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A duality scheme for a class of non convex problems in calculus of variations

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A non convex problem

Let Ω be a bounded domain of $\mathbb{R}^d$. We consider the problem

$$J(\Omega) := \inf_{u=u_0 \text{ on } \Gamma_0} \left\{ \int_\Omega (f(\nabla u) + g(x,u))\, dx + \int_{\Gamma_1} \gamma(u)\, dH^{d-1} \right\}$$

where:

- $f : \mathbb{R}^d \to \mathbb{R}$ convex continuous with p-growth $(1 \le p < +\infty)$
- g, γ are possibly non convex functions with suitable growth conditions
- (Γ_0, Γ_1) is a partition of $\partial\Omega$.

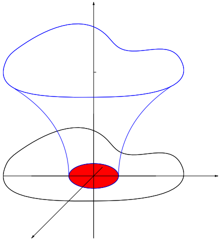
If $p > 1$, existence of a global minimizer
If $p = 1$, relaxed minimizer in $BV(\Omega)$.

Typical examples

- Cahn Hilliard fluid
  $f(z) = \varepsilon\,|z|^2$, $g(x,t) = W(t) - p(x)\,t$, $\gamma \in Lip(\mathbb{R})$, $\Gamma_1 = \partial\Omega$
  (W two wells potential , $p(x)$ pressure , γ wetting potential)

- Free boundary Pb
  $f(z) = \sqrt{1 + |z|^2}$, $g(t) = 1_{(0,+\infty)}$, $\Gamma_0 = \partial\Omega$, $u_0 = 1$

  (The free boundary coincides with $\partial\{u > 0\}$)



Free boundary Pb

The case $f(z) = |z|^2$ and $g(x,t) = \omega(x)\,1_{\{t>0\}}$ has been studied by Alt-Cafarelli (viscosity solutions of Euler equation)

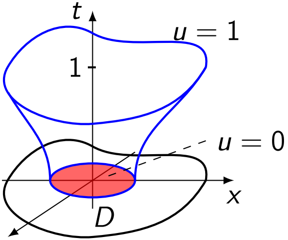
$$J(\Omega) := \inf\left\{\int_\Omega \frac{|\nabla u|^2}{2} + \lambda\,|u>0| \;\; : u = 1 \text{ on } \partial\Omega\right\}.$$

$\rightsquigarrow$ Free boundary Pb in term of $D = \{u > 0\}$,
u solves

$$\begin{cases} -\Delta u_D = 0 \text{ in } D \\ u_D = 1 \text{ on } \partial\Omega. \end{cases}$$

$\rightsquigarrow$ Shape functional

$$J : D \to \lambda|D| + \frac{1}{2}\int_\Omega |\nabla u_D|^2.$$



Questions

- How to characterize global minimizers ?
- Approximation scheme ruling out local minimizers and compatible with multiple solutions.
- Derivability of $J(\Omega)$ as a shape functional ?

Outline

1. Dual problem and $\inf = \sup$ result.
2. Case $d = 1$. Construction of explicit calibrations.
3. Min-Max scheme and identification of global minimizers
4. Case $d > 1$. Existence of calibrations for linear growth functionals
5. An open issue related to Munford-Shah functional.

1. Dual Problem

Assume $u_0 = 0$ and $\gamma(0) = 0$. Recall:

$$J(\Omega) := \inf_{u=0 \text{ on } \Gamma_0} \left\{ \int_\Omega (f(\nabla u) + g(x,u))\, dx + \int_{\Gamma_1} \gamma(u)\, dH^{d-1} \right\}$$

