Bottom detection and optimal control problem on water waves

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Partial differential equations, optimal design and numerics

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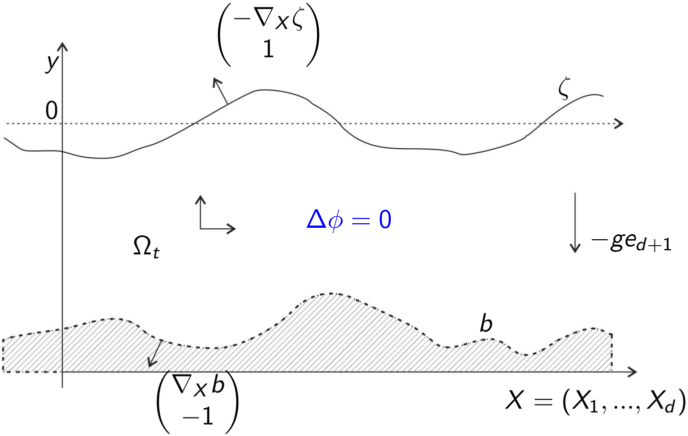
We are considering an incompressible, inviscid fluid + irrotational.



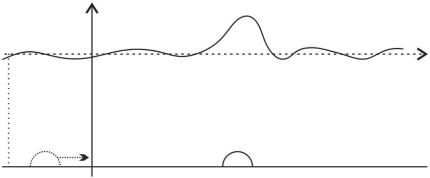
Water-waves equation

The general water-waves formulation for $(X, y) \in \mathbb{R}^d \times \mathbb{R}$, with $d = 1, 2$, $V = \nabla\phi$

$$
(1) \qquad \begin{cases}
\Delta_{X,y}\phi = 0, & b \leq y \leq \zeta, \\
\partial_t \zeta + \nabla\zeta \cdot \nabla\phi = \partial_y \phi, & y = \zeta, \\
\partial_t \phi + \dfrac{1}{2}|\nabla_{X,y}\phi|^2 + g\zeta = 0, & y = \zeta, \\
\dfrac{\partial\phi}{\partial n} = 0, & y = b.
\end{cases}
$$

Some phenomena involving water-waves

- Waves generation by bottom disturbances (Zuazua '14, Peregrine '67, Wu '84)

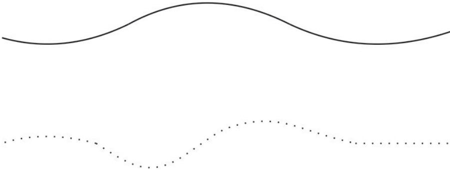


Some phenomena involving water-waves

- Waves generation by bottom disturbances (Zuazua '14, Peregrine '67, Wu '84)
- Tsunamis (generated by displacement of plates, (Iguchi '11, Wu '84)

Some phenomena involving water-waves

- Waves generation by bottom disturbances (Zuazua '14, Peregrine '67, Wu '84)
- Tsunamis (generated by displacement of plates, (Iguchi '11, Wu '84)
- Bottom detection through surface measurements (Taroudakis '01, Nicholls-Taber '08, Vasan-Deconinck '13)



Zakharov 1967; Craig, Sulem 1992

- $\psi(t,X) := \phi(t,X,\zeta(t,X))$
- If at time t one knows ψ, it is possible to solve on Ω_t

$$\begin{cases} \Delta_{X,y}\phi = 0, & \Omega_t \\ \phi|_{y=\zeta} = \psi, & \partial_n\phi|_{y=b} = 0 \end{cases}$$

$$G(\zeta,b)\psi = \sqrt{1+|\nabla\zeta|^2}\partial_n\phi\Big|_{y=\zeta}$$

System (1) becomes

$$(2) \qquad \begin{cases} \partial_t\zeta - G(\zeta,b)\psi = 0, \\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2(1+|\nabla\zeta|^2)}\left(G(\zeta,b)\psi + \nabla\zeta\cdot\nabla\psi\right)^2 = 0. \end{cases}$$

Well-posedness within a sobolev class

- Nalimov '74, Yosihara '82, Craig '85: Existence 1-D locally in time with almost flat bottom, around small perturbations of still water.
- S. Wu '97, '99, '11: Local existence 1-D, 2-D and global existence 2-D in the case of a layer of fluid of infinite depth.
- Lannes '05, '13: Local existence in time, 1,2-D, uneven fixed bottom.
- Germain, Masmoudi, Shatah '12: Global existence in time, 2-D with infinite depth, small initial data.

