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An abstract on finite-time stabilizability of controllable dynamical systems

C. JAMMAZI

Ecole Polytechnique de Tunisie

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A Introduction

1. Control systems
2. Motivations and bibliography
3. Obstruction to finite-time stabilization

B Main results

1. Robustness results
2. Extensions to control systems
3. Applications

❶ Control systems

Control systems :

 

$$\dot{x} = f(x, u), \ f(0,0) = 0.$$

$x \in \mathbb{R}^n$ the state, and $u \in \mathbb{R}^m$ the control.

Controllability : $\exists ?\ u : [0, T] \to \mathbb{R}^m, u \in L^\infty :$

$$(\dot{x} = f(x, u), x(0) = a) \Rightarrow x(T) = b.$$

Stabilizability : $\exists ?\ u \in \mathscr{C}^0(\mathbb{R}^n, \mathbb{R}^m), u(0) = 0,$

$$\dot{x} = f(x, u(x)),$$

is locally (resp. globally) asymptotically stable (L A S).

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1 Motivations and bibliography

1. If u is available, is possible to, stabilize in finite-time controllable systems by $\mathscr{C}^0$– feedback law $u(x)$?
2. No ! Sontag & Sussmann (1979), Brockett (1983), Zabczyk (1989), Coron (1990), Coron & Rosier (1994).
3. In dimension 1 : $n = m = 1$.

Theorem (Sontag & Sussmann (1979))

The control system $\dot{x} = f(x, u)$ is A. S if, and only if there exists $u \in \mathscr{C}^0(\mathbb{R}, \mathbb{R})$

$$\forall\, x \in \mathbb{R} \backslash \{0\},\; x f(x, u(x)) < 0.$$

- $\dot{x} = x + u|x|$ is not $\mathscr{C}^0$-stabilizable.
  The condition $x(x + u(x)|x|) < 0$, $x \neq 0$ fails.

❷ Brockett's obstruction

Solution : Sontag & Susmann proposed the stabilization by discontinuous feedbacks.

Theorem (Brockett (1983))

If the system $\dot{x} = f(x, u)$ is $\mathscr{C}^0$-L A S, then $f(\mathbb{R}^{n+m})$ contains a neighborhood of zero .

Brockett integrateur or Heseinberg system :

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_1 u_1 - x_2 u_2.$$

Theorem (Jammazi CDC 2013))

If the system $\dot{x} = f(x, u)$ is $\mathscr{C}^0$-FTS, then $f(\mathbb{R}^{n+m})$ contains a neighborhood of zero .

What is the solution ? What do we do ?

❸ Stabilization strategies

1. **Stabilization by discontinuous feedbacks** Susmann, Coron & Rosier, Ceragioli, Bacciotti, Tsiotras, Prieur, Sontag, Clarke, ...

   Stabilization by discontinuous time-varying feedbacks Susmann, Coron Samson et Morin, Pettersen et al., Canudas de Wit et Sørdalen , Jammazi,...

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2. Stabilization by time-varying periodic feedback laws $u(x, t)$; Sontag & Susmann, Samson, Coron, Morin, Pomet, M'Closky & Murray, Pettersen, Abichou & Beji,…

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3. What do we do if some states dont't converges ?

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4. Could you ignore them ?

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5. **Finite-time partial stabilization** Jammazi, CRAS, IMA, COCV, SIAM, CDC, ECC.

4 Partial finite-time stability and stabilization

In $\mathbb{R}^n$, we consider the system $\dot{x} = X(x)$, where X is only continuous.

Partition of the state : We decompose the state $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$.

New system

$$\dot{x}_1 = f_1(x_1, x_2), \dot{x}_2 = f_2(x_1, x_2). \tag{1}$$

Partial equilibrium point : We assume that

$$f_1(0, x_2) = 0 \text{ and } f_2(0, x_2) = 0 \ \forall x_2 \in \mathbb{R}^{n-p}. \tag{2}$$

Definition

The system (1) is sait to be p-finite time partially stable if :

1. The equilibrium is Lyapunov stable.
2. $\exists\, r > 0$ and, $T = T(x(0)) > 0$ called settling-time function, such that, if $\dot{x} = X(x)$ and $|x(0)| < r$, then $x_1(t) = 0$ for every $t \geqslant T$.
3. The solution $x(t)$ starting from the initial condition $x(0)$ is defined and unique in forward time for $t \in [0, T)$.

5 Discontinuous case

In $\mathbb{R}^n$, we consider the system $\dot{x} = X(x)$, where X is not continuous.

