Study of an extremal problem for eigenvectors of some Sturm-Liouville problems

Thibault Liard, Pierre Lissy, Yannick Privat

January 2015

Let $T > 0$, $L > 0$ and $\omega \subset (0, L)$, measurable.

$$\begin{array}{ll} \partial_{tt}\varphi(t,x) - \partial_{xx}\varphi(t,x) + a(x)\varphi(t,x) = 0, & (t,x) \in (0,T) \times (0,L), \\ \varphi(t,0) = \varphi(t,\pi) = 0, & t \in [0,T], \\ \varphi(0,x) = \varphi_0(x),\ \partial_t\varphi(0,x) = \varphi_1(x), & x \in [0,L], \end{array} \tag{Eq-wave}$$

where the potential $a(\cdot) \in L^\infty(0, L)$ is non-negative.

Definition

The equation (Eq-wave) *is said to be observable on* ω *in time* T *if there exists a positive constant* C *such that*

$$C \int_0^\pi \left(\varphi_1(x)^2 + \varphi_0'(x)^2 + a(x)\varphi_0(x)^2\right)\, dx \leq \int_0^T \int_\omega \partial_t\varphi(t,x)^2\, dxdt, \tag{Obs-wave}$$

for all $(\varphi_0, \varphi_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$.

We denote by $C_{T,\mathrm{obs}}(\omega)$ the largest constant in the previous inequality

Let $L > 0$. Let $a(\cdot) \in L^\infty(0, L)$ such that $a \geq 0$ a.e. We consider the operator

$$A_a := -\partial_{xx} + a(\cdot)\,\mathrm{Id}$$

defined on $\mathcal{D}(A_a) = H_0^1(0, L) \cap H^2(0, L)$. Let us denoted by $e_{a,j} \in \mathcal{D}(A_a)$ a Hilbert basis of eigenfunctions in $L^2(0, L)$, such that $e_{a,j}$ solves the eigenvalue problem

$$\begin{cases} -e''_{a,j}(x) + a(x)e_{a,j}(x) = \lambda_{a,j}^2 e_{a,j}(x), \; x \in (0, L), \\ e_{a,j}(0) = 0, \\ e_{a,j}(L) = 0. \end{cases} \tag{Pb-vp}$$

We choose $\int_0^L e_{a,j}^2(x)dx = 1$.

Theorem

We have

$$C_{T,\mathrm{obs}}(\omega) \sim T \inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}(x)^2 dx, \quad \text{as } T \to \infty$$

There exists $T_1 > 0$ such that for all $T > T_1$,

$$C_{T,\mathrm{obs}}(\omega) \geq K(T) \inf_{j\in\mathbb{N}^*} \int_\omega e_{a,j}(x)^2 \, dx > 0,$$

with $K(T) > 0$.

Let $M > 0$. We introduce

$$\mathcal{A}_M = \{a \in L^\infty(0,L) \text{ such that } 0 \leq a \leq M \text{ a.e. on } (0,L)\},$$

Problem 1 : (L^∞-constraint on a)

$$\inf_{a \in \mathcal{A}_M} \inf_{\substack{\omega \subset (0,L) \\ \text{s.t. } |\omega| = rL}} \inf_{j \in \mathbb{N}^*} \int_\omega e_{a,j}(x)^2 \, dx, \qquad (\text{Pb-}\mathcal{A}_M)$$

Remarks (when we fix j)

- *If we do not restrict our search to those subsets ω verifying $|\omega| = rL$, the problem is trivial.*
- *If we fix the potential a, it is well known that there exists $\tau \in \mathbb{R}_+$ such that*

$$\inf_{\substack{\omega\subset(0,L)\\ s.t.\ |\omega|=rL}} \int_\omega e_{a,j}(x)^2\,dx = \int_{\omega^*} e_{a,j}(x)^2\,dx,$$

where $\omega^* = \{e_{a^*,j}(x)^2 < \tau\}$ up to a set of zero Lebesgue measure.

