV}(\Omega)$$

Problem: $\mathrm{BV}(\Omega)$ is not separable.

Facts for $\mathrm{BV}(\Omega)$

- $\mathrm{BV}(\Omega)$ is the dual of a separable Banach space X_o, i.e.

$$\mathrm{BV}(\Omega) = \mathrm{X}_o^*$$

- Elements of X_o can be written as

$$g - \operatorname{div} G \quad \text{with } g \in C_0^0(\Omega) \text{ and } G \in C_0^0(\Omega, \mathbb{R}^n).$$

Time dependent function spaces

- $v\colon (0,T) \to \mathrm{BV}(\Omega)$ is weak* measurable, iff

  $$(0,T) \ni t \mapsto \langle v(t), \varphi \rangle \quad \text{is measurable for any } \varphi \in \mathrm{X}_o$$

  Here $\langle \cdot , \cdot \rangle$ denotes the natural pairing on $\mathrm{BV}(\Omega) \times \mathrm{X}_o$.
- Natural spaces for the total variation flow:

  $$L^p_{w^*}\big(0,T;\mathrm{BV}(\Omega)\big) \quad \text{with } 1 \le p \le \infty,$$

  i.e. weak* measurable mappings $v\colon (0,T) \to \mathrm{BV}(\Omega)$ with

  $$\int_0^T \|v(t)\|^p_{\mathrm{BV}(\Omega)}\, dt < \infty.$$

Boundary values for $\mathrm{BV}(\Omega)$

- Trace operator $T_\Omega : \mathrm{BV}(\Omega) \to L^1(\partial\Omega)$ is not continuous with respect to weak* convergence; for example take $\Omega_j \Subset \Omega$ with

$$\chi_{\Omega_j} \uparrow \chi_\Omega.$$

- Idea: Consider a larger reference domain Ω^* with $\Omega \Subset \Omega^*$.
- Given $u_o \in \mathrm{BV}(\Omega^*)$ the Dirichlet boundary condition $u = u_0$ on $\partial\Omega$ is defined by requiring for $u \in \mathrm{BV}(\Omega^*)$ that

$$u = u_o \quad \text{a.e. on } \Omega^* \setminus \overline{\Omega}$$

- The affine space of all these functions is denoted by

$$\mathrm{BV}_{u_o}(\Omega) = u_o + \mathrm{BV}_0(\Omega).$$

Data

- **Initial datum:**

$$u_o \in L^2(\Omega^*) \cap \mathrm{BV}(\Omega^*)$$

- **Obstacle function:**

$$\psi \in L^2(\Omega_T^*) \cap L^1_{w^*}\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big), \quad \psi(0)\text{exists}$$

- **Compatibility:**

$$u_o \geq \psi(0) \quad \text{a.e. on } \Omega^*$$

- **Extension of u_o:**

There exists $g \in L^1_{w^*}\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big)$ with $\partial_t g \in L^2(\Omega_T^*)$

$g(0) = u_o$ a.e. on Ω and $g \geq \psi$ a.e.on Ω_T^*

Variational solutions

Definition: A map

$$u \in L^{\infty}\big(0,T;L^2(\Omega^*)\big) \cap L^1_{w^*}\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big)$$

with $u \geq \psi$ a.e. on Ω_T is called variational solution to the obstacle problem for the total variation flow if

$$\int_0^{\tau} \|Du\|(\Omega^*)\,dt \leq \int_0^{\tau} \|Dv\|(\Omega^*)\,dt + \iint_{\Omega_\tau} \partial_t v(v-u)\,dxdt$$

$$- \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2} + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2}$$

holds true for a.e. $\tau \in [0,T]$ and any $v \in L^1_{w^*}\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big)$ with $\partial_t v \in L^2(\Omega^*_T)$ and $v(0) \in L^2(\Omega^*)$, $v \geq \psi$ a.e. in Ω_T.