The dual problem reads as a linear program on $\Omega \times \mathbb{R}$:

$$J^*(\Omega) := \sup_{\sigma \in K(\Omega)} \left\{ \begin{array}{ll} -\int_\Omega \sigma^t(x,0)\, dx \ , & \operatorname{div}\sigma = 0 \ \text{ on } \Omega \times \mathbb{R} \\ & \\ \sigma^x \cdot \nu_\Omega = -\gamma'(t) & \text{a.e. on } \Gamma_1 \times \mathbb{R} \end{array} \right\}$$

where ν_Ω unit exterior normal on $\partial\Omega$ and competitors $\sigma = (\sigma^x, \sigma^t) \in L^\infty(\Omega \times \mathbb{R}; \mathbb{R}^{d+1})$ belong to convex set

$$K(\Omega) = \left\{ \sigma \in C^0(\overline{\Omega} \times \mathbb{R}; \mathbb{R}^{d+1}) \ : \ f^*(\sigma^x) \le g + \sigma^t \text{ in } \Omega \times \mathbb{R} \right\}$$

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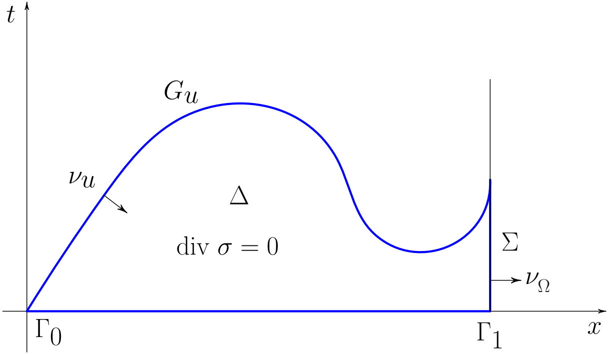


$$\begin{cases} f^*(\sigma^x) \leq g + \sigma^t & \text{on } \Omega \times \mathbb{R} \\ \sigma^x \cdot \nu_\Omega = \gamma'(t) & \text{a.e. on } \Sigma := \Gamma_1 \times \mathbb{R} \end{cases}$$

Inequality $J(\Omega) \geq J^*(\Omega)$

Let u a competitor and divergence free σ such that $\sigma^t \geq -f^*(\sigma^x) + g(t)$. By Fenchel:

$$\int_\Omega (f(\nabla u) + g(x,u))\,dx \;\geq\; \int_\Omega (\sigma^x(x,u(x)) \cdot \nabla u - \sigma^t(x,u(x)))\,dx$$

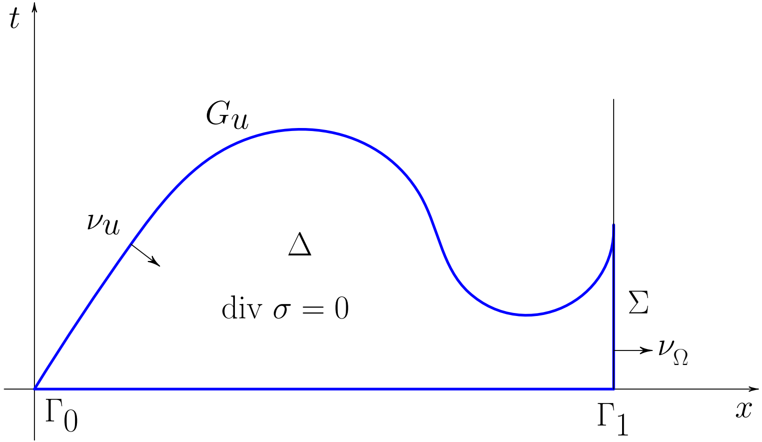
$$\int_{\Gamma_1} \gamma(u)\,dH^{d-1} \;=\; \int_{\Gamma_1} \left(\int_0^{u(x)} \gamma'(t)dt \right) dH^{d-1}$$

Thus, the total energy $E(u)$ satisfies

$$E(u) \;\geq\; \int_{G_u} \sigma \cdot \nu_u\,dH^d + \int_{\Gamma_1 \times \mathbb{R}} \sigma^x \cdot \nu_\Omega dH^d dt \;\geq\; \int_\Omega -\sigma^t(x,0)\,dx$$

where $\nu_u = \frac{1}{\sqrt{1+|\nabla u|^2}}(\nabla u, -1)$ unit normal to th graph G_u, and we apply Stokes formula on $\Delta = \{(x, s\,u(x)) : x \in \Omega, s \in (0,1)\}$

