Resolvent estimates for high-contrast elliptic problems with periodic coefficients

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Partial differential equations, optimal design and numerics

UNIVERSITY OF BATH

EPSRC Engineering and Physical Sciences Research Council

Key ingredients:

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- Homogenisation of periodic elliptic DO

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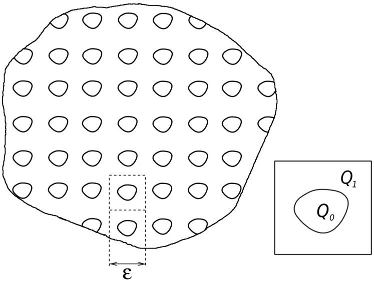
- Homogenisation of periodic elliptic DO
- High-contrast periodic media
- Gelfand transform
- Multiscale analysis, matched asymptotics

Homogenisation setting



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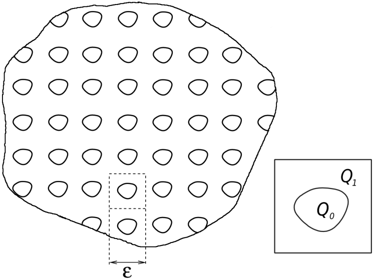
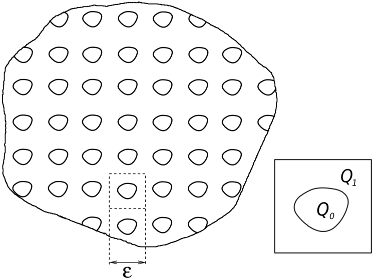


"Classical" homogenisation

$$-\mathrm{div}\Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla u\Big)+u=f,\qquad f\in L^2(\mathbb{R}^d),$$

$$A\geq \nu I>0$$



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Convergence (two-scale expansions, compensated compactness, two-scale convergence, periodic unfolding, Bloch decomposition):

$$u=u_\varepsilon\rightharpoonup u_0\ \ \text{in}\ \ H^1(\mathbb{R}^d),\qquad -\mathrm{div}A^{\mathrm{hom}}\nabla u_0+u_0=f.$$

Problem under study: “high contrast”

We study the problem

$$-\mathrm{div}\Big(A^\varepsilon\Big(\frac{x}{\varepsilon}\Big)\nabla u\Big) + u = f, \qquad f \in L^2(\mathbb{R}^d).$$

Here:

$$A^\varepsilon = A_1 + \varepsilon^2 A_0$$

$$A_0,\ A_1 \in [L^\infty(Q)]^{d\times d}, \quad Q-\text{periodic},\ \text{symmetric}$$

$$A_1 \geq \nu I,\ \nu > 0 \text{ on } Q_1 \subset Q := [0,1)^d,\ (\text{“stiff}'' \text{ component})$$

$$A_0 \geq \nu I,\ A_1 = 0 \text{ on } Q_0 = Q \setminus Q_1\ (\text{“soft}'' \text{ component})$$

$$\cup_{n\in\mathbb{Z}^d}\big(Q_1 + n\big) \text{ is connected in } \mathbb{R}^d$$

Notation

$$\mathfrak{b}^{\mathrm{hom}}_{\varepsilon,\theta}\big((c,u),(d,v)\big) := A^{\mathsf{hom}}\theta\cdot\theta c\overline{d} + \int_Q A_0(\nabla+\mathrm{i}\varepsilon\theta)u\cdot\overline{(\nabla+\mathrm{i}\varepsilon\theta)v}, \qquad (c,u),(d,v)\in\mathcal{H}_0,$$

$$A^{\mathrm{hom}}\xi\cdot\xi = \min_{u\in H^1_\#(Q)}\int_Q A_1(\xi+\nabla u)\cdot\overline{(\xi+\nabla u)}, \quad \xi\in\mathbb{R}^d,$$

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$$\mathcal{L} := \{c+\tilde{u} : c\in\mathbb{C},\ \tilde{u}\in L^2(Q),\ \tilde{u}|_{Q_1}=0\}\subset L^2(Q).$$

"Identification" map

$$\mathcal{I} : \mathbb{C}\times L^2(Q_0)\ni(c,u)\mapsto c+\tilde{u}\in\mathcal{L}, \qquad \tilde{u}=u \textit{ on } Q_0, \quad \tilde{u}=0 \textit{ on } Q_1.$$

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Operators $\mathcal{B}^{\rm hom}_{\varepsilon,\theta}$ in $\mathbb{C}\times L^2(Q_0)$:

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$\mathcal{P}$ orthogonal projection of $L^2(\varepsilon^{-1}Q'\times Q)$ onto

$$\big\{c+g: c\in L^2(\varepsilon^{-1}Q'),\ g\in L^2(\varepsilon^{-1}Q'\times Q),\ g(\theta,y)=0 \text{ a.e. } (\theta,y)\in\varepsilon^{-1}Q'\times Q_1\big\}.$$

Gelfand transform

"Scaled" version:

$$(\mathcal{G}_\varepsilon f)(\theta, z) := \left(\frac{\varepsilon}{2\pi}\right)^{d/2} \sum_{n\in\mathbb{Z}^d} f(z+\varepsilon n)\mathrm{e}^{-\mathrm{i}\theta(z+\varepsilon n)}, \quad \theta \in \varepsilon^{-1}[0, 2\pi)^d.$$

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Consider scaling of space variable: $(\mathcal{T}_\varepsilon h)(\theta, y) := \varepsilon^{d/2} h(\theta, \varepsilon y)$
and unitary

$$(\mathcal{U}_\varepsilon f)(\theta, y) := \left(\frac{\varepsilon^2}{2\pi}\right)^{d/2} \sum_{n\in\mathbb{Z}^d} f\big(\varepsilon(y+n)\big)\mathrm{e}^{-\mathrm{i}\varepsilon\theta(y+n)}.$$

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Lemma

Define

$$(\mathcal{B}_{\varepsilon,\theta} u, v) = \int_Q (\varepsilon^{-2}A_1 + A_0)\big(\nabla + \mathrm{i}\varepsilon\theta\big)u \cdot \overline{\big(\nabla + \mathrm{i}\varepsilon\theta\big)v}, \qquad u, v \in H^1_\#(Q)$$

Note:

$$u_\theta^\varepsilon = (\mathcal{B}_{\varepsilon,\theta} + 1)^{-1}F, \qquad F \in L^2(Q),$$

$$-\varepsilon^{-2}\left(\nabla + \mathrm{i}\varepsilon\theta\right) \cdot A^\varepsilon(y)\left(\nabla + \mathrm{i}\varepsilon\theta\right) u_\theta^\varepsilon + u_\theta^\varepsilon = F,$$

Then

$$\mathcal{U}_\varepsilon(\mathcal{A}^\varepsilon + 1)^{-1}\mathcal{U}_\varepsilon^{-1} = \int_{\varepsilon^{-1}Q'}^{\oplus} (\mathcal{B}_{\varepsilon,\theta} + 1)^{-1} d\theta.$$

The main convergence result

Theorem

The resolvents of the operator family $\mathcal{A}^\varepsilon$ *are asymptotically close as* $\varepsilon \to 0$ *to the family*

$$\mathcal{R}^\varepsilon := \mathcal{U}_\varepsilon^{-1} \int_{\varepsilon^{-1}Q'}^{\oplus} \mathcal{I}\big(\mathcal{B}_{\varepsilon,\theta}^{\rm hom} + I\big)^{-1}\mathcal{I}^{-1}\, \mathrm{d}\theta\, \mathcal{P}\mathcal{U}_\varepsilon,$$

where the corresponding approximation error is of order $O(\varepsilon)$. *More precisely, there exists a constant* $C > 0$, *independent of* ε, *such that*

$$\big\|(\mathcal{A}^\varepsilon + 1)^{-1} - \mathcal{R}^\varepsilon\big\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le C\varepsilon$$

An asymptotic expansion

$$u_\theta^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n u_\theta^{(n)}, \qquad\qquad u_\theta^{(n)} \in H^1_\#(Q), \quad n = 0, 1, 2, \dots$$

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$$\nabla \cdot A_1 \nabla u_\theta^{(0)} = 0, \qquad \text{hence} \;\; A_1 \nabla u_\theta^{(0)} = 0.$$

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$$-\nabla \cdot A_1 \nabla u_\theta^{(2)} = \mathrm{i}\nabla \cdot A_1 \theta u_\theta^{(1)} + \nabla \cdot A_0 \nabla u_\theta^{(0)} - \theta \cdot A_1 \theta u_\theta^{(0)} + u_\theta^{(0)}.$$

$$u_\theta^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n u_\theta^{(n)}, \qquad u_\theta^{(n)} \in H^1_\#(Q), \quad n = 0, 1, 2, \ldots$$