Filippov solutions : The Filippov solutions, are maps x of the interval $I \subset \mathbb{R} \to \mathbb{R}^n$

$$\dot{x}(t) \in F(x(t)) \quad p.p.\, t \in I, \tag{3}$$

$$F(x) := \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{N})=0} \bar{co} f(B(x,\delta) \setminus \mathcal{N}) \tag{4}$$

Definition

Let $X : \mathbb{R}^n \to \mathbb{R}^n$ mesurable, L^∞. The equilibrium 0 is said LFTS for $\dot{x} = X(x)$ if

1. The equilibrium $(0,0)$ is stable in Lyapunov sense.
2. The convergence of the state x_1

$$\lim_{t\to+\infty} |x_1(t, t_0, x_0)| = 0,$$

   holds for all $(x_1(t_0), x_2(t_0))$ such that $|x_1(t_0)| + |x_2(t_0)| < \delta$
3. there exists $T = T(x(t_0)) > 0$ called settling-time function, such that, $x_1(t) = 0$ for every $t \geqslant t_0 + T$.

Some results

- For the unicycle system and chained systems.
- **Theorem (COCV 2012)** Let $\alpha \in (0,1)$, then under the feedbacks

$$u_1 = |x_1|^{\alpha_1} + |x_2|^{\alpha_1(2-\alpha)},\ \ with\ 0 < \alpha_1 < \tfrac{1-\alpha}{2-\alpha}$$

$$u_2 = -\big(sgn(x_2)\,|x_2|^{\alpha} + sgn(x_3)|x_3|^{\frac{\alpha}{2-\alpha}}\big)u_1.$$

  we have $(x_2, x_3) = (0,0) \in \mathbb{R}^2$ in finite-time and $x_1(t)$ is constant for t is large enough.
- **Theorem (SIAM 2014)**

$$\dot{x}_1 = x_2\, u_1,\, \dot{x}_2 = x_3\, u_1, \ldots \dot{x}_{n-2} = x_{n-1}\, u_1,\, \dot{x}_{n-1} = u_2,\, \dot{x}_n = u_1,$$

  is $(n-1)$ partially finite-time stabilizable by means of continuous or discontinuous feedback laws.

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B Main results

1. Links between local controllability and finite-time stability
2. Applications to control systems
3. Examples

B-1 Links between local controllability and finite-time stability

Perturbed system : In $\mathbb{R}^n$, we consider

$$\dot{x} = g(x) + h(x),\ g, h \in \mathscr{C}^0 \ and\ g(0) = f(0) = 0. \tag{5}$$

Nominal system : The system $\dot{x} = g(x)$ is said nominal system.

Question : How the FTS of $\dot{x} = g(x)$ can affect on (5) ?

Theorem 1 (IMA, 2015)

Consider the system (5) and assume that

1. $x = 0$ is finite-time stable for the nominal system $\dot{x} = g(x)$,
2. the settling time $T : x \mapsto T(x)$ is a continuous map on $\mathbb{R}^n$,
3. for all $i = 1, \ldots, n$, $\lim_{|x| \to 0} \dfrac{|h_i(x)|}{|g_i(x)|} = 0$,
   $h = (h_1, \ldots, h_n)'$ and $g = (g_1, \ldots, g_n)'$.

Then, the origin of the perturbed system (5) is locally finite-time stable.

B-1 Exentension of Bhat and Bernstein condition

Corollary (IMA, 2015)

Consider the system (5) and assume that

1. $x = 0$ is finite-time stable for the nominal system $\dot{x} = g(x)$, with Lyapunov function V satisfies

$$\dot{V} + c\,V^{\alpha} \leqslant 0,$$

where $\alpha \in (0, 1)$ and $c > 0$.

2. For all $i = 1, \ldots, n$, $\lim_{|x| \to 0} \frac{|h_i(x)|}{|x|} = 0$.

Then, the origin of the perturbed system (5) is locally finite-time stable.

Generalization of Bhat and Bernstein result, without restriction on α.

Proof. Since the nominal system $\dot{x} = g(x)$ is finite-time stable with with Lyapunov function V satisfies $\dot{V} + cV^{\alpha} \leqslant 0$, then, the solutions are defined in forward time [Section 2, p. 752]Bernstein :2000, and Rosier :05, and the vector field g with $g(0) = 0$ satisfies the inequality

$$|g(x)| \leqslant k|x|^{\beta}, \beta \in (0,1), k > 0. \tag{6}$$

Let be β' such that $0 < \beta' < \beta$, then we get $|g(x)| = o(|x|^{\beta'})$. Then, from (6) and the above remark, is not hard to obtain

$$\lim_{|x|\to 0} \frac{|h_i(x)|}{|g_i(x)|} = \lim_{|x|\to 0} \frac{|h_i(x)|}{|x|} \frac{|x|}{|g_i(x)|} = 0.$$

The local finite-time stability follows then from Theorem 1.

