Null controllability of the Kolmogorov equation in the whole space

joint work with J. Le Rousseau (Université d'Orléans)

Iván Moyano

CMLS, École Polytechnique

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The Kolmogorov equation in $\mathbb{R}^{2d}$

$$\begin{cases} (\partial_t + v \cdot \nabla_x - \Delta_v) f(t,x,v) = 1_\omega(x,v) u(t,x,v), & \text{in } (0,T) \times \Omega, \\ f(0,x,v) = f_0(x,v), & \text{in } \Omega, \end{cases}$$

where $\omega \subset \Omega = \mathbb{R}^{2d}$, $d \geq 1$.

- STATE: $f(t,x,v)$
- CONTROL: $u(t,x,v)$, supported in ω.

GOAL Null controllability in L^2-setting:

$$\forall T > 0, \ \forall f_0 \in L^2(\mathbb{R}^{2d}), \ \exists u \in L^2((0,T) \times \mathbb{R}^{2d}) \text{ such that } f(T,\cdot,\cdot) = 0.$$

DIFFICULTIES

1. HYPOELLIPTICITY AND DEGENERACY
2. UNBOUNDED DOMAIN

Previous work on Kolmogorov-type equations

$$\left(\partial_t + v^\gamma \partial_x - \partial_v^2\right) f(t,x,v) = 1_\omega(x,v) u(t,x,v), \quad \text{in } (0,T) \times \Omega,$$

Null-controllability results in an L^2-setting:

- K. Beauchard and E. Zuazua (Ann. IHP, 2009) in the case $\gamma = 1$, $\Omega = \mathbb{R}^2$, $\omega = \mathbb{R} \times (\mathbb{R} - [a,b])$,
- K. Beauchard (MCSS, 2014)
  - in the case $\gamma = 1$, $\Omega = \mathbb{T} \times (-1,1)$ with *periodic* boundary conditions and ω an arbitrary open subset,
  - in the case $\gamma = 1$ with Dirichlet conditions and $\omega = \mathbb{T} \times (a,b)$, $-1 < a < b < 1$.
  - in the case $\gamma = 2$ with Dirichlet conditions and $\omega = \mathbb{T} \times (a,b)$, with $-1 < a < 0 < b < 1$. However, if $0 < a < b < 1$, there is a minimal time.
- K. Beauchard, B. Helffer, R. Henry and L. Robbiano (ESAIM COCV, 2014) in the case $\Omega = \mathbb{T} \times (-1,1)$ with Dirichlet boundary conditions. In that case, arbitrary control regions do not seem suitable $\Rightarrow$ GCC and minimal time

DIFFICULTY Observability in unbounded domains: the obrservability region ω must be 'large enough'.

NEGATIVE AND POSITIVE RESULTS:
S. Micu, E. Zuazua, (Trans. AMS, 2001), V.R. Cabanillas, S.B. de Menezes, E. Zuazua (J. Opt. Appl., 2001). L. Escauriaza, G. Seregin, V. Sverak (ARMA, 2003).

INTRODUCTION OF WEIGHTS
P. Cannarsa, P. Martinez, J. Vancostenoble (ESAIM, 2004).

NECESSARY OR SUFFICIENT CONDITIONS IN DOMAINS WITH BOUNDARY
L. Miller (Bull. Sci. Math, 2005). M. González-Burgos, L. de Teresa, (Adv, Diff, Eq., 2007). V. Barbu (ESAIM COCV, 2014).

A sufficient geometric condition for observability

We give a geometric condition which is sufficient for observabiity in the whole space:

Definition

Let $\omega \subset \mathbb{R}^d$, $d \geq 1$. We say that ω is an observability open set on the whole space if there exist $\delta, r > 0$ such that

$$\forall y \in \mathbb{R}^d, \exists y' \in \omega \text{ such that } B_{\mathbb{R}^d}(y', r) \subset \omega \text{ and } |y - y'| \leq \delta.$$

EXAMPLES:

- $\omega = \left(\mathbb{R}^N - B_{\mathbb{R}^N}(x_1, r_1)\right) \times \left(\mathbb{R}^N - B_{\mathbb{R}^N}(x_2, r_2)\right)$, $d = 2N$,
- Periodic structure. $\omega = \prod_{i=1}^d \bigcup_{k \in \mathbb{Z}} (a_i + h_i k, b_i + h_i k)$, with $h_i > b_i - a_i > 0$, $\forall i = 1, \ldots, d$.
- More general structures not necessarily periodic.

