On the cost of null controllability of a linear KdV equation

Partial Differential Equations, Optimal Design and Numerics
Centro de Ciencias de Benasque Pedro Pascual
Thematic session on "Fluid Mechanics"

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August 26, 2015

Outline

A linear KdV equation on a bounded domain

- $T > 0$, $M \in \mathbb{R} \setminus \{0\}$ (transport coefficient), $\varepsilon > 0$ (dispersion coefficient), $Q := (0,T) \times (0,L)$.

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases}$$

- This kind of boundary conditions have been introduced by Colin and Ghidaglia (1997,2001).
- Null controllability for every $T > 0$ was proved by Guilleron (2014).
- We are interested in the behaviour of the cost of null controllability with respect to ε.

$$C^{\varepsilon}_{cost} := \sup_{y_0 \in L^2(0,L)} \left\{ \min_{v \in L^2(0,T)} \frac{\|v\|_{L^2(0,T)}}{\|y_0\|_{L^2(0,L)}} : y_{|t=0} = y_0, y_{|t=T} = 0 \text{ in } (0,L) \right\}.$$

Examples

- Heat equation:

$$\begin{cases} y_t - \varepsilon y_{xx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0 & \text{in } (0,T). \end{cases}$$

Coron, Guerrero (2005): $C_{cost}^{\varepsilon,heat} \leq C_0 \exp\left(C(T,M)\varepsilon^{-1}\right)$.

- (Classic) KdV equation:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0, \quad y_{x|x=L} = 0 & \text{in } (0,T). \end{cases}$$

Glass, Guerrero (2009): $C_{cost}^{\varepsilon,KdV} \leq C_0 \exp\left(C(T,M)\varepsilon^{-1/2}\right)$.

- (Our) KdV equation:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T). \end{cases}$$

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An estimate of the cost of null controllability

Theorem (Guerrero, C.)

Let $T > 0$, $M \in \mathbb{R}$ and $\varepsilon > 0$ be fixed. Then,

$$C^{\varepsilon}_{cost} \leq C_0 \exp\left(C(\varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2} + MT)\right), \quad \text{if } M > 0, \text{ and}$$

$$C^{\varepsilon}_{cost} \leq C_0 \exp\left(C(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2})\right), \quad \text{if } M < 0,$$

where $C > 0$ is a constant independent of T, M and ε, and $C_0 > 0$ depends polynomially on ε^{-1}, T^{-1} and $|M|^{-1}$.

- In particular, if ε is small enough

$$C^{\varepsilon}_{cost} \leq C_0 \exp\left(C(T, M)\varepsilon^{-1/2}\right).$$

The Hilbert Uniqueness Method (HUM)

- The proof is based on an observability inequality

$$\|\varphi_{|t=0}\|_{L^2(0,L)} \leq C_{obs} \|\varphi_{xx|x=0}\|_{L^2(0,T)},$$

where φ satisfies (adjoint equation)

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } Q, \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0,T). \end{cases}$$

- We consider the function $\phi := \varepsilon\varphi_{xx} - M\varphi$, which solves

$$\begin{cases} -\phi_t - \varepsilon\phi_{xxx} + M\phi_x = 0 & \text{in } Q, \\ \phi_{x|x=0} = 0, \quad \phi_{xx|x=0} = 0, \quad \phi_{|x=L} = 0 & \text{in } (0,T) \end{cases}$$

and we prove (Carleman estimate)

$$\iint_Q e^{-2s\alpha}\alpha^5|\phi|^2 \leq C_0 \int_0^T e^{-2s\alpha}\alpha^5|\phi_{|x=0}|^2, \quad \alpha = \frac{p(x)}{t^{1/2}(T-t)^{1/2}}.$$

- We recover φ from ϕ and $\varphi_{|x=0} = \varphi_{x|x=0} = 0$ (O.D.E.).

