Stabilization of a system of dispersive equations modelling water waves

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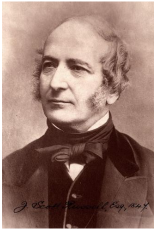
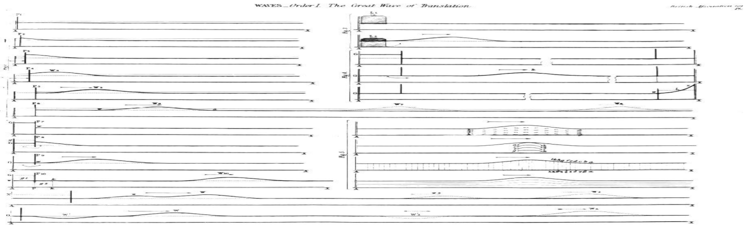
Water waves



John Scott Russell (1808 –1882):

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."



J. Scott Russell: Report on waves, Fourteenth meeting of the British Association for the Advancement of Science, 1844.

- Scott Russell's experimental work seemed at contrast with Isaac Newton's and Daniel Bernoulli's theories of hydrodynamics. George Biddell Airy and George Gabriel Stokes had difficulty to accept Scott Russell's experimental observations because Scott Russell's observations could not be explained by the existing water-wave theories. His contemporaries spent some time attempting to extend the theory but it would take until the 1870s before an explanation was provided.
- Lord Rayleigh published a paper in Philosophical Magazine in 1876 to support John Scott Russell's experimental observation with his mathematical theory. In his 1876 paper, Lord Rayleigh mentioned Scott Russell's name and also admitted that the first theoretical treatment was by Joseph Valentin Boussinesq in 1871.
- Joseph Boussinesq mentioned Scott Russell's name in his 1871 paper. Thus Scott Russell's observations on solitary waves were accepted as true by some prominent scientists within his own lifetime.
- Korteweg and de Vries did not mention John Scott Russell's name at all in their 1895 paper but they did quote Boussinesq's paper in 1871 and Lord Rayleigh's paper in 1876. Although the paper by Korteweg and de Vries in 1895 was not the first theoretical treatment of this subject, it was a very important milestone in the history of the development of soliton theory.
- It was not until the 1960s and the advent of modern computers that the significance of Scott Russell's discovery in physics, electronics, biology and especially fibre optics started to become understood, leading to the modern general theory of solitons. The pioneering computer simulation by Martin David Kruskal and Norman Zabusky from 1965 show a "solitary wave" solution of the KdV equation that propagates nondispersively and even regains its shape after a collision with other such waves. Because of the particle-like properties of such a wave, they named it a "soliton," a term that caught on almost immediately.

Many types of waves

Many types of waves

Many types of waves

Many types of waves

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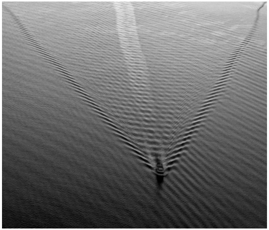
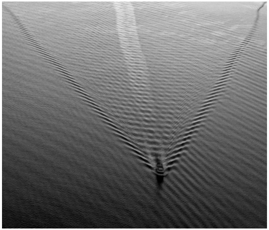
Many types of waves







Many types of waves



The linear water wave theory

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial y^2} + \dfrac{\partial^2\varphi}{\partial x^2} = 0 & -h_0 < y < \eta(t,x) \\[2ex] \dfrac{\partial\varphi}{\partial y} = 0 & y = -h_0 \\[2ex] \dfrac{\partial\varphi}{\partial y} = \dfrac{\partial\eta}{\partial t} & y = \eta \\[2ex] \dfrac{\partial\varphi}{\partial t} + g\eta = 0 & y = \eta \end{cases} \tag{1}$$

x is the horizontal variable
y is the vertical variable
$\eta(t,x)$ is the elevation of the wave (free surface)
$\varphi(t,x,y)$ is the velocity potential

