PDEs, Optimal Design and Numerics

Minimization of the Ground State of the Mixture of Two Conductors Materials

(“State of the Art")

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Optimal Design Problem

- Bdd domain $\Omega \subset \mathbb{R}^d$, $\quad 0 < \alpha < \beta$, $\quad 0 < m < |\Omega|$
- $B \subset \Omega$ measurable; $A = \Omega \setminus B$; $|B| = m$.
- $$\begin{cases} \inf \{\lambda(B) \mid B \in \mathscr{B}\} \\ \mathscr{B} = \{B \subset \Omega \text{ measurable}, |B| = m\} \end{cases} \tag{1}$$

  $$\begin{aligned} -\operatorname{div}(\sigma \nabla u) &= \lambda(B) u \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$
- $\sigma = \alpha \chi_A + \beta \chi_B$; $\lambda(B)$ the first eigenvalue (ground state)
- Simplicity : u is unique up to a multiplication constant
- Unique by normalisation : $u > 0$ in Ω; $\int_\Omega u^2 dx = 1$

Setting of the problem (II)

A useful characterisation of the target

- Rayleigh quotient

$$\lambda(B) = \min_{u \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega (\alpha\chi_A + \beta\chi_B)|\nabla u|^2 dx}{\displaystyle\int_\Omega |u|^2 dx}$$

Questions of interest

1. Does there exists a minimiser for this problem?
2. If yes, how does it looks like?
   (Characterisation of minimisers)
3. Can we find some explicit solutions?
4. If we don't have satisfactory answers,
   - What can we do?
   - What do numerics and computational simulations can tell / teach us?

Survey of previous results

Classical solutions - Existence

- General domains - Open
- 1D case
  - [1] M.G. Kreĭn - *AMS Translations Series* 1955

Proof exploits the equivalence between (1) and a similar vibrating membrane problem involving the target

$$\mu(B) = \min_{v \in H_0^1(]0,L[)} \frac{\displaystyle\int_0^L |\nabla v|^2 dy}{\displaystyle\int_0^L \rho(y)|v|^2 dy}.$$

where $y = y(x) = \int_0^x \frac{1}{\sigma(s)} ds$ and $\rho(y) = \sigma(x)$; $\Omega =]0, 1[$

  - Precise minimiser consists in taking β in the middle; shown by symmetrisation.
  - The equivalence does not hold in higher dimensions.

Survey of previous results ... continued

Classical solutions - Existence of a radially symmetric solution

- Case of a ball
  - [1] A. Alvino, G. Trombetti & P.L. Lions - *Nonlinear Anal.* 1989

Proof based on Schwarz symmetrisation and a tricky adaptation of the proof of existence for $\mu(B)$.

  - [2] C^2, R. Mahadevan, L. Sanz - *Appl. Math. Opt.* 2009

Proof based on Schwarz symmetrisation, which reduces things to 1D, and the fact that explicit formulae for the homogenisation process are available in 1D; the classical solution sought is retrieved as an extremal point of a non-empty, weak* compact convex set of L^∞(ball).

Uniqueness

Open question

- [3] J. Casado-Díaz - *Siam J. Control Opt.* *2015*

Theorem 1 (Case of a rectangle)

There exists $\varepsilon_0 > 0$ *such that* $\forall \varepsilon \in (0, \varepsilon_0)$, *problem (1) with* $m = |\Omega| - \varepsilon$ *has no solution.*

Proof based on sharp smoothness properties for a suitable relaxation of (1), namely

$$\begin{cases} \min \int_\Omega \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \right)^{-1} |\nabla u|^2 dx \\ \theta \in L^\infty(\Omega; [0,1]), \ \int_\Omega \theta dx \leq m, \ u \in H_0^1(\Omega), \ \int_\Omega |u|^2 dx = 1 \end{cases} \tag{2}$$

Asymptotic expansion - Low contrast regime

- Define $\epsilon = \beta - \alpha$, write $\beta = \alpha + \varepsilon$, $\varepsilon > 0$ small (low contrast)
- $\sigma^\varepsilon = \alpha + \varepsilon \chi_B$

Theorem 2 (Rellich - 1969)

The first eigenvalue λ^ε of

$$\begin{aligned} -\operatorname{div}(\sigma^\varepsilon \nabla u^\varepsilon) &= \lambda^\varepsilon u^\varepsilon \quad \text{in} \quad \Omega \\ u^\varepsilon &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

is an analytic function of ε in a neighbourhood of $\varepsilon = 0$ and the positive eigenfunction u^ε satisfying the normalization condition

$$\int_\Omega (u^\varepsilon)^2 dx = 1$$

is analytic with respect to ε.