Remarks

- **Why L^∞–L^2 instead of C^0–L^2?** Solutions will be in general not in this better functions space. The question whether or not solutions are in C^0–L^2 is connected to uniqueness.
- **Why $\tau \in [0, T]$ a.e. ?** This is also connected to the missing C^0–L^2 regularity. If this held true (and if ψ was more regular), one could formulate the variational inequality on Ω_T and conclude (by a localization argument) that the variational inequality holds for all $\tau \in [0, T]$.

Further remarks

- **Why the extension property?** This condition ensures that the class of admissible testing functions is non-empty.
- Testing the variational inequality with g leads to energy-estimates for solutions.
- As a consequence one also obtains that u attains the initial datum in the usual L^2-sense, i.e.

$$\lim_{\tau \downarrow 0} \frac{1}{\tau} \int_0^\tau \|u(t) - u_o\|_{L^2}^2 \, dt = 0.$$

Existence of variational solutions

Theorem (Bögelein, Duzaar, Scheven).

Let $\Omega \Subset \Omega^*$ be a bounded Lipschitz domain and u_o, ψ, g as before. Then there exists a variational solution to the obstacle problem for the total variation flow in the sense of the Definition from before, which attains the initial datum u_o in the usual $L^2(\Omega^*)$-sense.

History (existence)

A non-complete list:

- **Stationary case** $p = 1$:
  De Giorgi, De Giorgi-Colombini-Piccinini,
  Carriero-Dal Maso-Leaci-Pascali
- **Parabolic** p**-Laplacian:**
  Lions, Brezis, Alt-Luckhaus, Kinnunen, Lindqvist,
  Bögelein-D-Mingione, Scheven
- **porous medium equation:**
  Alt-Luckhaus, Bögelein-Lukkari-Scheven

Related problems

Our method of proof is stable enough to treat

- initial data in $L^2(\Omega^*)$,
- the Cauchy-Dirichlet problem for the total variation flow with time dependent boundary values, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } \Omega_T, \\ u(0) = u_o & \text{on } \Omega, \\ u = \phi & \text{on } \partial\Omega \times (0,T). \end{cases}$$

The initial datum $u_o \colon \Omega \to \mathbb{R}$ and the lateral boundary values $\phi \colon \partial\Omega \times (0,T) \to \mathbb{R}$ are given.

Idea of proof

Building block: An existence result for regular obstacles:

$$\begin{cases} \psi \in W^{1,1}(\Omega_T^*),\ \partial_t\psi \in L^2(\Omega_T^*),\ \partial_t D\psi \in L^1(\Omega_T^*), \\ \psi = u_o \text{ on } \left(\Omega^* \setminus \overline{\Omega}\right) \times (0,T). \end{cases}$$

Time discretization method: Subdivide $(0,T]$:

$$(0,T] = \bigcup_{j=1}^{\ell} \big((j-1)h, jh\big] \quad h := \frac{T}{\ell}.$$

Let

$$\psi_j := \psi(jh),\ g_j := g(jh) \quad j \in \{0,1,\dots,\ell\}.$$

Minimizing movements I

- Start with u_o.
- Suppose that $u_{j-1} \in L^2(\Omega^*) \cap \mathrm{BV}_{u_o}(\Omega)$, $j \geq 1$ has already be constructed.
- Minimize

$$\mathbf{F}[v] := \|Dv\|(\Omega^*) + \frac{1}{2h} \int_{\Omega^*} |v - u_{j-1}|^2 \, dx$$

  in the class of functions

$$v \in L^2(\Omega^*) \cap \mathrm{BV}_{u_o}(\Omega), \quad v \geq \psi_j \text{ a.e. on } \Omega^*.$$

  Note: g_j is admissible.
- Denote the minimizer by u_j.