The linear theory for water waves is accurate for small perturbations of water initially at rest (η and φ small). If the ratio of the water depth to wave length is small we have shallow water waves for which we have several good models:

The linear theory for water waves is accurate for small perturbations of water initially at rest (η and φ small). If the ratio of the water depth to wave length is small we have shallow water waves for which we have several good models:

- $\eta_t(t,x) + \eta_x(t,x) = 0$
  (transport equation)

- $\eta_t(t,x) + \eta_x(t,x) + \eta_{xxx}(t,x) = 0$
  (linearized Korteweg-de Vries (KdV) equation, 1895)

- $\eta_t(t,x) + \eta_x(t,x) - \eta_{txx}(t,x) = 0$
  (linearized Benjamin-Bona-Mahony (BBM) equation, 1972)

  Benjamin, T.B., Bona, J.L. and Mahony, J.J., Model equations for long waves in nonlinear dispersive systems, Philis. Trans. Roy. Soc. London, Ser. A, 272 (1972).

Dispersion relation for linearized water wave problem:
$\omega^2 = gk\tanh(kh_0)$.

In shallow waters $h_0 k \ll 1$ and $\omega \sim k\sqrt{gh_0}$.

- $\omega = k - k^3$ (KdV)
- $\omega = \dfrac{k}{1+k^2}$ (BBM)

The unbounded dispersion relation and computational difficulties of the KdV equation make the BBM equation a valuable alternative model.
The (nonlinear) BBM equation has solitons of the form:

$$\eta(t,x) = \frac{3c^2}{1-c^2}\mathsf{sech}^2\left(\frac{1}{2}\left(cx - \frac{ct}{1-c^2} + x_0\right)\right) \qquad c \in (0,1).$$

Stabilization and controllability results

For the KdV equation:

- D. Russell, B.-Y. Zhang, Exact controllability and stabilizability of the Korteweg-de Vries equation, Trans. Amer. Math. Soc. 348 (1996) 3643-3672.
- L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var., 2 (1997), 33-55.
- G. Perla Menzala, C. F. Vasconcellos, and E. Zuazua, *Stabilization of the Korteweg-de Vries equation with localized damping,* Quarterly of Applied Mathematics, **15** (2002), 111-129.
- J.-M. Coron and E. Crépeau, *Exact boundary controllability of a nonlinear KdV equation with a critical length*, J. Eur. Math. Soc., **6** (2004), 367-98.
- A. F. Pazoto, *Unique continuation and decay for the Korteweg-de Vries equation with localized damping,* ESAIM Control Optim. Calc. Var., **11** (2005), 473-486.
- L. Rosier, B.-Y. Zhang, Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain, SIAM J. Control Optim. 45 (2006), 927-956.

For the BBM equation:

- S. M., On the controllability of the linearized Benjamin-Bona-Mahony equation, SIAM J. Control Optim. 39 (2001), no. 6, 1677-1696.
- M. Yamamoto, *On unique continuation for a linearized Benjamin-Bona-Mahony equation*, J. Inverse Ill-Posed Probl., **11** (2003), 537-543.
- O. Glassa, *Controllability and asymptotic stabilization of the Camassa-Holm equation*, J. Diff. Eq., 245(6) (2008), 1584-1615.
- V. Perrollaz, *Initial boundary value problem and asymptotic stabilization of the Camassa-Holm equation on an interval*, J. Funct. Analysis, 259(9), (2010), 2333-2365.
- L. Rosier, B.-Y. Zhang, *Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain*, Journal of Differential Equations 254 (2013), 141-178.