So, we can introduce the series expansions

$$u^\varepsilon = v_0 + \varepsilon v_1 + \ldots,$$
$$\lambda^\varepsilon = \lambda_0 + \varepsilon \lambda_1 + \ldots,$$

in the equations above and gather terms of similar order in ε :

$$\left\{ \begin{array}{rcl} -\mathrm{div}(\alpha \nabla v_0) & = & \lambda_0 v_0 \quad \text{in} \quad \Omega \\ v_0 & = & 0 \quad \text{on} \quad \partial\Omega \end{array} \right. \tag{3}$$

$$\left\{ \begin{array}{rl} -\mathrm{div}(\alpha \nabla v_1) - \lambda_0 v_1 = & \mathrm{div}(\chi_B \nabla v_0) + \lambda_1 v_0 \quad \text{in} \quad \Omega \\ v_1 = & 0 \quad \text{on} \quad \partial\Omega \end{array} \right. \tag{4}$$

Due to the Fredholm alternative, equation (6) has a solution if and only if

$$\int_\Omega \mathrm{div}(\chi_B \nabla v_0) v_0 + \lambda_1 \int_\Omega v_0^2 = 0.$$

As $\int_\Omega v_0^2 = 1$, straightforward calculations using (5) & (6) yields

$$\lambda_1 = \lambda_1(B) = \int_B |\nabla v_0|^2$$

Theorem 3 (C², Laurain & Mahadevan, *Siam J. Appl. Math.* 2012)

Let us denote by

$$\tilde{\lambda}^\varepsilon(B) = \lambda^\varepsilon(B) - \lambda_0 - \varepsilon\lambda_1(B)$$

the remainder in the ansatz for λ^ε. For $\varepsilon > 0$ sufficiently small, there exists a constant C independent of ε and B such that

$$|\tilde{\lambda}^\varepsilon(B)| \leq C\varepsilon^{\frac{3}{2}} \quad \forall B \in \mathscr{B}.$$

Hence,

$$\left| \inf_{B\in\mathscr{B}} \lambda^\varepsilon(B) - \lambda_0 - \varepsilon \inf_{B\in\mathscr{B}} \lambda_1(B) \right| \leq C\varepsilon^{\frac{3}{2}}$$

Corollary 4 (An asymptotic approximation for $\lambda_1(\cdot)$)

If $B_\varepsilon^\star \in \mathscr{B}$ is a minimizer of $\lambda^\varepsilon(\cdot)$ then

$$\left|\lambda_1(B_\varepsilon^\star) - \inf_{B\in\mathscr{B}} \lambda_1(B)\right| \leq 2C\varepsilon^{\frac{1}{2}}$$

Proof

$$\begin{aligned}
&\varepsilon \left|\lambda_1(B_\varepsilon^\star) - \inf_{B\in\mathscr{B}} \lambda_1(B)\right| \\
&\quad = \left|(\lambda^\varepsilon(B_\varepsilon^\star) - \lambda_0 - \varepsilon\lambda_1(B_\varepsilon^\star)) - (\lambda^\varepsilon(B_\varepsilon^\star) - \lambda_0 - \varepsilon \inf_{B\in\mathscr{B}} \lambda_1(B))\right| \\
&\quad \leq |\lambda^\varepsilon(B_\varepsilon^\star) - \lambda_0 - \varepsilon\lambda_1(B_\varepsilon^\star)| + \left|\inf_{B\in\mathscr{B}} \lambda^\varepsilon(B) - \lambda_0 - \varepsilon \inf_{B\in\mathscr{B}} \lambda_1(B)\right| \\
&\quad \leq 2C\varepsilon^{\frac{3}{2}}.
\end{aligned}$$

Some other consequences and remarks

1. Corollary 4 gives us to understand that a minimizer for $\lambda^{\varepsilon}(\cdot)$ is approximately a minimizer for $\lambda_1(\cdot)$.

2. (Conversely) A similar argument proves that a minimizer for $\lambda_1(\cdot)$ is approximately a minimizer for $\lambda^{\varepsilon}(\cdot)$: If B^* is a minimizer of $\lambda_1(\cdot)$, then

$$\left| \lambda^{\varepsilon}(B^{\star}) - \inf_{B \in \mathscr{B}} \lambda^{\varepsilon}(B) \right| \leq 2C\varepsilon^{\frac{3}{2}}$$

($\longrightarrow$ numerical approximation for $\inf_{B \in \mathscr{B}} \lambda^{\varepsilon}(B)$)

Level sets of $|\nabla v_0|$

- Theorem 3 tells us that, asymptotically, the minimum value of $\lambda^\varepsilon(\cdot)$ can be calculated approximately minimizing $\lambda_1(\cdot)$.
- This later problem is easier since

Theorem 5 (Characterization of the minimizers of $\lambda_1(\cdot)$)

There exists $c^* \geq 0$ such that whenever B is a measurable subset of Ω satisfying

$$\{x \mid |\nabla v_0(x)| < c^*\} \subset B \subset \{x \mid |\nabla v_0(x)| \leq c^*\}$$

and $|B| = m$, then B is an optimal solution for the problem of minimizing $\lambda_1(B)$ over $B \in \mathscr{B}$.