Our main result

Let $\omega := \omega_x \times \omega_v$ be a subset of $\mathbb{R}^{2d}$ such that

ω_x and ω_v are open observability sets in the whole space in $\mathbb{R}^d$.

Our main result is the following

Theorem

For every $T > 0$ and $f_0 \in L^2(\mathbb{R}^{2d}, \mathbb{R})$, there exists a control $u \in L^2((0, T) \times \mathbb{R}^{2d}, \mathbb{R})$ such that the solution of

$$\begin{cases} (\partial_t + v \cdot \nabla_x - \Delta_v) f(t, x, v) = 1_\omega u(t, x, v), & \text{in } (0, T) \times \mathbb{R}^{2d}, \\ f(0, x, v) = f_0(x, v), & \text{in } \mathbb{R}^{2d}, \end{cases}$$

satisfies $f(T, \cdot, \cdot) \equiv 0$.

Difficulty of the proof: HUM

NULL CONTROLLABILITY $\Leftrightarrow$ OBSERVABILITY of the adjoint system

$$\begin{cases} (\partial_t - v\cdot\nabla_x - \Delta_v)\, g(t,x,v) = 0, & (t,x,v)\in(0,T)\times\mathbb{R}^{2d},\\ g(0,x,v) = g_0(x,v), & (x,v)\in\mathbb{R}^{2d}, \end{cases}$$

i.e. $\exists C>0$ such that

$$\int_{\mathbb{R}^{2d}} |g(T)|^2\,\mathrm{d}x\,\mathrm{d}v \le C\int_0^T\int_\omega |g(t)|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v,\ \forall g_0\in L^2(\mathbb{R}^{2d}).$$

DIFFICULTY: Appropriate 2D Carleman inequalities are not known.

IDEA: Taking the Fourier transform with respect to x i.e.,

$$\hat{g}(t,\xi,v) := \int_{\mathbb{R}^d} g(t,x,v)e^{-i\xi\cdot x}\,\mathrm{d}x,$$

the 2d-dimensional equation reads

$$(\partial_t - iv\cdot\xi - \Delta_v)\,\hat{g}(t,\xi,v) = 0,$$

a family of d-dimensional heat equations indexed by $\xi\in\mathbb{R}^d$.

Detailed strategy: separating x and v

What to do in VARIABLE v?
GLOBAL CARLEMAN inequality for $\partial_t - iv \cdot \xi - \Delta_v$ in $\mathbb{R}^d$, observing from ω_v.

What to do in VARIABLE x?

- In the unbounded case by K. Beauchard-E. Zuazua, the observation was made for all $x \in \mathbb{R}$.
- In the bounded with periodic case by K. Beauchard, the observation is made from $\omega = (a, b) \times (c, d)$, thanks to a Lebeau-Robbiano spectral inequality, following A. Benabdallah, Y.Dermenjian and J. Le Rousseau (J. Math. Anal. Appl, 2007).

CONCLUSION We must obtain a Lebeau-Robbiano *spectral* inequality in the unbounded case, taking ω_x as an observability set in $\mathbb{R}^d$.

Key point of the Lebeau-Robbiano strategy

Proposition

Let $c, d \in \mathbb{R}$ be such that $0 < d - c < 2\pi$. There exists $C > 0$ such that, for every $N \in \mathbb{N}^*$ and $(b_n)_{|n| \leq N} \in \mathbb{C}^{2N+1}$, the following inequality holds

$$\sum_{n=-N}^{N} |b_n|^2 \leq e^{CN} \int_c^d \left| \sum_{n=-N}^{N} b_n e^{inx} \right|^2 \mathrm{d}x.$$

This allows to construct a semiexplicit control combining two ingredients:

- the DISSIPATION of the system without control
- an OBSERVABILITY INEQUALITY for initial data whose Fourier transform is compactly supported.

DIFFICULTY The original proof of this inequality makes an essential use of the boundedness of the domain (interpolation inequalities).

$$(K)\begin{cases}(\partial_t + v\cdot\nabla_x - \Delta_v)f(t,x,v) = 1_\omega u(t,x,v), & \text{in } (0,T)\times\mathbb{R}^{2d},\\ f(0,x,v) = f_0(x,v), & \text{in } \mathbb{R}^{2d}.\end{cases}$$

Following K. Beauchard and E. Zuazua we prove the $L^2(\mathbb{R}^{2d})$-WP.

Proposition

$\forall f_0 \in L^2(\mathbb{R}^{2d},\mathbb{R}),\ T>0,\ u\in L^2((0,T)\times\mathbb{R}^{2d},\mathbb{R}),$
$\exists f \in C^0([0,T],L^2(\mathbb{R}^{2d}))$ unique *weak* solution of (K).