X. Zhang, E. Zuazua, *Unique continuation for the linearized Benjamin-Bona-Mahony equation with space dependent potential*, Math. Ann. 325 (2003), no. 3, 543-582.

$$\begin{cases} u_t - u_{txx} = [\alpha(x)u]_x + \beta(x)u = 0 & x \in (0,1),\ t > 0 \\ u(t,0) = u(t,1) = 0 & t > 0 \\ u(0,x) = u_0(x) & x \in (0,1). \end{cases} \quad (2)$$

Theorem

Let $\beta = \chi_{(a,b)}$, $\alpha \in W^{2,\infty}(0,1)$ *with* $\min_{x\in[0,1]} |\alpha(x)| > 0$ *and* $\alpha'(x) \le 0$ *a.e. in* $(0,1)$. *Then for any* $u_0 \in H_0^1(0,1)$ *the solution of* (2) *verifies*

$$\|u(t)\|_{H_0^1} \to 0 \text{ when } t \to \infty. \quad (3)$$

A Boussinesq-type system

In 2002, Bona, Chen and Saut study a system of dispersive equations, first derived by Boussinesq, to describe the two-way propagation of small-amplitude, long wavelength, gravity waves on the surface of water in a canal.

$$\begin{cases} \eta_t + w_x + (\eta w)_x + \tilde{b} w_{xxx} - b\eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + \tilde{d}\eta_{xxx} - d w_{xxt} = 0. \end{cases} \tag{4}$$

Bona J. L., Chen M. and Saut J.-S., *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear Sci., 12 (2002), 283-318.

In this system, η is the elevation from the equilibrium position and $w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid.
The parameters $\tilde{b}$, b, $\tilde{d}$, d, that one might choose in a given modelling situation, are required to fulfill the relations

$$\tilde{b} + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad \tilde{d} + d = \frac{1}{2}(1 - \theta^2) \geq 0,$$

where $\theta \in [0, 1]$ specifies which velocity the variable w represents.

Contrary to some classical wave models which assumes that the waves travel only in one direction, system (4) is free of the presumption of unidirectionality and may have a wider range of applicability.

- S. M., J. H. Ortega, L. Rosier and B.-Y. Zhang, *Control and stabilization of a family of Boussinesq systems*, Discrete Contin. Dyn. Syst. **24** (2009), 273-313.
- M. Chen and O. Goubet, *Long-time asymptotic behavior of two-dimensional dissipative Boussinesq systems*, Discrete Contin. Dyn. Syst. Ser. S **2** (2009), 37-53.
- A. Pazoto, L. Rosier, Stabilization of a Boussinesq system of KdV-KdV type, System & Control Letters 57 (2008), 595-601.
- R. C. Capistrano Filho, Control of dispersive equations for surface waves, PhD Thesis, Universidade Federal do Rio de Janeiro and Université de Lorraine, 2014.

The purely BBM system

We consider the linear Boussinesq system, when the parameters $\tilde{b} = \tilde{d} = 0$ (purely BBM case), and a localized damping term acts on the first equation only.

$$\begin{cases} \eta_t + w_x - b\eta_{txx} + \varepsilon a(x)\eta = 0 & \text{for } x \in (0, 2\pi), \quad t > 0 \\ w_t + \eta_x - dw_{txx} = 0 & \text{for } x \in (0, 2\pi), \quad t > 0 \\ \eta(t, 0) = \eta(t, 2\pi) = 0 & \text{for } t > 0 \\ w(t, 0) = w(t, 2\pi) = 0 & \text{for } t > 0 \\ \eta(0, x) = \eta^0(x) & \text{for } x \in (0, 2\pi) \\ w(0, x) = w^0(x) & \text{for } x \in (0, 2\pi), \end{cases} \tag{5}$$

where b, $d > 0$ and $\varepsilon > 0$ is a small parameter.

