Discretization error analysis of optimal control problems with PDEs

State Constraints

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Outline of the Talk

Content

An Extended Model

$$\begin{aligned}
\min_{Q\times V}\ & \frac{1}{2}\|u-u^d\|^2+\frac{\alpha}{2}\|q\|^2\\
\text{s.t.}\quad -\Delta u &= q \text{ in } \Omega,\\
u &= 0 \text{ on } \partial\Omega,\\
a \le u &\le b\\
|\nabla u|^2 &\le c \text{ in } \overline{\Omega}.
\end{aligned}$$

What is the challenge:

- Need to pick space for the constraint: L^2 has no interior points $\rightarrow L^\infty$.
- PDE needs to work in 'non natural spaces' L^∞ in addition to H^1.
- Lagrangemultiplier is now a measure – non separable space!
- The space $W^{1,\infty}$ makes the previous remarks more pronounced.
- $W^{1,\infty}$ regularity gives severe restrictions on the problem data.

An Extended Model

$$
\begin{aligned}
\min_{Q\times V}\ & \frac{1}{2}\|u-u^d\|^2+\frac{\alpha}{r}\|q\|_{L^r}^r\\
\text{s.t.}\quad & -\Delta u = q \text{ in } \Omega,\\
& u = 0 \text{ on } \partial\Omega,\\
& a \le u \le b\\
& |\nabla u|^2 \le c \text{ in } \overline{\Omega}.
\end{aligned}
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An abstract Viewpoint

Different constraints give different KKT-conditions!
The operators take the following form:

Box Control Constraints

$$\begin{pmatrix} -\Delta & -P_{\mathrm{ad}} \circ \frac{-1}{\alpha} I \\ -I & -\Delta \end{pmatrix}$$

Penalized State Constraints

$$\begin{pmatrix} -\Delta & -P_{\mathrm{ad}} \circ \frac{-1}{\alpha} I \\ -I - \mu_\gamma & -\Delta \end{pmatrix}$$

Penalized Gradient State Constraints

$$\begin{pmatrix} -\Delta & -P_{\mathrm{ad}} \circ \frac{-1}{\alpha} I \\ -I - \nabla^* \mu_\gamma \nabla & -\Delta \end{pmatrix}$$

An Extended Discretization Error Analysis

State constraints with variational discretization:

- Similar to the control constrained case, but we need to control the $L^2 \longrightarrow L^\infty$ error of the solution operator (instead of $L^2 \longrightarrow L^2$).
- Need a control that is 'interior' w.r.t. the state constraint to cope with the fact that the continuous optimal control is not discrete feasible and vice versa.
- For the control, the obtained convergence order is optimal both w.r.t the regularity as well as the observed convergence rates.
- For the state, the convergence rates (in L^2) are neither optimal w.r.t. regularity nor the observed convergence rates.

An Extended Discretization Error Analysis II

Optimal error estimates for the State:

- Finite Dimensionality - either the control space or the state constraint is finite dimensional.
- Penalty/Barrier Approaches - gives optimal estimates for fixed parameters, but the dependence on the parameter is gereraly suboptimal.
- The infitite dimensional case without regularization is an open problem.

An Extended Discretization Error Analysis III

Gradient-state constraints with variational discretization:

- Now, we need error control in $W^{1,\infty}$ - needs more regularity than L^2 for the control!
- On a general (non-convex polygonal) domain - the space L^p never admits a control beeing 'interior' (geometric singularities). Special care needs to be taken when comparing the continuous and discrete problem.
- For the control, the obtained convergence order is optimal both w.r.t the regularity as well as the observed convergence rates on 'smooth' domains.
- For the state (and the control on 'nonsmooth' domains, the convergence rates (in L^2) are neither optimal w.r.t. regularity nor the observed convergence rates.
- Penalty and barrier approaches fail to give optimal convergence rates (w.r.t the regularity of the solution) unless the singular behavior is countered appropriately.

R. Herzog and A. Rösch and S. Ulbrich and W. Wollner
OPTPDE — A Collection of Problems in PDE-Constrained Optimization
http://www.optpde.net

All literature of W. Wollner at
http://www.math.uni-hamburg.de/home/wollner/publications.html

Thank you for your attention!