Theorem, well-posedness of (2) (Lannes 2013)

Let $d_0 > \frac{d}{2}$ and $N \geq d_0 + max\{d_0, 2\} + 3/2$. Then let $U_0 = (\zeta_0, \psi_0) \in H^{d_0+2} \times H^2$, $\mathcal{E}^N \in L^\infty$, $b \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$. Moreover assume that

$$\exists h_{min} > 0,\ \exists a_0 > 0, \quad \zeta_0(X) - b(X) \geq h_{min} \text{ and } \mathfrak{a}(U_0) \geq a_0.$$

Then there exists $T > 0$ and a unique solution $U \in C([0, T]; H^{d_0+2} \times H^2)$, $\mathcal{E}^N \in L^\infty$, to (2) with initial data U_0.

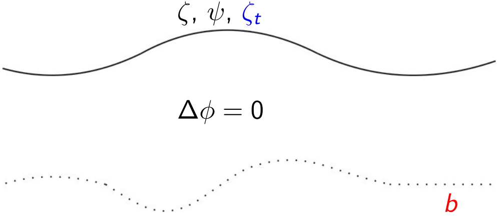
$$\mathfrak{a} = g + w_t + V \cdot \nabla w, \quad \mathcal{E}^N = |B\psi|^2_{H^{d_0+3/2}} + \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq N} |\zeta_{(\alpha)}|_2^2 + |B\psi_{(\alpha)}|_2^2,$$

$$\zeta_{(\alpha)} = \partial^\alpha \zeta, \quad \psi_{(\alpha)} = \partial^\alpha \psi - \underline{\omega} \partial^\alpha \zeta, \quad B = \frac{|D|}{(1+|D|)^{1/2}}.$$

$$\underline{\omega} = \frac{G(\zeta)\psi + \nabla_X \zeta \cdot \nabla_X \psi}{1 + |\nabla_X \zeta|^2}.$$

Identifiability of the bottom

What measurements can be made on the free surface to detect the bottom uniquely?



$$\begin{cases} \partial_t \zeta = G(\zeta, b)\psi = \sqrt{1+|\nabla\zeta|^2}\,\partial_n \phi\big|_{y=\zeta}, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2(1+|\nabla\zeta|^2)}\left(G(\zeta, b)\psi + \nabla\zeta\cdot\nabla\psi\right)^2 = 0. \end{cases}$$

Theorem

Let $T > 0$ and $d_0 > \frac{d}{2}$. Assume that for $j = 1, 2$, $(\zeta_j, \psi_j) \in C^1([0, T]; H^{d_0+1}(\mathbb{R}^d) \times H^2(\mathbb{R}^d))$ are solutions of (2), with $b_1, b_2 \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$, such that there exists $h_{min} > 0$, such that, for all $X \in \mathbb{R}^d$ and $t \in (0, T)$,

$$\zeta_j(t, X) - b_j(X) \geq h_{min}, \quad j = 1, 2.$$

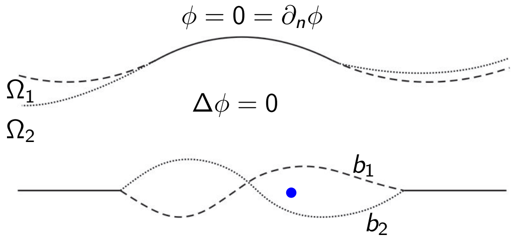
Let S be an open subset of $\mathbb{R}^d$ and $t_0 \in (0, T)$ a single time. If $\forall X \in S$,

$$\zeta_1(t_0, X) = \zeta_2(t_0, X), \quad \psi_1(t_0, X) = \psi_2(t_0, X), \quad \partial_t\zeta_1(t_0, X) = \partial_t\zeta_2(t_0, X),$$

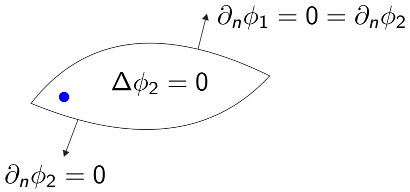
then

$$b_1(X) = b_2(X) \quad \forall X \in \mathbb{R}^d.$$

$\phi = \phi_1 - \phi_2$



- $\partial_t \zeta_1 = \sqrt{1 + |\nabla \zeta_1|^2} \partial_n \phi_1 = \sqrt{1 + |\nabla \zeta_2|^2} \partial_n \phi_2 = \partial_t \zeta_2$



One gets, $\phi_2 = C = \phi_1$, which we exclude (Still water).