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Solvability $\Longrightarrow$ $u_\theta^{(0)} = c_\theta^{(0)} + v_\theta^{(0)}$, where $(c_\theta^{(0)}, v_\theta^{(0)}) \in \mathcal{H}_0$ satisfies

$$\mathfrak{b}_{0,\varepsilon}^{\mathrm{hom}}\Big(\big(c_\theta^{(0)}, v_\theta^{(0)}\big), (d, \varphi)\Big) + \int_Q \big(c_\theta^{(0)} + v_\theta^{(0)}\big)\overline{(d+\varphi)} = \int_Q \mathcal{P}_{\mathrm{f}} F \overline{(d+\varphi)}, \qquad (d, \varphi) \in \mathcal{H}_0,$$

$$\Longleftrightarrow \ \big(c_\theta^{(0)}, v_\theta^{(0)}\big) = \big(\mathcal{B}_{0,\theta}^{\mathrm{hom}} + I\big)^{-1} \mathcal{I}^{-1} \mathcal{P}_{\mathrm{f}} F.$$

Justification of asymptotics: error estimates

Theorem

Solution u_θ^ε, denote

$$U_\theta^\varepsilon := u_\theta^{(0)} + \varepsilon \tilde{u}_\theta^{(1)},$$

Then there exists $C > 0$:

$$\left\| u_\theta^\varepsilon - U_\theta^\varepsilon \right\|_{H^1(Q)} \leq C\varepsilon \|F\|_{L^2(Q)} \qquad |\theta| \leq 1.$$

Corollary

$$\left\| u_\theta^\varepsilon - u_\theta^{(0)} \right\|_{L^2(Q)} \leq C\varepsilon \|F\|_{L^2(Q)} \qquad |\theta| \leq 1.$$

Main ingredients of the proof

Proposition

Define $V := \left\{ v \in H^1_\#(Q) \middle| A_1 \nabla v = 0 \right\}$. *There exists* $C > 0$:

$$\|w\|_{H^1(Q)} \le C \|A_1 \nabla w\|_{L^2(Q)} \qquad \forall w \in V^\perp.$$

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Lemma

$u_\theta^{(0)} = c_\theta^{(0)} + v_\theta^{(0)}$:

$$\big(c_\theta^{(0)}, v_\theta^{(0)}\big) = \big(\mathcal{B}^{\rm hom}_{0,\theta} + I\big)^{-1} \mathcal{I}^{-1} \mathcal{P}_{\rm f} F, \quad \text{corrector} \quad u_\theta^{(1)} = {\rm i} \sum_{j=1}^d N_j \theta_j u_\theta^{(0)}.$$

Then $\exists\ C > 0$:

$$\big\|u_\theta^{(0)}\big\|_{H^1(Q)} \le C\big(1 + |\theta|^2\big)^{-1} \|F\|_{L^2(Q)},$$

$$\big\|u_\theta^{(1)}\big\|_{H^1(Q)} \le C|\theta|\big(1 + |\theta|^2\big)^{-1} \|F\|_{L^2(Q)}.$$

Extension to all $\theta \in Q'$

Theorem

Consider $u^{(0)}_{\varepsilon,\theta} := c^{(0)}_{\varepsilon,\theta} + v^{(0)}_{\varepsilon,\theta}$ *where* $\big(c^{(0)}_{\varepsilon,\theta}, v^{(0)}_{\varepsilon,\theta}\big) \in \mathcal{H}_0$ *satisfies*

$$\mathfrak{b}^{\mathrm{hom}}_{\varepsilon,\theta}\Big(\big(c^{(0)}_{\varepsilon,\theta}, v^{(0)}_{\varepsilon,\theta}\big), (d,\varphi)\Big) + \int_Q \big(c^{(0)}_{\varepsilon,\theta} + v^{(0)}_{\varepsilon,\theta}\big)\overline{(d+\varphi)} = \int_Q \mathcal{P}_{\mathrm{f}} F\overline{(d+\varphi)}, \qquad (d,\varphi) \in \mathcal{H}_0.$$

Then $\exists\, C > 0: \quad \big\|u^{\varepsilon}_{\theta} - u^{(0)}_{\varepsilon,\theta}\big\|_{L^2(Q)} \leq C\varepsilon \|F\|_{L^2(Q)} \quad \forall \theta.$

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Additional ingredients in the proof:

Proposition

Define $V(\varkappa) := \big\{ v \in H^1_{\varkappa}(Q) \,\big|\, A_1 \nabla v = 0 \big\}$. *There exists* $C > 0$:

$$\|w\|_{H^1(Q)} \le C d(\varkappa) \|A_1 \nabla w\|_{L^2(Q)} \qquad \forall w \in V(\varkappa)^{\perp}, \qquad\qquad d(\varkappa) = \begin{cases} 1, & \varkappa = 0, \\ |\varkappa|^{-1}, & \varkappa \neq 0. \end{cases}$$

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Lemma

$H_{\varepsilon,\theta} \in H^{-1}_{\#}(Q)$, $\langle H_{\varepsilon,\theta}, \varphi\rangle = 0\ \forall \varphi \in H^1_0(Q_0) \implies \exists\ R_{\varepsilon,\theta} \in H^1_{\#}(Q)$:

$$-\big(\nabla + \mathrm{i}\varepsilon\theta\big)\cdot A_1 \big(\nabla + \mathrm{i}\varepsilon\theta\big) R_{\varepsilon,\theta} = H_{\varepsilon,\theta},$$

$$\big\|R_{\varepsilon,\theta}\big\|_{H^1(Q)} \le C\left[\frac{1}{|\varepsilon\theta|}\big\|H_{\varepsilon,\theta} - \langle H_{\varepsilon,\theta}, 1\rangle\big\|_{H^{-1}_{\#}(Q)} + \frac{1}{|\varepsilon\theta|^2}\big|\langle H_{\varepsilon,\theta}, 1\rangle\big|\right].$$

Theorem

Let $w_{\varepsilon,\varkappa} \in H^1_\varkappa(Q)$ *be solution to*

$$-\nabla \cdot \left(\varepsilon^{-2} A_1 + A_0\right) \nabla w_{\varepsilon,\varkappa} + w_{\varepsilon,\varkappa} = F, \qquad F \in L^2(Q).$$

Then $\forall\, N \;\exists C_N > 0:$

$$\left\| w_{\varepsilon,\varkappa} - U^{(N)}_{\varepsilon,\varkappa} \right\|_{H^1(Q)} \le C_N \left(\frac{\varepsilon}{|\varkappa|}\right)^{2(N+1)} \|F\|_{L^2(Q)}.$$

Truncation of formal series: $\quad U^{(N)}_{\varepsilon,\varkappa} = \sum_{n=0}^{N} \varepsilon^{2n} w^{(n)}_\varkappa.$

In particular,

$$\left\| w_{\varepsilon.\varkappa} - w^{(0)}_\varkappa \right\|_{H^1(Q)} \le C_1 \left(\frac{\varepsilon}{|\varkappa|}\right)^{2} \|F\|_{L^2(Q)},$$

$$\int_{Q_0} \left(A_0 \nabla w^{(0)}_\varkappa \cdot \nabla \varphi + w^{(0)}_\varkappa \varphi\right) = \langle F, \varphi \rangle \quad \forall \varphi \in H^1_0(Q_0).$$

Matched asymptotic expansions "in the Bloch space"

"Inner":
$$\Bigl\|u_\theta^\varepsilon-\sum_{n=0}^N\varepsilon^n u_\theta^{(n)}\Bigr\|\le C_N|\varepsilon\theta|^{N+1},\qquad |\theta|\le\varepsilon^{-\alpha};$$

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"Outer": $$\left\| w_{\varepsilon,\varkappa} - \sum_{n=0}^{N} \varepsilon^{2n} w_\varkappa^{(n)} \right\| \le D_N \left| \frac{\varepsilon}{\varkappa} \right|^{2(N+1)}, \qquad |\varkappa| > \varepsilon^{1-\alpha}.$$

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Error: $C_N \varepsilon^{(1-\alpha)(N+1)} + D_N \varepsilon^{2\alpha(N+1)}$. "Optimal" for $\alpha = 1/3$.

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“Matching region”: $\varepsilon^{-\gamma} \le |\theta| \le \varepsilon^{\beta-1} \qquad (0 < \gamma < \gamma + \beta < 1)$

(Equivalently, $\varepsilon^{1-\gamma} \le |\varkappa| \le \varepsilon^\beta$.)

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(Equivalently, $\varepsilon^{1-\gamma} \le |\varkappa| \le \varepsilon^{\beta}$.)

$\mathcal{F}_N$ "common part" of $\sum_{n=1}^N \varepsilon^n u_\theta^{(n)}$ and $\sum_{n=1}^N \varepsilon^{2n} w_\varkappa^{(n)}$.

Both replaced by $\sum_{n=1}^N \varepsilon^n u_\theta^{(n)} + \sum_{n=1}^N \varepsilon^{2n} w_\varkappa^{(n)} - \mathcal{F}_N$.