Behaviour of the cost in the vanishing dispersion limit

- We are now interested in the behaviour of C^{ε}_{cost} as $\varepsilon \to 0^+$.
- Consider the transport equation ($\varepsilon = 0$)

$$\begin{aligned} &y_t - My_x = 0 &&\text{in } Q := (0,T)\times(0,L),\\ &y_{|t=0} = y_0 &&\text{in } (0,L) \end{aligned}$$

with controls:

$$\begin{aligned} &y_{|x=0} = v_1(t) &&\text{if } M < 0,\\ &y_{|x=L} = v_2(t) &&\text{if } M > 0. \end{aligned}$$

- The transport equation is controllable if only if $T \geq L/|M|$.
- Then, it is natural to expect for KdV:
- $\lim_{\varepsilon\to 0^+} C^{\varepsilon}_{cost} = +\infty$ if $T < L/|M|$.
- $\lim_{\varepsilon\to 0^+} C^{\varepsilon}_{cost} = 0$ if $T \geq L/|M|$.

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An explosion result of the cost

For the classic KdV equation:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0, \quad y_{x|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L) \end{cases}$$

Glass, Guerrero (2009) proved that

1. $T < L/|M| \, : \, C_{cost}^{\varepsilon,KdV} \geq \exp(C\varepsilon^{-1/2})$ if $M \neq 0$.
2. $T \geq KL/M \, : \, C_{cost}^{\varepsilon,KdV} \leq \exp(-C\varepsilon^{-1/2})$ if $M > 0, K > 0$ large.

- The idea is to reproduce these results for the boundary conditions

$$y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0.$$

An explosion result of the cost $(M < 0)$

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases}$$

Theorem

Let $M < 0$. Then, for every $T < L/|M|$ there exist $C > 0$ (independent of ε) and $\varepsilon_0 > 0$ such that

$$C^{\varepsilon}_{cost} \geq \exp\left(C\varepsilon^{-1/2}\right), \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Idea of proof ($M < 0$)

We construct a particular solution $\hat{\varphi}$ of

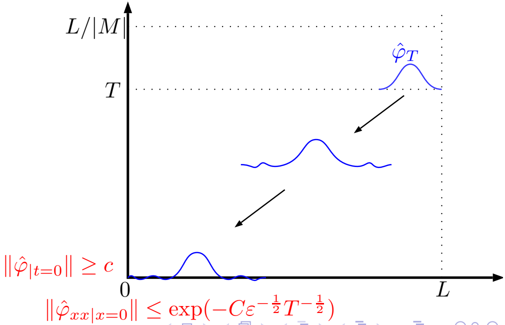
$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } Q, \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \hat{\varphi}_T & \text{in } (0,L), \end{cases}$$

where $0 \leq \hat{\varphi}_T \in \mathcal{C}_0^\infty(0,L)$, $\|\hat{\varphi}_T\|_{L^2(0,L)} = 1$.

We prove:

- $\|\hat{\varphi}_{xx|x=0}\|_{L^2(0,T)} \leq \exp\big(-C\varepsilon^{-1/2}T^{-1/2}\big)$
- $\|\hat{\varphi}_{|t=0}\|_{L^2(0,L)} \geq c > 0$

and we can conclude.



Uniform controllability? $(M > 0)$

- $C^{\varepsilon}_{cost} \leq \exp(-C(T,M)\varepsilon^{-1/2})$, T large?
- A possible strategy is to combine an observability inequality:

$$\|\varphi_{|t=T/2}\|_{L^2(0,L)} \leq \exp\left(C\varepsilon^{-1/2}\right)\|\varphi_{xx|x=0}\|_{L^2(0,T)}$$

with an exponential dissipation estimate (T large enough):

$$\|\varphi_{|t=0}\|_{L^2(0,L)} \leq \exp\left(-CT\varepsilon^{-1/2}\right)\|\varphi_{|t=T/2}\|_{L^2(0,L)}.$$

- In our case: we do not know how to prove such a dissipation estimate...