- When $a \equiv 0$, system (5) is a pair of linear BBM equations coupled through first order terms.
- The function a is nonnegative and has the property that there exists an open set $\Omega \subset (0, 2\pi)$ such that

$$a(x) \geq a_0 > 0 \qquad (x \in \Omega). \tag{6}$$

The energy of the solution $\begin{pmatrix} \eta \\ w \end{pmatrix}$ of (5) is defined as follows

$$E(t) = \frac{1}{2}\left(\|\eta_x(t)\|_{L^2}^2 + \|\eta(t)\|_{L^2}^2 + \|w_x(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\right). \quad (7)$$

The term $\varepsilon a \eta$ represents a dissipation. Indeed, multiplying the first equation by η and the second one by w, integrating over $(0, 2\pi)$ and adding the two relations we get that

$$\frac{d}{dt}E(t) = -\varepsilon \int_0^{2\pi} a(x)\eta^2(t,x)\,\mathrm{d}x \leq 0, \quad (8)$$

which means that the energy of (5) is non increasing.

Our purpose is to study the behavior of $E(t)$, as t goes to infinity, under the assumption (6) and supposing that ε is sufficiently small.

Main result

Theorem

Let $a \in W^{2,\infty}(0, 2\pi)$ *with* $a(0) = a'(0) = 0$ *be a nonnegative function which verifies* (6)*. Then, there exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$ *and any* $\begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \in (H_0^1(0, 2\pi))^2$*, the solution* $\begin{pmatrix} \eta \\ w \end{pmatrix}$ *of* (5) *verifies that*

$$\lim_{t\to\infty} E(t) = 0. \tag{9}$$

Moreover, the decay of the energy of (5) *is not exponential, i. e., there exists no positive constants* M *and* ω *such that*

$$E(t) \leq Me^{-\omega t} \qquad (t \geq 0). \tag{10}$$

- It is the first study of the stability properties of (5) when a localized damping term, acting in one equation only, is present in the model.
- The stability issue of (5) is addressed by Chen and Guobet, when the system is possed on the whole real axis and the dissipation acts in the entire domain.

  M. Chen and O. Goubet: *Long-time asymptotic behavior of two-dimensional dissipative Boussinesq systems*, Discrete Contin. Dyn. Syst. Ser. S (2009).

We expect that the main difficulties described above for the single BBM equation to be inherited by system (5).

- The lack of the relative compactness of trajectories does not allow to reduce the study of stability to the unique continuation property.
- The coupling of variables η and w, as well as the fact that the dissipation is localized on one equation only, are additional difficulties, specific to the system, one has to deal with.

Main ideas of the proof of the main theorem

- The spectrum of the differential operator corresponding to (5) is located in the left open half-plane of the complex plane. Moreover, we obtain the asymptotic behavior of the spectrum.
- There exists a Riesz basis of $(H_0^1(0, 2\pi))^2$ consisting of generalized eigenvectors of the corresponding differential operator.
  - we obtain the asymptotic behavior of the high eigenfunctions and prove that they are quadratically close to a Riesz basis formed by eigenvectors of a well chosen dissipative differential operator with constant coefficients. This is done by using less common two dimensional versions of the shooting method and Rouché's Theorem.
  - to control the low frequencies we use a result due to Guo, originally proved for a unbounded operator and extended by Zhang and Zuazua to the bounded case.

Main ideas of the proof of the main theorem

These properties allow us to conclude the proof of Theorem 2. If $\bigcup_{m=1}^{P_0}\{\widehat{\Phi}_{m,k}\}_{1\le k\le k_m}\bigcup\{\Phi_n^j\}_{|n|>M_0,\,j=1,2}$ is the Riesz basis in the space $(H_0^1(0,2\pi))^2$ consisting of generalized eigenfunctions of the corresponding differential operator, given any initial data $\left(\begin{array}{c}\eta^0\\ w^0\end{array}\right)\in(H_0^1(0,2\pi))^2$ such that

$$\left(\begin{array}{c}\eta^0\\ w^0\end{array}\right)=\sum_{m=1}^{P_0}\sum_{k=1}^{k_m}a_{m,k}\widehat{\Phi}_{m,k}+\sum_{|n|>M_0,\,j=1,2}a_n^j\Phi_n^j,$$

