Null controllability for parabolic equations with dynamic boundary conditions

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In this work, we are concerned with the null controllability of the following Parabolic equation with dynamic boundary conditions

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = 1_\omega v(t,x) & \text{in } \Omega_T := (0,T)\times\Omega, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d\partial_\nu y + b(t,x)y_\Gamma = 0 & \text{on } \Gamma_T := (0,T)\times\Gamma, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases} \tag{1}$$

- $y_\Gamma = y|_\Gamma$.
- The term $\partial_t y_\Gamma - \Delta_\Gamma y_\Gamma$ models the tangential diffusive flux on the boundary which is coupled to the equation on the bulk by the normal derivative $\partial_\nu y = (\nu \cdot \nabla y)|_\Gamma$.
- Sometimes these boundary conditions are called of Wentzell type.

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This type of boundary conditions arises for many known equations of mathematical physics.
They are motivated by :

- problems in diffusion phenomena,
- Reaction-diffusion systems in phase-transition phenomena.
- Special flows in hydrodynamics (the flow of heat for a solid in contact with a fluid),
- Models in climatology, ...

References :
C. Gal, Favini, J. and G. Goldstein, Gresselli, Miranville, Meyeries, Romanelli, Vazquez, Zellik, ...

G. R. Goldstein, Derivation of dynamical boundary conditions, Adv. Differential Equations, 11 (2006), 457–480.

The operator Δ_Γ on Γ is given here by the surface divergence theorem

$$\int_\Gamma \Delta_\Gamma y\, z\, dS = -\int_\Gamma \langle \nabla_\Gamma y, \nabla_\Gamma z \rangle_\Gamma \, dS, \ y \in H^2(\Gamma), \ z \in H^1(\Gamma),$$

where ∇_Γ is the surface gradient.

The operator $(\Delta_\Gamma, H^2(\Gamma))$ is self-adjoint and negative on $L^2(\Gamma)$. Thus it generates an analytic C_0-semigroup on $L^2(\Gamma)$.

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Consider the following inhomogeneous parabolic problem with dynamic boundary conditions

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = f(t,x), & \text{in } \Omega_T, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + b(t,x)y_\Gamma = g(t,x), & \text{on } \Gamma_T \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases} \quad (2)$$

On $\mathbb{L}^2 := L^2(\Omega) \times L^2(\Gamma)$, we consider the linear operator

$$A = \begin{pmatrix} d\Delta & 0 \\ -d\partial_\nu & \delta\Delta_\Gamma \end{pmatrix}, \qquad D(A) = \mathbb{H}^2,$$

where $\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) : y|_\Gamma = y_\Gamma\}$ for $k \in \mathbb{N}$, Consider the following inhomogeneous parabolic problem with dynamic boundary conditions

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where $\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) \, : \, y|_\Gamma = y_\Gamma\}$ for $k \in \mathbb{N}$, Our wellposedness and regularity results for the underlying evolution equations rely on this fact.

Proposition

The operator A is densely defined, self-adjoint, negative and generates an analytic C_0-semigroup $(e^{tA})_{t\geq 0}$ on $\mathbb{L}^2$. We further have $(\mathbb{L}^2, \mathbb{H}^2)_{1/2,2} = \mathbb{H}^1$.

Proof :

Introduce on $\mathbb{L}^2$ the densely defined, closed, symmetric, positive sesquilinear form

$$\mathfrak{a}[y, z] = \int_\Omega d\, \nabla y \cdot \nabla \overline{z}\, dx + \int_\Gamma \delta \, \langle \nabla_\Gamma y, \nabla_\Gamma \overline{z} \rangle_\Gamma \, dS, \qquad D(\mathfrak{a}) = \mathbb{H}^1.$$

It induces a positive self-adjoint sectorial operator $\tilde{A}$ on $\mathbb{L}^2$ and one can show that $A \subset \tilde{A}$.

For the other inclusion, we need to show that $\lambda - A$ is surjective for some "large" λ.

The following perturbed system

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = 0 & \text{in } \Omega_T := (0,T)\times\Omega, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + b(t,x)y_\Gamma = 0 & \text{on } \Gamma_T := (0,T)\times\Gamma, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases}$$

has also a solution which is an evolution family $S(t,s)$ on $\mathbb{L}^2$ depending strongly continuously on $0 \le s \le t \le T$ such that

$$S(t,\tau)y_0 = e^{(t-\tau)A}y_0 - \int_\tau^t e^{(t-s)A}(a(s,\cdot), b(s,\cdot))S(s,\tau)y_0\, ds$$

Definition

Let $f \in L^2(\Omega_T)$, $g \in L^2(\Gamma_T)$ and $y_0 \in \mathbb{L}^2$.