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Explosion result for arbitrary T

Theorem

Let $T, L, M > 0$ and $\delta \in (0,1)$. Then, there exists $\varepsilon_0 > 0$ such that

$$C^{\varepsilon,0}_{cost} \geq C \sinh\left((1-\delta)LM^{1/2}\varepsilon^{-1/2}\right), \quad \forall \varepsilon \in (0,\varepsilon_0)$$

where C depends polynomially on ε^{-1} and ε.

- Here:

$$C^{\varepsilon,0}_{cost} := \sup_{\substack{y_0 \in H^3_n(0,L) \\ y_0 \neq 0}} \min_{\substack{v \in L^2(0,T) \\ y_{|t=T}=0}} \frac{\|v\|_{L^2(0,T)}}{\|y_0\|_{H^3_n(0,L)}}$$

  and

$$H^3_n(0,L) := \{h \in H^3(0,L) : h'(L) = h''(L) = 0\}.$$

- In particular, since $C^{\varepsilon}_{cost} \geq C^{\varepsilon,0}_{cost}$ for any $\kappa \in (0,1)$:

$$C^{\varepsilon}_{cost} \geq \exp((1-\kappa)LM^{1/2}\varepsilon^{-1/2}), \quad \forall \varepsilon \in (0,\varepsilon_0).$$

An auxiliary problem

Find $u \in L^2(0,T)$ such that:

$$\begin{cases} w_t + \varepsilon w_{xxx} - M w_x = 0 & \text{in } (0,T)\times(\delta L, L), \\ w_{xx|x=\delta L} = u(t), \quad w_{x|x=L} = 0, \quad w_{xx|x=L} = 0 & \text{in } (0,T), \\ w_{|t=0} = w_0, \quad w_{|t=T} = 0 & \text{in } (\delta L, L). \end{cases}$$

We define its cost: $K^{\varepsilon}_{cost} := \sup\limits_{\substack{w_0 \in H^3_n(\delta L, L) \\ w_0 \neq 0}} \min\limits_{\substack{u \in L^2(0,T) \\ w_{|t=T}=0}} \dfrac{\|u\|_{L^2(0,T)}}{\|w_0\|_{H^3_n(\delta L, L)}}$.

- We prove that $K^{\varepsilon}_{cost} \geq C \sinh\left((1-\delta)LM^{1/2}\varepsilon^{-1/2}\right)$.
- By setting $u := y_{xx|x=\delta L}$, we can prove that $K^{\varepsilon}_{cost} \lesssim C^{\varepsilon,0}_{cost}$.
  - We show

    $$\|y_{xx|x=\delta L}\|_{L^2(0,T)} \leq C\left(\|v\|_{L^2(0,T)} + \|y_0\|_{H^3_n(0,L)}\right),$$

    where C depends polynomially on ε^{-1} and ε.

Particular solution for the adjoint equation

The adjoint equation is given by

$$\begin{cases} -\psi_t - \varepsilon\psi_{xxx} + M\psi_x = 0 & \text{in } (0,T)\times(\delta L, L), \\ \psi_{x|x=\delta L} = (\varepsilon\psi_{xx} - M\psi)_{|x=\delta L} = (\varepsilon\psi_{xx} - M\psi)_{|x=L} = 0 & \text{in } (0,T), \\ \psi_{|t=T} = \psi_T & \text{in } (\delta L, L). \end{cases}$$

- $\displaystyle\sup_{h\in H^3_n(\delta L,L)} \frac{\int_{\delta L}^{L} \psi_{|t=0} h}{\|h\|_{H^3_n(\delta L,L)}} \leq \varepsilon K^{\varepsilon}_{cost} \|\psi_{|x=\delta L}\|_{L^2(0,T)}$ (observability ineq.).
- $\hat{\psi}(x) := \cosh\left((x-\delta L)M^{1/2}\varepsilon^{-1/2}\right)$ is a solution.