the solution $\left(\begin{array}{c}\eta\\ w\end{array}\right)$ of (5) is given by

$$\left(\begin{array}{c}\eta\\ w\end{array}\right)=\sum_{m=1}^{P_0}e^{\widehat{\lambda}_m t}\sum_{k=1}^{k_m}a_{m,k}\sum_{s=1}^{k}\frac{t^{k-s}}{(k-s)!}\widehat{\Phi}_{m,s}+\sum_{|n|>M_0,\,j=1,2}a_n^j e^{\lambda_n^j t}\Phi_n^j.$$

Main ideas of the proof of the main theorem

From the Riesz basis property we deduce that

$$C_1 \sum_{|n|>M_0,\, j=1,2} \left|a_n^j\right|^2 e^{2\Re(\lambda_n^j)t} \le \left\| \begin{pmatrix} \eta(t) \\ w(t) \end{pmatrix} \right\|^2_{(H_0^1)^2}$$

$$\le C_2 \left(\sum_{m=1}^{P_0} e^{2\Re(\widehat{\lambda} m)t} \sum_{k=1}^{k_m} \left| \sum_{s=k}^{k_m} a_{m,s} \frac{t^{s-1}}{(s-1)!} \right|^2 + \sum_{|n|>M_0,\, j=1,2} \left|a_n^j\right|^2 e^{2\Re(\lambda_n^j)t} \right). \tag{11}$$

From the second inequality in (11) and since

$$\Re(\lambda_n)| < 0 \qquad (\forall n), \tag{12}$$

we deduce that the energy $E(t)$ tends to zero as t goes to infinity.

From the first inequality in (11) and the fact that

$$|\Re(\lambda_n)| \le \frac{C}{|n|^2} \qquad (|n| > M_0), \tag{13}$$

it follows that the decay rate of the energy is not exponential.

Similar strategies have been successfully used by

- S. Cox and E. Zuazua, *The rate at which energy decays in a damped string*, Comm. Partial Differential Equations **19** (1994), 213-243.
- A. Benaddi and B. Rao, *Energy decay rate of wave equations with indefinite damping*, J. Differential Equations **161** (2000), 337-357.
- X. Zhang and E. Zuazua, *Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential*, Math. Ann. **325** (2003), 543-582.

The simplest system

$$\begin{cases} -b\varphi_{xx} + \widetilde{\mu} v_x = 0 & \text{for } x \in (0, 2\pi) \\ -d v_{xx} + \widetilde{\mu}\varphi_x = 0 & \text{for } x \in (0, 2\pi) \\ \varphi(0) = \varphi(2\pi) = 0 \\ v(0) = v(2\pi) = 0. \end{cases} \tag{14}$$

- It has a family of double eigenvalues $(\widetilde{\mu}_n)_{n\in\mathbb{Z}^*}$,

$$\widetilde{\mu}_n = \sqrt{bd}n\, i \qquad (n \in \mathbb{Z}^*).$$

- The family of corresponding eigenfunctions $(\widetilde{\Phi}_n^j)_{n\in\mathbb{Z}^*,\, j\in\{1,2\}}$,

$$\widetilde{\Phi}_n^1 = \frac{b}{\widetilde{\mu}_n}\begin{pmatrix} \sqrt{\frac{d}{b}}\sinh\left(\frac{\widetilde{\mu}_n x}{\sqrt{bd}}\right) \\ \cosh\left(\frac{\widetilde{\mu}_n x}{\sqrt{bd}}\right) - 1 \end{pmatrix},\ \widetilde{\Phi}_n^2 = \frac{d}{\widetilde{\mu}_n}\begin{pmatrix} \cosh\left(\frac{\widetilde{\mu}_n x}{\sqrt{bd}}\right) - 1 \\ \sqrt{\frac{b}{d}}\sinh\left(\frac{\widetilde{\mu}_n x}{\sqrt{bd}}\right) \end{pmatrix},$$

forms an orthonormal basis in $(H_0^1)^2$.