- *If $a = 0$ and $L = \pi$, so $e_{0,j}(x) = \sin(jx)$. We have the following inequality*

$$\int_\omega \sin(jx)^2\,dx \geq \frac{|\omega| - \sin|\omega|}{2}, \quad \textit{for every measurable } \omega.$$

Theorem

Let $r \in (0,1)$ and $M \in \mathbb{R}_+^*$.

1. *Problem* (Pb-$\mathcal{A}_M$) *has a solution* (j_0, ω^*, a^*). *In particular, there holds*

$$m(L,r) \quad = \quad \min_{a \in \mathcal{A}_M(0,L)} \min_{\omega \in \Omega_r(0,L)} \int_\omega e_{a,j_0}(x)^2 \, dx,$$

   and the solution a^* of Problem (Pb-$\mathcal{A}_M$) *is bang-bang, equal to* 0 *or M a.e. in* $(0,L)$.

2. *Assume that $M = \pi^2/L^2$. Then, ω^* is the union of $j_0 + 1$ intervals, and a^* has at most $3j_0 - 1$ and at least j_0 switching points Moreover, one has the estimate*

$$\gamma r^3 \leq m(L,r) \leq r - \frac{\sin(\pi r)}{\pi}, \tag{1}$$

   with $\gamma = \frac{7\sqrt{3}}{8}(3 - 2\sqrt{2}) \simeq 0.2600$.

Let $r \in (0,1)$ and $M \in (0, \pi^2/L^2]$. There holds

$$m_j(L,r) := \inf_{a \in \mathcal{A}_M(0,L)} \inf_{\omega \in \Omega_r(0,L)} \int_\omega e_{a,j}(x)^2 \, dx \geq \underline{m}_j,$$

for every $j \in \mathbb{N}^*$, where the sequence $(\underline{m}_j)_{j \in \mathbb{N}^*}$ is defined by

$$\underline{m}_j = \begin{cases} \frac{1}{2} & \text{if } j = 1, \\ \dfrac{(2j^2-1)(j^2-1)^{\frac{3}{2}}\left(\sqrt{\frac{j^2}{j^2-2}}-1\right)^2}{3j^3\left(\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}}-1\right)^2} & \text{if } j \geq 2. \end{cases}$$

What about the L^∞ constraint

The constraint of the L^∞-norm is mandatory, in the following meaning :

Theorem

Let $r \in (0,1)$, $j \in \mathbb{N}^*$ and $V > 0$. The optimal design problem

$$\inf_{a \in \mathcal{A}_\infty} \inf_{\substack{\omega \subset (0,L) \\ s.t.\ |\omega| = rL}} \int_\omega e_{a,j}(x)^2 \, dx, \qquad \text{(PbInf)}$$

where $\mathcal{A}_\infty = \cup_{M>0} \mathcal{A}_M$, has no solution.

In the case : $j = 1$

Since the eigenfunction $e_{a,1}$ is normalized in $L^2(0, L)$, there holds

$$e_{a,1}^2(x_{max}) \geq \frac{3}{2L}. \tag{2}$$

Since $e_{a,1}$ is concave one has the successive inequalities

$$e_{a,1}(x) \geq Tr_{a,1}(x) \geq \triangle_1(x), \tag{3}$$

for every $x \in [0, L]$.

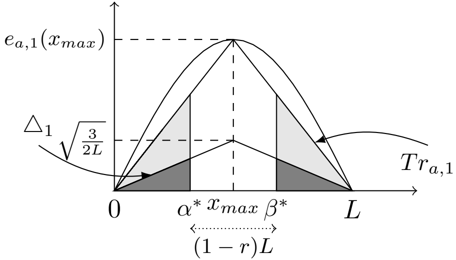

FIGURE : Graphs of the functions $e_{a,1}$, $Tr_{a,1}$ and $\triangle_1$.