(a) A *strong solution of* (2) is a function $y \in \mathbb{E}_1 := H^1(0,T;\mathbb{L}^2) \cap L^2(0,T;D(A))$ fulfilling (2) in $L^2(0,T;\mathbb{L}^2)$.

(b) A *mild solution of* (2) is a function $y \in C([0,T];\mathbb{L}^2)$ satisfying, in $\mathbb{L}^2$, $\quad \forall t \in [0,T]$,

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-\tau)A}[f(\tau) - a(\tau)y(\tau), g(\tau) - b(\tau)y(\tau)]\, d\tau.$$

Definition

(c) A *distributional solution of* (2) is a function $y \in L^2(0, T; \mathbb{L}^2)$ such that for all $\varphi \in \mathbb{E}_1$ with $\varphi(T, \cdot) = 0$ we have

$$\begin{aligned} &\int_{\Omega_T} y(-\partial_t\varphi - d\Delta\varphi + a\varphi)\, dx\, dt \\ &+ \int_{\Gamma_T} y(-\partial_t\varphi - \delta\Delta_\Gamma\varphi + d\partial_\nu\varphi + b\varphi)\, dS\, dt \\ &= \int_{\Omega_T} f\varphi\, dx\, dt + \int_{\Gamma_T} g\varphi\, dS\, dt + < y_0, \varphi(0,\cdot) >_{\mathbb{L}^2} . \end{aligned} \qquad (3)$$

(d) We call $y \in L^2(0, T; \mathbb{L}^2)$ a *distributional solution of* (2) *with vanishing final value* if y satisfies (3) for all $\varphi \in \mathbb{E}_1$.

Proposition

Let $f \in L^2(\Omega_T)$, $g \in L^2(\Gamma_T)$ *and* $y_0, \varphi_T \in \mathbb{L}^2$.

(a) *There is a unique mild solution* $y \in C([0,T];\mathbb{L}^2)$ *of* (2). *The solution map* $(y_0, f, g) \mapsto y$ *is linear and continuous from* $\mathbb{L}^2 \times L^2(\Omega_T) \times L^2(\Gamma_T)$ *to* $C([0,T];\mathbb{L}^2)$.
Moreover, y *belongs to* $\mathbb{E}_1(\tau, T) := H^1(\tau, T; \mathbb{L}^2) \cap L^2(\tau, T; D(A))$ *and solves* (2) *strongly on* (τ, T) *with initial* $y(\tau)$, *for all* $\tau \in (0,T)$ *and it is given by*

$$y(t) = S(t,0)y_0 + \int_0^t S(t,s)(f(s), g(s))\, ds, \qquad t \in [0,T],$$

Proposition

(b) *Given* $R > 0$, *there is a constant* $C = C(R) > 0$ *such that for all* a *and* b *with* $\|a\|_\infty, \|b\|_\infty \leq R$ *and all data the mild solution of* y *of* (2) *satisfies*

$$\|y\|_{C([0,T];\mathbb{L}^2)} \leq C(\|y_0\|_{\mathbb{L}^2} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}).$$

(c) *If* $y_0 \in \mathbb{H}^1$, *then the mild solution* y *of* (2) *is the strong one, i.e.,* $y \in \mathbb{E}_1 := H^1(0,T;\mathbb{L}^2) \cap L^2(0,T;D(A))$ *and solves* (2) *strongly on* $(0,T)$ *with initial data* y_0.

(d) *A function* y *is a distributional solution of* (2) *if and only if it is a mild solution.*

(e) *A distributional solution of* y (2) *with vanishing end value satisfies* $y(T,\cdot) = 0$.