Consider the control system

$$\dot{x} = f(x,u),\ f(0,0) = 0 \ and \ f \in \mathscr{C}^1. \tag{7}$$

Linearized system The linearized system around the equilibrium point $(x_e, u_e) = (0,0)$ is defined by $\dot{x} = Ax + Bu$ where

$$A = \frac{\partial f}{\partial x}(0,0), \quad B = \frac{\partial f}{\partial u}(0,0).$$

In this case, we have the expansion :

$$\dot{x} = f(x,u) = Ax + Bu + g(x,u),$$

with

$$\lim_{|(x,u)| \to (0,0)} \frac{g(x,u)}{|(x,u)|} = 0, \tag{8}$$

Theorem 2 (IMA, 2015)

If the linearized system , $\dot{x} = Ax + Bu$, is controllable then (7) is locally stabilizable in finite-time by means of explicit continuous or discontinuous feedback laws.

Proof of continuous case :

- If $\dot{x} = Ax + Bu$ is controllable, then, thanks to [Bernstein :2005], this is globally finite-time stable by continuous state feedback laws.
- Controllability condition - and Brunovsky transformation - lead to seen the system as a collection of decoupled independently controlled of chains of integrators [Sontag :1990].
- Let us consider an example of this family of chains which is presented as follows [Moulay :2009]

$$\dot{z}_1 = z_2, \dot{z}_2 = z_3, \ldots, \dot{z}_n = v. \tag{9}$$

$$\bar{v}(z(t)) = -k'_1 [z_1]^{\alpha_1} - \ldots - k'_n [z_n]^{\alpha_n}, \ [\delta]^r := sgn(\delta)|\delta|^r, \tag{10}$$

where $\alpha_2, ..., \alpha_n$ satisfy

$$\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, \ i = 2, \ldots, (n-1), \tag{11}$$

with $\alpha_{n+1} = 1$, $\alpha_n = \alpha$.

B-2 Proof of Theorem 2

We write $f(x, u) = Ax + Bu + g(x, u)$ with assumption (8) ($g(x, u) = o((x, u))$), then, if we denote by $\bar{u}(x)$, the continuous finite-time stabilizing feedback laws of the collection of chains of integrators, it follows that, by the continuity of $\bar{u}$ at zero,

$$\lim_{|x|\to 0} \frac{g(x, \bar{u}(x))}{|(x, \bar{u}(x))|} = \lim_{(|x, \bar{u}(x))|\to 0} \frac{g(x, \bar{u}(x))}{|(x, \bar{u}(x))|} = 0.$$

Proof of discontinuous case

Controllability of linear system $\dot{x} = Ax + Bu$ leads to finite-time stabilization of (7) (Jammazi :2014).

We define the function

$$s(z) := z_n + h(z_1, z_2, ..., z_{n-1}), \tag{12}$$

where $h(z_1, z_2, ..., z_{n-1}) = -k_1' \lceil z_1 \rfloor^{\alpha_1} - \ldots - k_{n-1}' \lceil z_{n-1} \rfloor^{\alpha_{n-1}}$.

Then the feedback

(13)

$$\overline{v}(z) = \begin{cases} -k\Big(1 + \sum_{i=1}^{n-1} |z_i|\Big) sgn(s), \; if \;\; s \neq 0, \text{k is large enough,} \\ 0, \;\; if \;\; s = 0, \end{cases}$$

makes the studied system reaches the set $S := \{z \in \mathbb{R}^n : s(z) = 0\}$ in finite-time, and in the sliding surface, the state gets to the equilibrium in finite-time.

Thus, the system in closed loop is FTS. If we denotes by $\bar{u}(x)$ the sliding finite-time stabilizing feedback laws of the collection of chains of integrators. Then is not hard to see that $g(x, \overline{u}(x)) = o(|(x, \overline{u}(x))|)$ when $x \to 0$. This achieves the proof.

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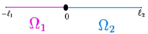
Application : Finite-time of controlled PDF

$$\rho_1 u_{tt} = \sigma_1 u_{xx}, \qquad x \in \Omega_1, \qquad t > 0$$
$$\rho_2 v_{tt} = \sigma_2 v_{xx}, \qquad x \in \Omega_2, \qquad t > 0.$$



- Works of Hansan & Zuazua, Carlos & Zuazua (93, 2000-2007) on vibrating structures.
- The position of the mass $M > 0$ attached to the point $x = 0$ is described by the function $z = z(t)$ for $t > 0$.
  The dynamic of point charge is given by : $Mz_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0$.
- **Problem :** How to build feedbacks to remove the vibration of strings in finite time ? and what about the point mass ?
- The inequality $\dot{E} \leqslant -c\,E^{\alpha}$ is hard for this problem !
- Only possible for Schrödinger equation with damping (Work of R. Carles and C. Gallo, 2011-2013)
- The solution will be presented in the talk of Ghada Ben Belgacem !

Thank you for your attention !