We readily obtain

$$\inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,1}(x)^2\,dx \geq \inf_{\omega\in\Omega_r(0,L)} \int_\omega \triangle_1(x)^2 dx = \int_{\hat{\omega}} \triangle_1(x)^2 dx, \qquad (4)$$

with $\hat{\omega} = (0,\alpha^*) \cup (\beta^*, L)$ verifying

$$\triangle_1(\alpha^*) = \triangle_1(\beta^*) \qquad \text{and} \qquad |\omega^*| = L - \beta^* + \alpha^* = rL.$$

one computes

$$\int_{\hat{\omega}} \triangle_1(x)^2\,dx = \frac{r^3}{2}. \qquad (5)$$

In the general case

Let $0 = x_j^0 < x_j^1 < x_j^2 < L = x_3^3$ be the 4 zeros of the 3-th eigenfunction $e_{a,3}$.

First step : for every $i \in \{1, 2, 3\}$, there exist $A_i > 0$ such that $e_{a,j}(x_{max}^i) = \max_{x \in \Omega^i} e_{a,j}(x) \geq \sqrt{A_i}$.

Second step :

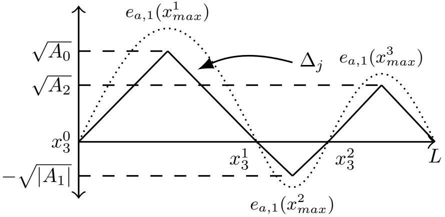

FIGURE : Graphs of the functions $e_{a,1}$ and $\triangle_j$.

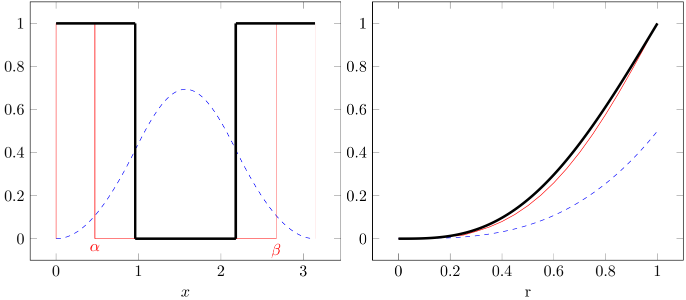

FIGURE : $L = \pi$ and $M = 1$. Left : plots of the optimal set $\omega(-)$, a(-) and $e_{a,1}^2$ $(\dots)$ with respect to the space variable with $r = 0.3$. Right : plot of $r \mapsto m_1(L, r)$ $(-)$, $r \mapsto r - \frac{\sin(\pi r)}{\pi}$ (-) and $r \mapsto r^3/2\,(\cdots)$.

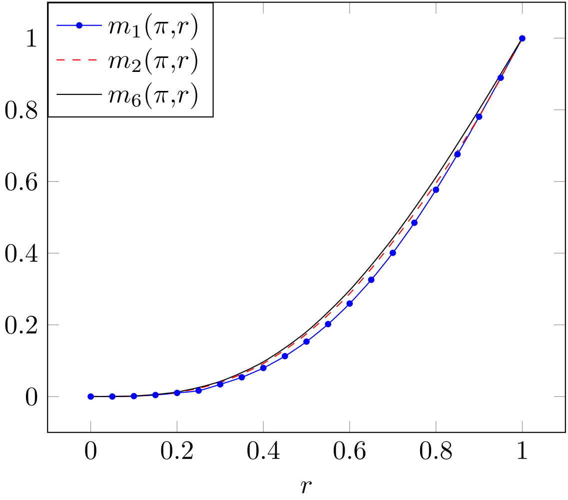

FIGURE : $L = \pi$ and $M = 1$. Plots of $m_j(\pi,r)$ for $j = 1(\circ)$, $j = 2(\text{- -})$ and $j = 6$ with respect to r.

Thank you for you attention