We study the null controllability of the linear system

$$
\begin{aligned}
\partial_t y - d\Delta y + a(t,x)y &= v(t,x)1_\omega && \text{in } \Omega_T, && (4)\\
\partial_t y - \delta\Delta_\Gamma y + d\partial_\nu y + b(t,x)y &= 0 && \text{on } \Gamma_T, && (5)\\
y(0,\cdot) &= y_0 && \text{in } \overline{\Omega}, && (6)
\end{aligned}
$$

Definition

The system (1) is said to be null controllable at time $T > 0$ if for all given $y_0 \in L^2(\Omega)$ and $y_{0,\Gamma} \in L^2(\Gamma)$ we can find a control $v \in L^2((0,T) \times \omega)$ such that the solution satisfies

$$y(T, \cdot) = 0 \qquad \text{on } \overline{\Omega}.$$

The Null controllability of parabolic equations was studied in the literature in the case of Dirichlet, Neumann or mixed boundary conditions (also called **Robin or Fourier boundary** conditions).

- Russel
-Lebeau-Robbiano
- Fursikov-Imanuvilov

- Albano, Cannarsa, Zuazua, Yamamoto, Zhang, Guerrero, Fernandez-Cara, Puel, Benabdellah, Dermenjian, Le Rousseau, ... Ammar-Khodja, González-Burgos, De teresa, Dupaix, ...

1. I.I. Vrabie, the approximate controllability of (1), ($\omega = \Omega$).

2. D. Höomberg, K. Krumbiegel, J. Rehberg, Optimal Control of (1), ($\omega = \Omega$.)
3. G. Nikel and Kumpf, Approximate controllability of dynamic boundary control problems, (one-dimension heat equation)

To show the null controllability of the linear system

$$\partial_t y - d\Delta y + a(t,x)y = v(t,x)1_\omega \qquad \text{in } \Omega_T, \tag{7}$$

$$\partial_t y - \delta\Delta_\Gamma y + d\partial_\nu y + b(t,x)y = 0 \qquad \text{on } \Gamma_T, \tag{8}$$

$$y(0,\cdot) = y_0 \qquad \text{in } \overline{\Omega}, \tag{9}$$

we write its mild solution as

$$y(T,\cdot) = S(T,0)y_0 + \mathcal{T}v, \qquad \mathcal{T}v = \int_0^T S(T,\tau)(1_\omega v(\tau),0)\,d\tau.$$

$$\forall y_0, \exists v : y(T,\cdot) = 0 \iff ImS(T,0) \subset Im\mathcal{T}$$

$$\iff \exists C : \|S(T,0)^*\varphi_T\|_{\mathbb{L}^2} \le C\|\mathcal{T}^*\varphi_T\|_{\mathbb{L}^2}, \quad \varphi_T \in \mathbb{L}^2. \tag{10}$$

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Lemme

1. The function $\varphi(t) = S(T,t)^*\varphi_T$ *is the solution of the backward adjoint system*

$$\begin{aligned} -\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= 0 && \text{in } \Omega_T, \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= 0 && \text{on } \Gamma_T \\ \varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega}, \end{aligned}$$

2. The adjoint of the operator $\mathcal{T}$ *is given by*

$$\mathcal{T}^*\varphi_T = 1_\omega\varphi.$$

3. The estimate (10) *can then be written as the Observability Inequality*

$$\|\varphi(0,\cdot)\|_{\mathbb{L}^2} \le C\int_0^T\int_\omega |\varphi|^2\,dx\,dt.$$

The crucial way to show the observabity inequality is to show a Carleman estimate for the backward adjoint linear problem

$$\begin{aligned} -\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= f(t,x) && \text{in } \Omega_T, \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= g(t,x) && \text{on } \Gamma_T \quad (11) \\ \varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega}, \end{aligned}$$

for given φ_T in $H^1(\Omega)$ or in $L^2(\Omega)$, $f \in L^2(\Omega_T)$ and $g \in L^2(\Gamma_T)$.

Theorem

There are constants $C > 0$ and $\lambda_1, s_1 \geq 1$ such that, $\forall \lambda \geq \lambda_1, s \geq s_1$ and every mild solution φ of (11), we have

$$
\begin{aligned}
& s\lambda^2 \int_{\Omega_T} e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt + s^3\lambda^4 \int_{\Omega_T} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\
& + s\lambda \int_{\Gamma_T} e^{-2s\alpha}\xi|\nabla_\Gamma\varphi|^2 + s^3\lambda^3 \int_{\Gamma_T} e^{-2s\alpha}\xi^3|\varphi|^2\,dS\,dt \\
& + s\lambda \int_{\Gamma_T} e^{-2s\alpha}\xi|\partial_\nu\varphi|^2\,dS\,dt \\
\leq \quad & Cs^3\lambda^4 \int_0^T \int_\omega e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\
& + C\int_{\Omega_T} e^{-2s\alpha}|f|^2\,dx\,dt + C\int_{\Gamma_T} e^{-2s\alpha}|g|^2\,dS\,dt.
\end{aligned}
$$