Optimal control problem

Given a target function $\tau(X) \in L^2(\mathbb{R}^d)$, we are considering
$F : H^{N+max\{d_0,1\}+2}(\mathbb{R}^d) \to \mathbb{R}$

$$F(b) = \frac{1}{2} \int_{\mathbb{R}^d} |\partial_t \zeta(X)|_{t=t_0} - \tau(X)|^2 dX$$

Problem

To find $b^{min} \in \mathcal{B}_{ad}$, such that,

$$F(b^{min}) = \min_{b \in \mathcal{B}_{ad}} F(b).$$

- $\mathcal{B}_{ad} = \{b \in H^{N+max\{d_0,1\}+2}(\mathbb{R}^d) : \mathrm{supp}(b) \subset K, |b|_{H^{N+max\{d_0,1\}+2}(\mathbb{R}^d)} \leq C\}$

Theorem, existence of minimizers

Let $\zeta \in H^{d_0+2}(\mathbb{R}^d)$ and $\psi \in H^2(\mathbb{R}^d)$. Assume that $\tau(X) \in L^2(\mathbb{R}^d)$. Then the minimization problem,

$$\min_{b \in \mathcal{B}_{ad}} F(b),$$

has one minimizer $b^m \in \mathcal{B}_{ad}$.

- Let $\{b_n\} \subset \mathcal{B}_{ad}$ be a minimizing sequence of F
- $\exists$ subsequence, such that $b_n \rightharpoonup \bar{b}$ in $H^{N+max\{d_0,1\}+2}(\mathbb{R}^d)$, with $\bar{b} \in \mathcal{B}_{ad}$
- $b_n \to \bar{b}$ strongly in $H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$
- Let ϕ_n and $\bar{\phi}$ the corresponding solutions with bottom b_n and $\bar{b}$
- $G(\zeta, b_n)\psi \to G(\zeta, \bar{b})\psi, \quad \text{strongly in } L^2(\mathbb{R}^d)$
- $\inf_{b\in\mathcal{B}_{ad}} F(b) = \lim_{n\to\infty} F(b_n) = F(\bar{b})$
- Uniqueness of this minimum follows from the identifiability.

Shape derivative of F

Given ζ_0, ψ_0

$$F(b) = \frac{1}{2}\int_{\mathbb{R}^d} \left|\partial_t \zeta(X)|_{t=t_0} - \tau(X)\right|^2 dX = \frac{1}{2}\int_{\mathbb{R}^d} \left|G(\zeta_0, b)\psi_0 - \tau(X)\right|^2 dX$$

For all $h \in H^{N+max\{d_0,1\}+2}(\mathbb{R}^d)$, one has

$$F'(b)\cdot h = -\int_{\Gamma_b} h \frac{\nabla\phi\cdot\nabla\psi}{\sqrt{1+|\nabla_X b|^2}}$$

with ϕ and ψ being the solutions of

$$\left\{\begin{array}{ll} \Delta\phi = 0, \quad \Omega, & \Delta\psi = 0, \quad \Omega, \\ \phi = \psi_0, \quad \Gamma_{\zeta_0} & \psi = G - \tau, \quad \Gamma_{\zeta_0} \\ \frac{\partial\phi}{\partial n} = 0, \quad \Gamma_b, & \frac{\partial\psi}{\partial n} = 0, \quad \Gamma_b. \end{array}\right.$$

$\phi_\delta = \phi(\Omega + \delta u)$ is the solution of the problem

$$\begin{cases} \Delta\phi_\delta = 0, & \Omega_\delta \\ \phi_\delta = \psi_0, & \Gamma_{\zeta_0} \\ \dfrac{\partial \phi_\delta}{\partial n^\delta} = 0, & \Gamma_b^\delta, \end{cases}$$

$\phi' = lim_{\delta\to 0} \dfrac{\phi_\delta - \phi_0}{\delta}$ one has

$$\begin{cases} \Delta\phi' = 0, & \Omega \\ \phi' = 0, & \Gamma_{\zeta_0} \\ \dfrac{\partial \phi'}{\partial n} = \nabla u_n \cdot \nabla\phi - u_n \dfrac{\partial^2 \phi}{\partial n^2}, & \Gamma_b. \end{cases}$$

Then

$$\begin{aligned} F'(b)\cdot h &= \int_{\mathbb{R}^d} (G(\zeta_0,b)\psi_0 - \tau)\sqrt{1+|\nabla_X\zeta_0|^2}\ \frac{\partial\phi'}{\partial n}\Big|_{\zeta_0} dX \\ &= \int_{\Gamma_{\zeta_0}} (G-\tau)\frac{\partial\phi'}{\partial n} dA, \\ &= -\int_{\Gamma_b} \psi\frac{\partial\phi'}{\partial n} dA \\ &= \int_{\Gamma_b} \psi\left[\nabla\left(\frac{h}{\sqrt{1+|\nabla_X b|^2}}\right)\cdot\nabla_X\phi - \frac{h}{\sqrt{1+|\nabla_X b|^2}}\frac{\partial^2\phi}{\partial n^2}\right] dA \end{aligned}$$

After some calculations

$$F'(b)\cdot h = -\int_{\Gamma_b} h\frac{\nabla\phi\cdot\nabla\psi}{\sqrt{1+|\nabla_X b|^2}}.$$

Thanks