We also have the following parabolic decay.

Proposition

For every $f_0 \in L^2(\mathbb{R}^{2d},\mathbb{R})$, the solution of (K) with $u\equiv 0$ satisfies

$$\|\hat{f}(t,\xi,\cdot)\|_{L^2(\mathbb{R}^d)} \le \|\hat{f}_0(\xi,\cdot)\|_{L^2(\mathbb{R}^d)} e^{-\frac{|\xi|^2t^3}{12}},\ \ \forall\xi\in\mathbb{R}^d,\ \forall t\in\mathbb{R}^+.$$

A Lebeau-Robbiano type inequality

Let ω_x be an observability set in $\mathbb{R}^d$.

Proposition

There exists $C > 0$ such that $\forall N \geq 1$ and $f \in L^2(\mathbb{R}^d)$ such that $supp(\hat{f}) \subset \overline{B_{\mathbb{R}^d}(0, N)}$,

$$\|f\|_{L^2(\mathbb{R}^d)} \leq e^{C(N+1)} \|f\|_{L^2(\omega_x)}.$$

What about the proof?
We follow F. Boyer, F. Hubert and J. Le Rousseau (JMPA, 2010):
We derive the Lebeau-Robbiano type inequality from a GLOBAL ELLIPTIC CARLEMAN ESTIMATE for $-\partial_t^2 - \Delta_x$.
Weight in $\mathbb{R}^d$: $\exists \psi \in C^3 \cap W^{3,\infty}([0, S] \times \mathbb{R}^d, \mathbb{R}^+)$ s.t for some $C > 0$,

$$\begin{aligned} |\nabla_{s,x}\psi| &\geq C, \ \text{ in } [0, S] \times \mathbb{R}^d, \qquad & \partial_s\psi(S, x) &< -C, \ \text{ in } \mathbb{R}^d, \\ \partial_s\psi(0, x) &\geq C, \ \text{ on } \omega_x^c. \qquad & \psi(S, x) &= 0, \ \text{ in } \mathbb{R}^d. \end{aligned}$$

Proof of the Lebeau-Robbiano inequality

GOAL:

$$\frac{1}{2\pi}\int_{B_{\mathbb{R}^d}(0,N)} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \leq e^{2C(N+1)} \int_{\omega_x} |f(x)|^2 \,\mathrm{d}x.$$

Global elliptic Carleman estimate for $P = -\partial_s^2 - \Delta_x$

Let $\omega_x \subset \mathbb{R}^d$ be an observability open set on the whole space $\mathbb{R}^d$ and $Q = (0,S) \times \mathbb{R}^d$. $\exists \psi \in C^3 \cap W^{3,\infty}([0,S] \times \mathbb{R}^d)$ such that for $\varphi(s,x) = \exp\left(\lambda\psi(s,x)\right)$, there exist $C > 0$, $\tau_0 \geq 1$, and $\lambda_0 \geq 1$ such that

$$\begin{aligned}
&\tau^3\|e^{\tau\varphi}u\|^2_{L^2(Q)} + \tau\|e^{\tau\varphi}\nabla u\|^2_{L^2(Q)} + \tau\|e^{\tau\varphi(0)}\partial_s u_{|s=0}\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad + \tau e^{2\tau}\|\partial_s u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \tau^3 e^{2\tau}\|u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} \\
&\leq C\left(\|e^{\tau\varphi}Pu\|^2_{L^2(Q)} + \tau e^{2\tau}\|\nabla_x u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \tau\|e^{\tau\varphi(0)}\partial_s u_{|s=0}\|^2_{L^2(\omega_x)}\right),
\end{aligned}$$

for $\tau \geq \tau_0$, $\lambda \geq \lambda_0$, and $u \in C^2([0,S], \mathcal{S}(\mathbb{R}^d,\mathbb{C}))$ such that $u_{|s=0} \equiv 0$.

Let $u(t,x) = \frac{1}{(2\pi)^d}\int_{B_{\mathbb{R}^d}(0,N)} \frac{\sinh(\xi t)}{\xi}\hat{f}(\xi)e^{i\xi\cdot x}\,\mathrm{d}\xi$. The goal follows using Plancherel adequately.