Dissipation estimate for the adjoint equation

For the solutions of

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } Q, \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \varphi_T & \text{in } (0,L) \end{cases}$$

we can prove an exponential dissipation estimate of the kind:

$$\int_0^{\delta L} |\varphi_{|t=0}|^2 \leq \exp\big(-CT^{1/2}\varepsilon^{-1/2}\big)\int_0^L |\varphi_{|t=T/2}|^2 + \exp\big(-CT^{-1/2}\varepsilon^{-1/2}\big)\int_0^T |\varphi_{|x=L}|^2, \ \delta \in (0,1), T \text{ large.}$$

- $\exp\big(-CT^{1/2}\varepsilon^{-1/2}\big)$ counteracts observability constant (from Carleman).
- $\varphi_{|x=L}$ allows to define a control like $y_{xx|x=L} = v_2(t)$.
- Price to pay: initial conditions y_0 supported in $[0, \delta L)$.

First case

$y_0 \in L^2(0,L)$, $y_{0|(\delta L,L)} = 0$:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v_0(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = v_2(t) & \text{in } (0,T), \\ y_{|t=0} = y_0, \quad y_{|t=T} = 0 & \text{in } (0,L). \end{cases}$$

We are able to prove:

$$\|v_0\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} \leq C_0 \exp\left(-C(T,M)\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,\delta L)}.$$

- Observability inequality "for free" from previous case

$$\|\varphi_{|t=T/2}\|_{L^2(0,L)} \leq \exp\left(C\varepsilon^{-1/2}\right)\|\varphi_{xx|x=0}\|_{L^2(0,T)}.$$

- Combined with the dissipation estimate we obtain:

$$\|\varphi_{|t=0}\|_{L^2(0,\delta L)} \leq \exp\left(-C(T,M)\varepsilon^{-1/2}\right)\|\varphi_{xx|x=0}\|_{L^2(0,T)} + \exp\left(-CT^{-1/2}\varepsilon^{-1/2}\right)\|\varphi_{|x=L}\|_{L^2(0,T)}.$$

Second case

$y_0 \in L^2(0,L)$, $y_{0|(\delta L, L)} = 0$:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = 0, \quad y_{x|x=L} = v_1(t), \quad y_{xx|x=L} = v_2(t) & \text{in } (0,T), \\ y_{|t=0} = y_0, \quad y_{|t=T} = 0 & \text{in } (0,L). \end{cases}$$

We are able to prove:

$$\|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} \leq C_0 \exp\left(-C(T,M)\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,\delta L)}.$$

- New Carleman inequality:

$$\iint_Q e^{-2s\alpha}|\varphi|^2 \leq C_0 \int_0^T e^{-2s\alpha}\left(|\varphi_{x|x=L}|^2 + |\varphi_{|x=L}|^2\right), \quad \alpha = \frac{p(x)}{t^{1/2}(T-t)^{1/2}}.$$

- No need to use $\phi := \varepsilon\varphi_{xx} - M\varphi$.

Open problems

- Remaining case:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v_0(t), \quad y_{x|x=L} = v_1(t), \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0, \quad y_{|t=T} = 0 & \text{in } (0,L). \end{cases}$$

$$\|v_0\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)} \le C_0 \exp\big(-C\varepsilon^{-1/2}\big)\|y_0\|_{L^2(0,L)}?$$

or

$$\|v_0\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)} \ge C_0 \exp\big(C\varepsilon^{-1/2}\big)\|y_0\|_{L^2(0,L)}?$$

- Nonlinear case:

$$\begin{cases} y_t + \varepsilon y_{xxx} - y_x + y y_x = 0 & \text{in } Q, \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0, \quad y_{x|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0, \quad y_{|t=T} = 0 & \text{in } (0,L). \end{cases}$$

Uniform local null controllability?

Thank you for your attention