Proof based on an accurate analysis of the level function $f(c) \overset{(\mathrm{def})}{=} |\{x \in \Omega \mid |\nabla v_0(x)| \leq c\}|$, with $c \in \mathbb{R}$.

A descent algorithm – general α, β & domain

Variational formulation for λ

$$\lambda = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega \sigma |\nabla u|^2}{\int_\Omega u^2} = \min_{u \in H_0^1(\Omega),\, ||u||_2 = 1} \int_\Omega \sigma |\nabla u|^2.$$

A descent algorithm

- Initial measurable set B_0, $|B_0| = m$.
- $m(B_0, c) \overset{\text{(def)}}{=} |\{x \mid |\nabla u_{B_0}(x)| \leq c\}|$. Non-decreasing $m(B_0, c) \to 0$ as $c \to 0$ whereas, $m(B_0, c) \to |\Omega|$ as $c \to \infty$.
- $$c_0 \overset{\text{(def)}}{=} \inf \{c \mid m(B_0, c) \geq m\}.$$
- Under suitable conditions $|\{x \mid |\nabla u_{B_0}(x)| \leq c_0\}| = m$.
- Actualization $B_1 = \{x \mid |\nabla u_{B_0}(x)| \leq c_0\}$.

2nd order expansion

Let us recall the series expansions

$$u^\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 \ldots,$$
$$\lambda^\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 \ldots,$$

Plugging them in the corresponding equations and gathering terms of similar order in ε :

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\alpha\nabla v_0) & = & \lambda_0 v_0 & \text{in} \quad \Omega \\ v_0 & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{5}$$

$$\left\{ \begin{array}{rl} -\mathrm{div}(\alpha\nabla v_1) - \lambda_0 v_1 = & \mathrm{div}(\chi_B \nabla v_0) + \lambda_1 v_0 \quad \text{in} \quad \Omega \\ v_1 = & 0 \quad \text{on} \quad \partial\Omega \end{array} \right. \tag{6}$$

Due to the Fredholm alternative, equation (6) has a solution if and only if

$$\int_\Omega \mathrm{div}(\chi_B \nabla v_0) v_0 + \lambda_1 \int_\Omega v_0^2 = 0.$$

2nd order expansion ... continued

As $\int_\Omega v_0^2 = 1$, straightforward calculations using (5) & (6) yields

$$\lambda_2(B) = \int_B \alpha \nabla v_1(B) \cdot \nabla v_0, \qquad \int_\Omega v_0 v_1(B) = 0.$$

Proposition 6

There is a constant $C > 0$ independent of B such that :

$$|\lambda^\varepsilon(B) - (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2)| \leq C\varepsilon^3 \quad \forall B \in \mathscr{B}.$$

Hence,

$$\left| \inf_{B\in\mathscr{B}} \lambda^\varepsilon(B) - \inf_{B\in\mathscr{B}} (\lambda_0 + \varepsilon\lambda_1(B) + \varepsilon^2\lambda_2(B)) \right| \leq C\varepsilon^3$$

A 2nd order minimisation approximate problem

From the expressions for $\lambda_1(B), \lambda_2(B)$, we focus attention on

$$\inf F(\chi) \overset{(\text{def})}{=} \int_\Omega \chi(\nabla v_0 + \varepsilon \nabla v(\chi)) \cdot \nabla v_0 dx$$

over the class of admissible domains represented by their characteristic functions

$$\mathscr{A} \overset{(\text{def})}{=} \{\chi\ ;\ \chi = \chi_B,\ B \subseteq \Omega,\ |B| = m\} \subseteq L^\infty(\Omega),$$

and $v = v(\chi) \in H_0^1(\Omega)$ satisfies

$$-\alpha \Delta v - \lambda_0 v = \lambda_1(\chi) v_0 + \operatorname{div}(\alpha \chi \nabla v_0) \quad (7)$$

$$\lambda_1(\chi) := \int_\Omega \alpha \chi |\nabla v_0|^2 \quad (8)$$

$$v \perp v_0 \text{ in } L^2(\Omega)$$

We appeal to the usual relaxation procedures; it requires calculate the *lower semicontinuous envelope* of F, w.r.t. weak* topology, i.e.,

$$\bar{F}(\theta) \overset{\text{(def)}}{=} \inf \{\liminf F(\chi_n) \mid \chi_n \rightharpoonup \theta, \text{ in } L^\infty(\Omega)\text{-weak}^*\}$$

$$\theta \in \overline{\mathscr{A}} = \overline{\mathscr{A}}^{L^\infty(\Omega)^*} = \{\theta \in L^\infty(\Omega) \mid 0 \le \theta \le 1, \int_\Omega \theta = m\}$$