A dissipative system with constant coefficients

$$\begin{cases} -b\psi_{xx} + \sigma u_x + \varepsilon a_0 \sigma \psi = 0 & \text{for } x \in (0, 2\pi) \\ -du_{xx} + \sigma \psi_x = 0 & \text{for } x \in (0, 2\pi) \\ \psi(0) = \psi(2\pi) = 0 & \\ u(0) = u(2\pi) = 0. & \end{cases} \tag{15}$$

- It has a double indexed family of complex eigenvalues $(\sigma_n^j)_{n\in\mathbb{Z}^*,\, j\in\{1,2\}}$.
- The family of corresponding eigenfunctions $(\Psi_n^j)_{n\in\mathbb{Z}^*,\, j\in\{1,2\}}$ forms a Riesz basis in $(H_0^1)^2$.

Spectral results (constant potential)

In the case of a constant potential a_0 we have:

- The eigenvalues $(\sigma_n^j)_{n\in\mathbb{Z}^*,\, j\in\{1,2\}}$ are such that $(\sigma_n^j)^2 + \varepsilon a_0 d\sigma_n^j - bd(\nu_n^j)^2 = 0$ and

$$\nu_n^1 = i\, n \qquad (n \in \mathbb{Z}^*), \tag{16}$$

$$\nu_n^2 = \mathsf{sgn}(n)\frac{1}{2\pi}\ln\left(\frac{\sqrt{b} - \pi\varepsilon a_0\sqrt{d}}{\sqrt{b} + \pi\varepsilon a_0\sqrt{d}}\right) + n\, i + \mathcal{O}\left(\frac{1}{n}\right). \tag{17}$$

- The eigenfunctions $\{\Phi_n^j\}_{n\in\mathbb{Z}^*,\, j=1,2}$ are explicitly done

$$\Phi_n^j = \begin{pmatrix} \dfrac{1}{\nu_n^j}\sinh\left(\nu_n^j x\right) + \dfrac{\beta_n^j\sigma_n^j}{b(\nu_n^j)^2}\left(\cosh\left(\nu_n^j x\right) - 1\right) \\ \dfrac{\sigma_n^j}{d(\nu_n^j)^2}\left(\cosh\left(\nu_n^j x\right) - 1\right) + \dfrac{\beta_n^j(\sigma_n^j)^2}{bd(\nu_n^j)^3}\sinh\left(\nu_n^j x\right) + \dfrac{\beta_n^j\sigma_n^j\varepsilon a_0}{b(\nu_n^j)^2}x \end{pmatrix}. \tag{18}$$

Another way to see it

$$\begin{cases} -b\psi_{xx} + \sigma u_x + \varepsilon a_0 \sigma \psi = 0 & \text{for } x \in (0, 2\pi) \\ -d u_{xx} + \sigma \psi_x = 0 & \text{for } x \in (0, 2\pi) \\ \psi(0) = 0,\ \psi_x(0) = 1 & \\ u(0) = 0,\ u_x(0) = \beta. & \end{cases} \tag{19}$$

If we define the map $G : \mathbb{C}^2 \to \mathbb{C}^2$,

$$G(\sigma, \beta) = \begin{pmatrix} \psi(\sigma, \beta, 2\pi) \\ u(\sigma, \beta, 2\pi) \end{pmatrix},$$

we deduce that σ is an eigenvalue of (15) with corresponding eigenfunction $\begin{pmatrix} \psi \\ u \end{pmatrix}$ if and only if

$$G(\sigma, \beta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The spectrum of (15) is given by the zeros of the map G.