Lemma

For $f = g = 0$, we obtain the following fundamental estimates

$$\begin{aligned}\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega |\varphi|^2 \, dx \, dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Gamma |\varphi|^2 \, dS \, dt \\ \leq \quad C \int_0^T \int_\omega |\varphi|^2 \, dx \, dt\end{aligned}$$

and

$$\|\varphi(0,\cdot)\|_{\mathbb{L}^2}^2 \leq C\|\varphi(t,\cdot)\|_{\mathbb{L}^2}^2, \quad 0 \leq t \leq T.$$

Proposition

Let $T > 0$, a nonempty open set $\omega \Subset \Omega$ and $a \in L^\infty(\Omega_T)$ and $b \in L^\infty(\Gamma_T)$. Then there is a constant $C > 0$ (depending on $\Omega, \omega, \|a\|_\infty, \|b\|_\infty$) such that

$$\|\varphi(0,\cdot)\|_{\mathbb{L}^2} \leq C \int_0^T \int_\omega |\varphi|^2 \, dx \, dt$$

for every mild solution φ of the homogeneous backward problem

$$\begin{aligned}
-\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= 0 && \text{in } \Omega_T,\\
-\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= 0 && \text{on } \Gamma_T\\
\varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega},
\end{aligned}$$

Theorem

Let $T > 0$ and coefficients $d, \delta > 0$, $a \in L^\infty(\Omega_T)$ and $b \in L^\infty(\Gamma_T)$ be given. Then for each nonempty open set $\omega \Subset \Omega$ and for all data $y_0 \in \mathbb{L}^2$, there is a control $v \in L^2((0,T) \times \omega)$ such that the mild solution y of (7)–(9) satisfies $y(T,\cdot) = y_\Gamma(T,\cdot) = 0$.

$$\begin{aligned} \partial_t y - d\Delta y + F(y) &= v1_\omega, \quad \text{in } \Omega_T, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + G(y_\Gamma) &= 0, \quad \text{on } \Gamma_T, \\ y(0,\cdot) &= y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}. \end{aligned} \tag{12}$$

Theorem

Let $T > 0$, $\omega \Subset \Omega$ be open and nonempty, and $F, G \in C^1(\mathbb{R}.)$ satisfy

$$F(0) = G(0) = 0 \quad \textit{and} \quad |F(\xi)|+|G(\xi)| \leq C(1+|\xi|) \quad \textit{for} \ \ \xi \in \mathbb{R}.$$

Then for all $y_0 \in \mathbb{H}^1$, there is $v \in L^2(\omega_T)$ such that (12) has a unique strong solution $y \in \mathbb{E}_1$ with $y(T,\cdot) = y_\Gamma(T,\cdot) = 0$.

Consider the Parabolic equation with dynamic boundary conditions and a control on a part Γ_0 of the boundary Γ

$$\begin{aligned} \partial_t y - d\Delta y + F(y) = 0, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + G(y_\Gamma) = v 1_{\Gamma_0}, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}. \end{aligned} \tag{13}$$

Proposition

Let $y_0 \in \mathbb{H}^2$ with $y_0 \in W_p^{2-2/p}(\Omega)$ for some $p > (N+2)/2$. Then there is a control $v \in L^2((0,T); L^2_{\mathrm{loc}}(\Gamma_0))$ such that the solution y of (13) satisfies $y(T,\cdot) = 0$ on $\overline{\Omega}$.
This solution is contained in $H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$ and has a trace in $H^1(0,T; H^{1/2}(\Gamma')) \cap L^2(0,T; H^{5/2}(\Gamma'))$ where $\Gamma' = (\Gamma \setminus \Gamma_0) \cup \Gamma_1$ for any $\Gamma_1 \Subset \Gamma_0$.

In this work, the presence of the surface diffusion $\delta\Delta_\Gamma$ in the boundary equation played a crucial role to absorb some boundary integrals.

Now, if $\delta = 0$

$$
\begin{aligned}
\partial_t y - d\Delta y + F(y) &= v1_\omega, \quad \text{in } \Omega_T,\\
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\end{aligned}
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- Show a Carleman estimate ! !
- $\delta \longrightarrow 0$! !
- explicit constant C of the observability inequality.

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Thank you very much for your attention