Relaxation ... continued

Given $\chi_n \rightharpoonup \theta$, the calculation of the limit in both the objective and the state equation requires of fundamental results in *H*-mesures :

- [6] P. Gérard - *Comm. Partial Diff. Eqns.* 1991
- [7] L. Tartar - *Proc. Royal Soc. Edinburgh Sect. A* 1990

Theorem 7

For any $\theta \in \overline{\mathscr{A}}$, we have

$$\bar{F}(\theta) = \int_\Omega \theta \left[\nabla v_0 + \varepsilon \nabla v_\infty(\theta)\right] \cdot \nabla v_0 - \varepsilon\theta(1-\theta)|\nabla v_0|^2$$

where $v_\infty(\theta) \in H_0^1(\Omega)$ is solution of

$$-\alpha\Delta v - \lambda_0 v = \lambda_1(\theta)u_0 + \operatorname{div}(\alpha\theta\nabla v_0) \tag{9}$$

$$\lambda_1(\chi) \overset{\text{(def)}}{=} \int_\Omega \alpha\theta|\nabla u_0|^2 \tag{10}$$

$$v \perp u_0 \ \ in \ \ L^2(\Omega)$$

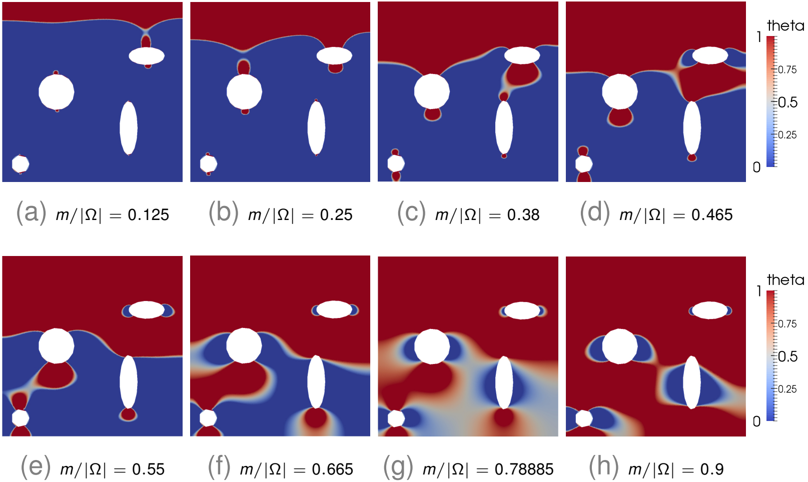
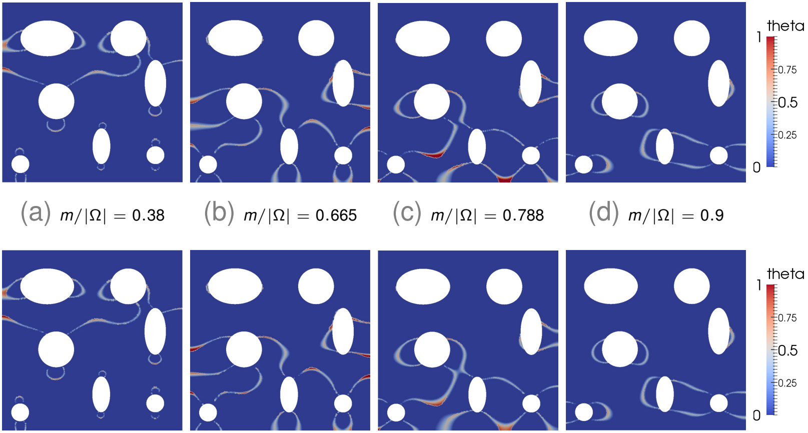


Figure: Optimal design for second order model for various fractions. The parameter ε takes the value 10^{-1}.



(a) $m/|\Omega| = 0.38$ (b) $m/|\Omega| = 0.665$ (c) $m/|\Omega| = 0.788$ (d) $m/|\Omega| = 0.9$

(e) $m/|\Omega| = 0.38$ (f) $m/|\Omega| = 0.665$ (g) $m/|\Omega| = 0.788$ (h) $m/|\Omega| = 0.9$

Figure: Absolute value of the gap between optimal design for first and second order models. The parameter ε takes the value 10^{-1} on the first line, 10^{-3} on the second line.

Path Forward

- It is still not clear why Problem (1) in smooth domains with partial symmetry should fail to have classical solutions.
- Existence for general domains