A two dimensional "shooting method"

We consider, for each $(\mu, \gamma) \in \mathbb{C}^2$, the IVP:

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0, 2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0, 2\pi) \\ \eta(0) = 0,\ \eta_x(0) = 1 \\ w(0) = 0,\ w_x(0) = \gamma. \end{cases} \tag{20}$$

If we define the map $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F(\mu, \gamma) = \begin{pmatrix} \eta(\mu, \gamma, 2\pi) \\ w(\mu, \gamma, 2\pi) \end{pmatrix}$, we deduce that μ is an eigenvalue of (20) with corresponding eigenfunction $\begin{pmatrix} \eta \\ w \end{pmatrix}$ if and only if

$$F(\mu, \gamma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The spectrum of the differential operator with variable potential is given by the zeros of the map F.

Rouché Theorem

Theorem

Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^N$ and f, g holomorphic maps of $\overline{\mathcal{D}}$ into $\mathbb{C}^N$ such that $\|g(z)\| < \|f(z)\|$ for $z \in \mathcal{D}$. Then f has finitely many zeros in $\mathcal{D}$, and f and $f + g$ have the same number of zeros in $\mathcal{D}$, counting multiplicity.

N. G. Lloyd: *Remarks on generalising Rouché's theorem*, J. London Math. Soc. **2** (1979), 259-272.

Given a zero (σ_n^j, β^j) of the map G, we define the domain

$$D_n^j(\delta) = \left\{ (\mu, \gamma) \in \mathbb{C}^2 \, : \, \sqrt{|\mu - \sigma_n^j|^2 + |\gamma - \beta_n^j|^2} \leq \frac{\delta}{|n|} \right\}.$$

- $\|G(\mu, \gamma)\| \geq \frac{C\delta}{n^2}$, when $(\mu, \gamma) \in \partial D_n^j(\delta)$.
- $\|F(\mu, \gamma) - G(\mu, \gamma)\| \leq \frac{C}{n^2}$, when $(\mu, \gamma) \in \partial D_n^j(\delta)$.

The inequality $\|F(\mu,\gamma) - G(\mu,\gamma)\| \leq \frac{C}{n^2}$ is the most difficult. We look for an ansatz $\begin{pmatrix} \varphi(\mu,\gamma,x) \\ z(\mu,\gamma,x) \end{pmatrix}$ for the solutions of the IVP:

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0,2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0,2\pi) \\ \eta(0) = 0,\ \eta_x(0) = 1 & \\ w(0) = 0,\ w_x(0) = \gamma, & \end{cases}$$

such that $\begin{pmatrix} \eta(\mu,\gamma,x) \\ w(\mu,\gamma,x) \end{pmatrix} = \begin{pmatrix} \varphi(\mu,\gamma,x) \\ z(\mu,\gamma,x) \end{pmatrix} + \mathcal{O}\left(\frac{1}{\mu^2}\right)$.

$$\begin{cases} \varphi(\mu,\gamma,x) = & \dfrac{\sqrt{bd}}{\mu}\sinh\left(\alpha(x)\right) + \dfrac{\gamma d}{\mu}\cosh\left(\alpha(x)\right) - \dfrac{\gamma d}{\mu + da(x)} \\ z(\mu,\gamma,x) = & \dfrac{b}{\mu}\left(\cosh\left(\alpha(x)\right) - 1\right) + \dfrac{\gamma\sqrt{bd}}{\mu}\sinh\left(\alpha(x)\right) + \dfrac{\gamma d}{\mu}\displaystyle\int_0^x a(s)ds, \end{cases}$$

where $\alpha(x) = \frac{\mu x}{\sqrt{bd}} + \frac{1}{2}\sqrt{\frac{d}{b}}\int_0^x a(s)ds$.

$$\|F(\mu,\gamma) - G(\mu,\gamma)\|$$

$$\leq \left\| F(\mu,\gamma) - \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} - G(\mu,\gamma) \right\|$$

$$\|F(\mu,\gamma) - G(\mu,\gamma)\|$$

$$\leq \left\| F(\mu,\gamma) - \left(\begin{array}{c} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{array} \right) \right\| + \left\| \left(\begin{array}{c} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{array} \right) - G(\mu,\gamma) \right\|$$

$$\leq \frac{C_1}{|\mu|^2} \text{ (from the ansatz property)}$$

$$\|F(\mu,\gamma) - G(\mu,\gamma)\|$$

$$\leq \left\| F(\mu,\gamma) - \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} - G(\mu,\gamma) \right\|$$