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"Outer": $\left\|w_{\varepsilon,\varkappa} - \sum_{n=0}^N \varepsilon^{2n} w_\varkappa^{(n)}\right\| \le D_N\left|\frac{\varepsilon}{\varkappa}\right|^{2(N+1)}, \qquad |\varkappa| > \varepsilon^{1-\alpha}.$

Error: $C_N\varepsilon^{(1-\alpha)(N+1)} + D_N\varepsilon^{2\alpha(N+1)}.$ "Optimal" for $\alpha = 1/3.$

"Matching region": $\varepsilon^{-\gamma} \le |\theta| \le \varepsilon^{\beta-1} \qquad (0 < \gamma < \gamma + \beta < 1)$

(Equivalently, $\varepsilon^{1-\gamma} \le |\varkappa| \le \varepsilon^\beta$.)

$\mathcal{F}_N$ "common part" of $\sum_{n=1}^N \varepsilon^n u_\theta^{(n)}$ and $\sum_{n=1}^N \varepsilon^{2n} w_\varkappa^{(n)}$.

Both replaced by $\sum_{n=1}^N \varepsilon^n u_\theta^{(n)} + \sum_{n=1}^N \varepsilon^{2n} w_\varkappa^{(n)} - \mathcal{F}_N.$

This yields error $E_N\varepsilon^{\sigma(N+1)}$ (σ to be determined)

Our result: "matching" with $\sigma = 1$ for $N = 0$.

$(c,u)\in\mathcal{H}_0$ is an eigenvector of $\mathcal{B}^{\rm hom}_{\varepsilon,\theta}$ with eigenvalue $\lambda \iff$

$$\iff A^{\mathsf{hom}}\theta\cdot\theta c\overline{d}+\int_Q A_0(\nabla+\mathrm{i}\varepsilon\theta)u\cdot\overline{(\nabla+\mathrm{i}\varepsilon\theta)v}=\lambda\int_Q(c+u)\overline{(d+v)}\qquad\forall(d,v)\in\mathcal{H}_0.$$

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Function β appeared in Zhikov (2000).

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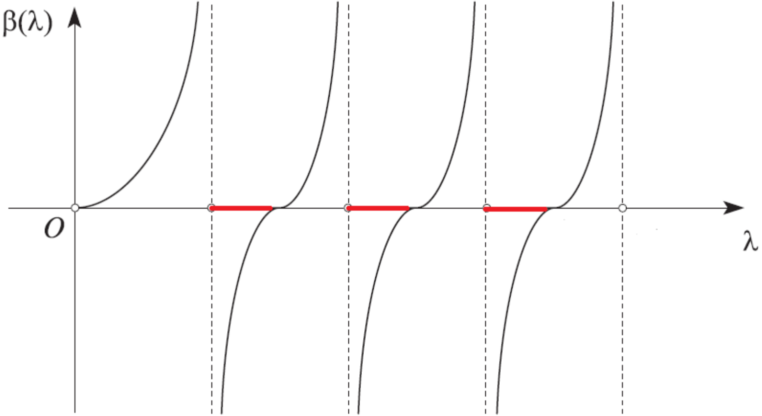
Theorem

The spectra of $\mathcal{A}^\varepsilon$ converge in the Hausdorff sense to the union of S_0 and

$$\lim_{\varepsilon\to0}\bigcup_{\theta\in\varepsilon^{-1}Q'}\bigl\{\lambda:\beta(\lambda)=A^{\mathrm{hom}}\theta\cdot\theta\bigr\}=\bigl\{\lambda:\beta(\lambda)\geq0\bigr\}.$$

The "Zhikov function" $\beta(\lambda)$



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Family $\mathcal{R}_\varepsilon$ consists of one element, the resolvent of $-\nabla \cdot A^{\text{hom}} \nabla$.

$\Longrightarrow$ We recover earlier results of Birman and Suslina (2004), Griso (2006).

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However: $\exists f^\varepsilon$ bounded in $L^2(\mathbb{R}^d)$ such that $C(f^\varepsilon) \to \infty$ as $\varepsilon \to 0$.

NB: Difference of spectral projections of $\mathcal{S}^\varepsilon(\mathcal{A}^{\rm dp} + I)^{-1}$ and $\mathcal{R}_\varepsilon$ does not go to zero near $(1 + \lambda_\infty)^{-1}$ such that $\beta(\lambda) \to \infty$ as $\lambda \to \lambda_\infty$.