Finite time stabilization : some particular examples

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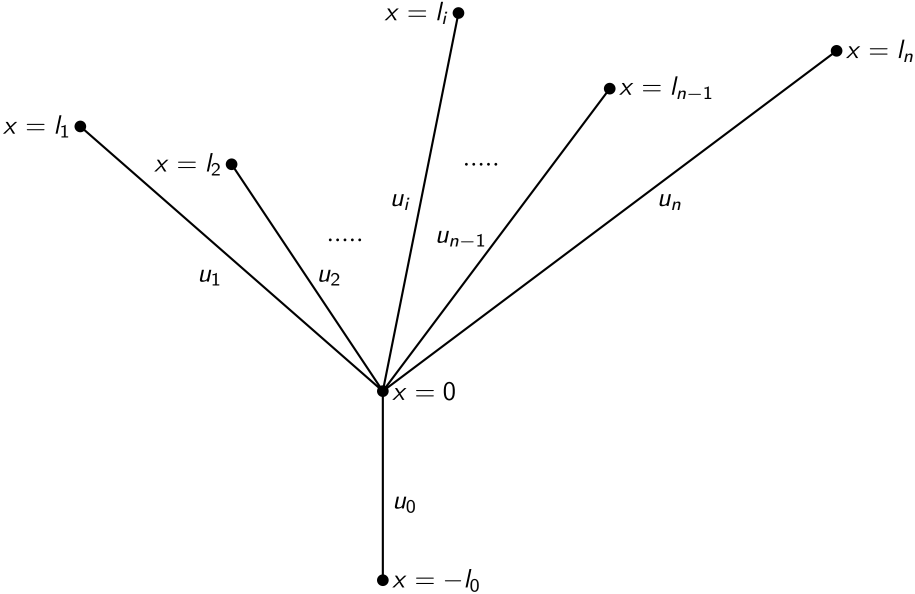
Outline


$x = l_i$
$x = l_n$
$x = l_{n-1}$
$x = l_1$
$x = l_2$
.....
u_i
u_n
.....
u_{n-1}
u_1
u_2
$x = 0$
u_0
$x = -l_0$

Simple Network of Strings

- On the top edges :

$$\forall i \in \{1, ..., n\} \qquad \partial_{tt}^2 u_i - c_i^2 \partial_{xx}^2 = 0, \qquad (t,x) \in \mathbb{R}^+ \times (0, l_i)$$
$$\partial_x u_i(t, l_i) = -\frac{\partial_t u_i}{c_i}, \qquad t \in \mathbb{R}^+$$

- On the bottom edge :

$$\partial_{tt}^2 u_0 - c_0^2 \partial_{xx}^2 u_0 = 0 \qquad (t,x) \in \mathbb{R}^+ \times (-l_0, 0)$$
$$u_0(t, -l_0) = 0 \qquad (\text{or} \quad \partial_x u_0(t, -l_0) = 0) \qquad t \in \mathbb{R}^+.$$

- At the middle node :

$$\forall t \geq 0, \quad \forall i \in \{1, ..., n\}, \qquad u_0(t, 0) = u_i(t, 0),$$

$$\sum_{i=1}^{n} c_i \partial_x u_i(t, 0) - c_0 \partial_x u_0(t, 0) = -\alpha \partial_t u(t, 0).$$

Finite time stabilization

Theorem (Alabau, P., Rosier)

The previous system generates a strongly continuous semigroup if and only if $\alpha \neq n+1$. This semigroup is finite time stable if and only if $\alpha = n-1$. More precisely :

$$\forall t \geq 2(l_0 + \max(l_1, ..., l_n)), \quad \forall i \in \{0, ..., n\}, \qquad u_i(t,.) = 0.$$

Remarks

- Theorem actually valid for all tree-shaped graphs.
- Main tool for existence : Banach fixed point in weighted space to decouple edges.
- Main tool for stabilization : new unknowns.

$$\forall i \in \{0, \ldots, n\}, \qquad \begin{aligned} P_i(t,x) &:= \partial_t u_i(t,x) - c_i \partial_x u_i(t,x), \\ N_i(t,x) &:= \partial_t u_i(t,x) + c_i \partial_x u_i(t,x). \end{aligned}$$