Observability of one Fourier mode

There exists a constant $C > 0$ such that for every $T > 0$, $g_{0,\xi} \in L^2(\mathbb{R}^d, \mathbb{C})$ and $\xi \in \mathbb{R}^d$, the solution of

$$\begin{cases} \partial_t g_\xi - i\xi \cdot v g_\xi - \Delta_v g_\xi = 0, & (t, v) \in (0.T) \times \mathbb{R}^d, \\ g_\xi(0, v) = g_{0,\xi}(v), & v \in \mathbb{R}^d, \end{cases}$$

satisfies

$$\int_{\mathbb{R}^d} |g_\xi(T, v)|^2 \,\mathrm{d}v \le e^{C(1+\frac{1}{T}+\sqrt{|\xi|})} \int_0^T \int_{\omega_v} |g_\xi(t, v)|^2 \,\mathrm{d}v \,\mathrm{d}t.$$

PROOF: global parabolic Carleman estimate for $P_\xi = \partial_t + i\xi \cdot v - \partial_v^2$ with weight $e^{\frac{\tau\varphi(v)}{t(T-t)}}$ and

$$\tau \ge C(T + T^2\sqrt{|\xi|}), \quad \varphi = e^{\lambda\tilde{\psi}(v) - 2\lambda\|\tilde{\psi}\|_\infty}$$

to deal with $i\xi \cdot v$ and $\tilde{\psi}$ similar to the elliptic case.

Observability of Fourier-mode packets

$$(K')\begin{cases} (\partial_t - v\cdot\nabla_x - \Delta_v)\, g = 0, & \text{in } (0,T)\times\mathbb{R}^{2d},\\ g(0,x,v) = g_0(x,v) \text{ in } \mathbb{R}^{2d}. \end{cases}$$

Observability of Fourier-mode packets

$\exists C>0$ s.t, $\forall T>0$, $N\in\mathbb{N}^*$ and $g_0\in L^2(\mathbb{R}^{2d})$ s.t $supp(\hat{g}_0)\subset\overline{B_{\mathbb{R}^d}(0,N)}\times\mathbb{R}^d$, the solution of (K') satisfies

$$\int_{\mathbb{R}^{2d}} |g(T,x,v)|^2\,\mathrm{d}x\,\mathrm{d}v \le e^{C(1+\frac{1}{T}+N)}\int_0^T\int_\omega |g(t,x,v)|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

By duality, we have that $\exists C>0$ s.t $\forall T>0$, $N\in\mathbb{N}^*$, $f_0\in L^2(\mathbb{R}^{2d})$ s.t $supp(\hat{f}_0)\subset\overline{B_{\mathbb{R}^d}(0,N)}\times\mathbb{R}^d$, $\exists u\in L^2((0,T)\times\mathbb{R}^{2d})$ s.t. the solution of (K) satisfies $f(T,\cdot,\cdot)=0$ and

$$\|u\|_{L^2((0,T)\times\mathbb{R}^{2d},\mathbb{R})}\le e^{C(1+\frac{1}{T}+N)}\|f_0\|_{L^2(\mathbb{R}^{2d})}.$$

Construction of the control

We split the time interval
$0 = a_0 < a_0 + T_0 < \cdots < a_j < a_j + T_j < a_{j+1} \to_{j\to\infty} T$.
Then,

- on $(a_j, a_j + T_j)$, we apply a control $\tilde{u}_j$ steering to zero the frequences $\xi \in B_{\mathbb{R}^d}(0, 2^j)$.
  COST: $e^{|\xi|}$
- on $(a_j + T_j, a_{j+1})$, we apply no control.
  DISSIPATION: $e^{-\frac{|\xi|^2 t^2}{12}}$.

Then, the control is

$$u(t) = \begin{cases} \tilde{u}_j(t - a_j), & \text{if } t \in (a_j, a_j + T_j), \\ 0, & \text{if } t \in (a_j + T_j, a_{j+1}], \end{cases}$$

KEY POINT $|\xi|^2 >>> |\xi| \Rightarrow u \in L^2((0, T) \times \mathbb{R}^2)$.

Conclusion and open questions

Achievements

- We give a null controllability result in an L^2-setting in any dimension $d \geq 1$
- We give a sufficient condition for observability in the whole domain.

Some questions

- What about other Kolmogorov-type operators?
- What happens if Ω is unbounded but $\partial\Omega \neq \emptyset$?
- Could ω be of finite measure, as in P. Cannarsa, P. Martinez and J. Vancostenoble (ESAIM COCV, 2004)? $\Rightarrow$ weights?

Work in progress

- Nonlinear control of kinetic equations: e.g. Fokker-Planck, Vlasov-Poisson-Fokker-Planck equations.

Thank you very much for your attention!