$\leq \dfrac{C_1}{|\mu|^2}$ (from the ansatz property)

$+\dfrac{C_2}{|\mu|^2}$ (by choosing conveniently the constant potential $a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} a(s)ds$)

$$\leq \frac{C}{|\mu|^2}.$$

Spectral results (variable potential)

We take $a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(x)\,\mathrm{d}x$. The differential operator with a variable potential a has the following spectral properties:

- A double family of eigenvalues $(\mu_n^j)_{n\in\mathbb{Z}^*,\ j\in\{1,2\}}$ such that

$$|\mu_n^j - \sigma_n^j| \le \frac{\delta}{|n|} \qquad (|n| > N_0,\ j = 1,2)\,. \tag{21}$$

- A family of generalized eigenfunctions $\bigcup_{m=1}^{P_0}\{\widehat{\Phi}_{m,k}\}_{1\le k\le k_m}\bigcup\{\Phi_n^j\}_{|n|>N_0,\,j=1,2}$ which forms a Riesz basis in $(H_0^1)^2$ and there exists a bounded operator in $(H_0^1)^2$, $\mathcal{K}_\varepsilon$, s. t.

$$\|\Phi_n^j - \mathcal{K}_\varepsilon\Psi_n^j\|_{(H_0^1)^2} \le \frac{C}{|n|}. \tag{22}$$

In particular, we have two properties needed in the main theorem:

- $|\Re(\lambda_n^j)| \le \frac{C}{|n|^2}$.
- A Riesz basis of generalized eigenfunctions.

Unique continuation principle

Theorem

Let $\begin{pmatrix} \eta \\ w \end{pmatrix}$ *be a finite energy solution of system* (5) *with* $a \equiv 0$ *and suppose that there exists an open set* $\Omega \subset (0, 2\pi)$ *and* $T > 0$ *such that*

$$\eta(t, x) = 0 \qquad ((t, x) \in (0, T) \times \Omega). \tag{23}$$

Then

$$\begin{pmatrix} \eta \\ w \end{pmatrix} \equiv 0 \text{ in } \mathbb{R} \times (0, 2\pi). \tag{24}$$

For the proof we use:

- Fourier decomposition of solutions (we have a Riesz basis formed by eigenfunctions)
- analyticity in time of solutions (property (23) holds for $t \in \mathbb{R}$
- unique continuation principle for each eigenfunction.

Open problems

- Less regularity for the potential a.
- Stronger dissipative mechanism, like $-[a(x)\eta_x]_x$, ensures the uniform decay?
- Stabilization results for the nonlinear problem.
- The mixed KdV-BBM system is exponentially stabilizable?
- Is the asymptotic stability true for periodic boundary conditions?

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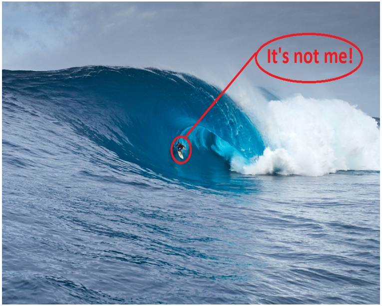

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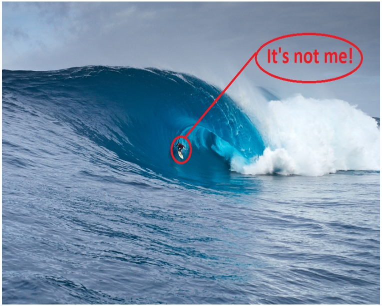

Thank you very much for your attention!