Minimizing movements II

- Define $u^{(h)}: (-h, T] \to \mathbb{R}$ by

$$u^{(h)}(x,t) = u_j(x) \quad \text{for } t \in ((j-1)h, jh],\, x \in \Omega^*,\, j \in \{0, 1, \dots, \ell\}.$$

- Goal: Prove energy estimates for $u^{(h)}$ independent of h which ensure (after passing to a subsequence) the convergence

$$u^{(h)} \rightharpoonup u \quad \text{weakly}^* \text{ in } L^\infty_{w^*}\big(0, T; \mathrm{BV}(\Omega^*)\big).$$

- This is the step where the regularity assumptions on ψ enter.

Energy estimates I

Compare the energy of u_j with the energy of $u_{j-1} - \psi_{j-1} + \psi_j$:

$$
\begin{aligned}
&\|Du_j\|(\Omega^*) + \frac{1}{2h}\int_{\Omega^*} \left|u_j - u_{j-1}\right|^2 dx \\
&\leq \|Du_{j-1}\|(\Omega^*) + \int_{\Omega^*} \left|D\psi_j - D\psi_{j-1}\right| dx + \frac{1}{2h}\int_{\Omega^*} \left|\psi_j - \psi_{j-1}\right|^2 dx \\
&\leq \|Du_{j-1}\|(\Omega^*) + \iint_{\Omega^* \times ((j-1)h, jh]} \left|\partial_t D\psi\right| + \frac{1}{2}\left|\partial_t \psi\right|^2 dx dt
\end{aligned}
$$

Energy estimates II

For $m \in \mathbb{N}$ with $mh \le T$ sum up the previous inequalities from $j = 1$ to $j = m$:

$$\|Du_m\|(\Omega^*) + \tfrac{1}{2h} \sum_{j=1}^{m} \int_{\Omega^*} \left|u_j - u_{j-1}\right|^2 dx \le \mathbf{E}(mh)$$

where

$$\mathbf{E}(\tau) = \|Du_o\|(\Omega^*) + \iint_{\Omega^*_\tau} \left|\partial_t D\psi\right| + \tfrac{1}{2}\left|\partial_t \psi\right|^2 dxdt$$

Note:

$$\frac{1}{2h} \sum_{j=1}^{m} \int_{\Omega^*} \left|u_j - u_{j-1}\right|^2 dx \le \frac{1}{2} \iint_{\Omega^*_{mh}} \Big| \underbrace{\frac{u^{(h)}(t) - u^{(h)}(t-h)}{h}}_{=[\Delta_{-h} u^{(h)}](t)} \Big|^2 dxdt$$

Energy estimates III

$$\sup_{t\in[0,T]} \|Du^{(h)}(t)\|(\Omega^*) + \iint_{\Omega_T^*} \left|[\Delta_{-h}u^{(h)}](t)\right|^2 dxdt \le 3\,\mathbf{E}(T)$$

After passing to a subsequence, this gives convergence

- $u^{(h)} \rightharpoondown u$ weak * in $L^\infty_{w^*}\big(0,T;\mathrm{BV}(\Omega^*)\big)$.
- $\Delta_{-h}u^{(h)} \rightharpoondown \partial_t u$ weakly in $L^2(\Omega_T^*)$.

for some $u \in L^\infty_{w^*}(0,T;\mathrm{BV}_{u_o}(\Omega^*))$ with $\partial_t u \in L^2(\Omega_T^*)$.

Minimality of $u^{(h)}$

The minimality of u_j, $j \in \{1, \ldots, \ell\}$, implies a minimality property of $u^{(h)}$. More precisely: $u^{(h)}$ minimizes the functional

$$\mathbf{F}^{(h)}[v] := \int_0^T \|Dv(t)\|(\Omega^*)\,dt + \frac{1}{2h} \iint_{\Omega_T^*} \left|v(t) - u^{(h)}(t-h)\right|^2 dxdt$$

in the class of mappings

$$v \in L^2(\Omega_T^*) \cap L^1_{w^*}\big(0, T; \mathrm{BV}_{u_o}(\Omega^*)\big) \quad v \geq \psi^{(h)} \text{ a.e. on } \Omega_T^*.[1]$$

[1] $\psi^{(h)}$ is defined similarly to $u^{(h)}$.