Evolution of new unknowns for $\alpha = n - 1$

- New equations :

$$\forall i \in \{0, ..., n\}, \qquad \begin{cases} \partial_t P_i + c_i \partial_x P_i = 0, \\ \partial_t N_i - c_i \partial_x N_i = 0. \end{cases}$$

- At bottom and top :

$$P_0(t, -l_0) = \pm N_0(t, -l_0), \qquad \forall i \in \{1, ..., n\}, \; N_i(t, l_i) = 0.$$

- At the middle node :

$$N_0(t,0) = \frac{\sum_{i=1}^{n} N_i(t,0)}{n},$$

$$P_k(t,0) = P_0(t,0) + \frac{\sum_{i \neq k} N_i(t,0)}{n} - (1 - \frac{1}{n}) N_k(t,0).$$

The case of the transport equation

$$\begin{aligned} \partial_t y + c\partial_x y &= 0, \qquad (t,x) \in (0,T) \times (0,L) \\ y(t,0) &= 0, \qquad t \in (0,T). \end{aligned}$$

Using the method of characteristics :

$$y(t,x) = \begin{cases} y_0(x-ct) & \text{if } x > ct, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \geq L$, $y(t,.) = 0$.

A Family of Lyapunov Functionals

- For $\nu > 0$:

$$J_\nu(t) := \int_0^L y^2(t,x) e^{-\nu x} dx.$$

- Formally at least :

$$\begin{aligned} \dot{J}_\nu(t) &= \int_0^L 2y_t(t,x) y(t,x) e^{-\nu x} dx \\ &= \int_0^L -2c y_x(t,x) y(t,x) e^{-\nu x} dx \\ &= [-c y^2(t,x) e^{-\nu x}]_0^L - c\nu J_\nu(t) \\ &\leq -c\nu J_\nu(t). \end{aligned}$$

- Using Gronwall :

$$J_\nu(t) \leq e^{-c\nu t} J_\nu(0).$$

Return on the L^2 norm

- Norm equivalence

$$\forall t \geq 0, \qquad e^{-\nu L}||y(t,.)||^2_{L^2(0,L)} \leq J_\nu(t) \leq ||y(t,.)||^2_{L^2(0,L)}.$$

- Inequality on L^2

$$||y(t,.)||^2_{L^2(0,L)} \leq e^{-\nu c(t-\frac{L}{c})}||y_0||^2_{L^2(0,L)},$$

- For $t \geq \frac{L}{c}$, letting $\nu \to +\infty$ we get $y(t,.) = 0$.

Remarks

- Can be adapted to general "transport" type equations (for instance quasilinear systems of conservation laws).
- Good for robustness estimate and perturbation :

$$y_t + cy_x = \epsilon g(y),$$

$$y_t + cy_x = \epsilon y_{xx}.$$

- In certain cases, useful for exact controllability to trajectory.

An abstract result

Theorem (Xu)

Let H be a separate Hilbert space. Let B be the infinitesimal generator of $S(t)$ a C_0 semigroup. If $R(\lambda, B)$ $(:= (\lambda Id - B)^{-1})$ is an entire function of finite exponential type i.e.

$$\exists \gamma, C > 0, \qquad s.t. \qquad \forall \lambda \in \mathbb{C}, \quad |||R(\lambda, B)||| \leq Ce^{\gamma|\lambda|},$$

then we have :

$$\forall t > \gamma, \qquad S(t) = 0.$$

Case of transport equation

$$\partial_t y + c\partial_x y = 0, \quad x \in (0, L), \qquad y(t, 0) = 0,$$

then $(\lambda Id - B)u = f$ becomes

$$\lambda u - c\dot{u} = f, \qquad u(0) = 0,$$

and so

$$u(x) = -\int_0^x e^{\frac{\lambda(x-r)}{c}} \frac{f(r)}{c} dr,$$

from which we get :

$$||u||_{L^2}^2 \leq \frac{L^2}{c^2} e^{2\frac{L|\lambda|}{c}} ||f||_{L^2}^2.$$

THANK YOU FOR YOUR ATTENTION