Exploiting Minimality I

Re-writing the minimality condition $\mathbf{F}^{(h)}[u^{(h)}] \le \mathbf{F}^{(h)}[v]$ gives:

$$
\begin{aligned}
&\int_0^T \|Du^{(h)}(t)\|(\Omega^*)\,dt \\
&\qquad \le \int_0^T \|Dv(t)\|(\Omega^*)\,dt \\
&\qquad\quad + \tfrac{1}{h} \iint_{\Omega_T^*} \left[\tfrac{1}{2}\big|v - u^{(h)}\big|^2 - \big(v - u^{(h)}\big)\big(u^{(h)} - u^{(h)}(\cdot - h)\big)\right] dxdt
\end{aligned}
$$

Now choose the comparison function in the form

$$
u^{(h)} + s\big(v - u^{(h)}\big), \quad s \in (0, 1],
$$

Re-arranging terms and dividing by $s > 0$ leads to

Exploiting Minimality II

$$\int_0^T \|Du^{(h)}(t)\|(\Omega^*)\,dt$$
$$\leq \int_0^T \|Dv(t)\|(\Omega^*)\,dt$$
$$+ \frac{1}{h} \iint_{\Omega_T^*} \left[\frac{s}{2}\left|v - u^{(h)}\right|^2 - \left(v - u^{(h)}\right)\left(u^{(h)} - u^{(h)}(\cdot - h)\right)\right] dxdt$$

Here send $s \downarrow 0$:

$$\int_0^T \|Du^{(h)}(t)\|(\Omega^*)\,dt$$
$$\leq \int_0^T \|Dv(t)\|(\Omega^*)\,dt - \iint_{\Omega_T^*} \left(v - u^{(h)}\right) \frac{u^{(h)} - u^{(h)}(\cdot - h)}{h}\,dxdt$$

Passing to the limit

Perform a partial integration in the second term of the right hand side:

$$\int_0^T \|Du^{(h)}(t)\|(\Omega^*)\,dt$$

$$\leq \int_0^T \|Dv(t)\|(\Omega^*)\,dt + \iint_{\Omega_T^*} \left(v - u^{(h)}\right) \frac{v - v(\cdot - h)}{h}\,dxdt$$

$$- \tfrac{1}{2h} \iint_{\Omega^* \times [T-h,T]} \left|v - u^{(h)}\right|^2 dxdt + \iint_{\Omega^* \times [-h,0]} \left|v - u_o\right|^2 dxdt$$

Here, v has been extended by $v(t) = v(0)$ for $t < 0$.

In the preceding inequality we can finally pass to the limit $h \downarrow 0$ to conclude that the variational inequality holds true.

Proof for irregular obstacles

The main result follows by a two-step approximation scheme.

- Firstly, a mollification with respect to time allows the reduction to obstacle functions with $\partial_t \psi \in L^2(\Omega^*)$.
- Secondly (much more involved) a mollification with respect to space allows the reduction to regular obstacles. Here the regularity assumption that Ω is a bounded Lipschitz domain enters. It is used to construct a mollification $\mathrm{M}_\varepsilon[\psi]$ of ψ such that it coincides with $\mathrm{M}_\varepsilon[u_o]$ in $\Omega^* \setminus \overline{\Omega}$ and that

$$\int_0^T \big\| D\mathrm{M}_\varepsilon[\psi] \big\| (\overline{\Omega})\, dt \longrightarrow \int_0^T \| D\psi \| (\overline{\Omega})\, dt \quad \text{as } \varepsilon \downarrow 0.$$

  Here we use techniques developed by Carriero & Dal Maso & Leaci & Pascali.

Thin obstacles

Theorem (Bögelein, Duzaar, Scheven).

Let $\Omega \Subset \Omega^*$ be a bounded Lipschitz domain, $u_o \in L^2 \cap W^{1,1}(\Omega^*)$.
For the obstacle $\psi\colon \Omega_T^* \to \mathbb{R}$ suppose that

$$\psi - u_o \text{ is upper semicontinuous on } \Omega_T, \ \operatorname{spt}(\psi - u_o) \Subset \Omega_T.$$

Then there exists $u \in L^\infty\big(0,T;L^2(\Omega^*)\big) \cap L^1_{w^*}\big(0,T;\mathrm{BV}_{u_o}(\Omega)\big)$ solving the relaxed obstacle problem, i.e.

$$\begin{aligned}
\int_0^\tau \|Du\|(\Omega^*)\,dt + \int_0^\tau \bigg[\int_\Omega \big(\psi - u^+\big)_+ d\sigma\bigg]dt \\
\le \int_0^\tau \|Dv\|(\Omega^*)\,dt + \iint_{\Omega_\tau} \partial_t v(v-u)\,dxdt \\
- \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2} + \tfrac{1}{2}\|v(0) - u_o\|^2_{L^2}
\end{aligned}$$

Thin obstacles

holds true

- for a.e. $\tau \in [0, T]$
- every $v \in L^1_{w^*}\big(0, T; \mathrm{BV}_{u_o}(\Omega)\big)$ with $\partial_t v \in L^2(\Omega_T^*)$, $v(0) \in L^2(\Omega^*)$ and $v \ge \psi$ on Ω_T, satisfying that

$$v - u_o \text{ is lower semicontinuous on } \overline{\Omega}_T.$$

The upper approximate limit

$u^+ \colon \Omega^* \to \mathbb{R}$ denotes the upper approximate limit of $u \in \mathrm{BV}(\Omega^*)$:

$$u^+(x_o) := \inf\left\{\lambda \in \mathbb{R} : \limsup_{\varrho \downarrow 0} \frac{\big|\{u > \lambda\} \cap B_\varrho(x_o)\big|}{|B_\varrho(x_o)|} = 0\right\}.$$

We have

- $u^+(x_o)$ = Lebesgue value of u at x_o in points where u is approximatively continuous;
- $u^+(x_o)$ = larger jump value in approximate jump points.

De Giorgi measure

- For $\varepsilon > 0$ let

$$\sigma_\varepsilon(E) = \inf\left\{\|\chi_E\|(\mathbb{R}^n) + \tfrac{1}{\varepsilon}|B| : B \text{ open, } E \subset B\right\}$$

- and then

$$\sigma(E) := \lim_{\varepsilon\downarrow 0} \sigma_\varepsilon(E) = \sup_{\varepsilon>0} \sigma_\varepsilon(E).$$

- σ Borel measure (not σ-finite)
- $\sigma(E) = 2\mathcal{H}^{n-1}(E)$ whenever E is a Borel set contained in a countable union of regular $(n-1)$-dimensional surfaces. In general $\sigma(E) \neq 2\mathcal{H}^n(E)$
- One always has the bounds:

$$C_1(n)\mathcal{H}^{n-1}(E) \leq \sigma(E) \leq C_2(n)\mathcal{H}^{n-1}(E).$$

Remark

The solution may violate the obstacle constraint $u \geq \psi$. This is penalized in the variational inequality by the integral on the left-hand side containing the De Giorgi measure. As a consequence of the variational solution and $\sigma \approx \mathcal{H}^{n-1}$ the exceptional set $\{u^+ < \psi\}$ is small in the sense that there holds

$$\mathcal{H} - \dim\big(E \cap \mathbb{R}^{n-1} \times \{t\}\big) \leq n - 1 \quad \text{for a.e. } t \in [0, T].$$

Thank you for your attention!