Integration by parts over Δ



$$\Sigma := \Gamma_1 \times \mathbb{R} \quad , \quad \partial\Delta = G_u \cup (\Omega \times \{0\}) \cup \Sigma$$

Calibration fields $(\sigma^x, \sigma^t) \in L^\infty(\Omega \times \mathbb{R}; \mathbb{R}^d \times \mathbb{R})$

- requiring continuity of σ is to stringent for being optimal in $J^*(\omega)$.
- $f^*(\sigma^x) \leq g + \sigma^t$ holding merely a.e. can't handle possible discontinuities of $g(u)$
  (at $u = 0$ in the free-boundary problem)
- σ has a normal trace well defined a.e. on every Lipschitz hypersurface S, in particular on every $S_t = \Omega \times \{t\}$, $t \in \mathbb{R}$
  (by Anzellotti , it exists on $d - 1$ -rectifiable subsets)

Class $\mathcal{B}$ of admissible fields for problem $J^*(\Omega)$

We extend the class of competitors for $J^*(\Omega)$ to the class $\mathcal{B}_0$ of fields $\sigma = (\sigma^x, \sigma^t) \in (L^\infty(\Omega \times \mathbb{R}))^{d+1}$ such that:

(s0) $\operatorname{div} \sigma = 0$ in $\Omega \times \mathbb{R}$;

(s1) $f^*(\sigma^x) \leq g(t) + \sigma^t$ a.e. $(x,t) \in \Omega \times \mathbb{R}$;

(s2) $\forall s \in D_g$, $-f(0) \leq g(s) + \sigma^t(\cdot, s)$ a.e. in Ω

(s3) $\sigma^x \cdot \nu_\Omega = \gamma'(t)$ a.e. on $\Gamma_1 \times \mathbb{R}$

Here D_g is the set of discontinuities of g (assumed to be negligible).

Lemma: $J(\Omega) \geq \int_\Omega -\sigma^t(x,0)\, dx \quad \forall \sigma \in \mathcal{B}_0$.

Note that in (s3):
- horizontal component σ^x is missing
- $f^*(\sigma^x)$ has been replaced by $\inf f^* = -f(0)$.

Duality result

Theorem It holds

$$J(\Omega) = \sup \left\{ \int_\Omega -\sigma^t(x,0)\, dx \;:\; \sigma \in \mathcal{B}_0 \right\} = J^*(\Omega)\,.$$

• The theortical existence of a maximizer $\overline{\sigma}$ (calibration) is a difficult issue: no control on the positive part of σ^t.
(in Neumann case, can show L^1 estimate and existence of relaxed solutions with possible singular part in σ^t_+).

• No uniqueness in general

• If $J(\Omega)$ admits a bounded minimizer $\overline{u}$ such $\alpha \le \overline{u} \le \beta$, then in dual problem $J^*(\Omega)$ we may take the supremum on the larger class $\mathcal{B}_{\alpha,\beta}$ obtained by requiring conditions (s0–s3) merely on $\Omega \times [\alpha, \beta]$.

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Optimality conditions for primal-dual problem

Let $C(t) := \left\{ q \in \mathbb{R}^{d+1} : \ q^t \geq f^*(q^x) - g(t) \right\}$, with support function

$$h(t,p) := \begin{cases} -p^t \Big(f\big(-\frac{p^x}{p^t}\big) + g(t) \Big) & \text{if } p^t < 0 \\ +\infty & \text{otherwise} \end{cases}$$

Theorem An admissible pair $(\overline{u}, \overline{\sigma}) \in W^{1,2}(\Omega) \times \mathcal{B}$ is optimal iff

$$\overline{\sigma} \cdot \nu_{\overline{u}} \ = \ h(t, \nu_{\overline{u}}) \quad \text{holds } H^d \text{ a.e. in } G_{\overline{u}} \ . \qquad (*)$$

If D_g is finite and $\overline{\sigma}$ continuous in $\Omega \times (\mathbb{R} \setminus D_g)$, $(*)$ amounts to check that:

$$\begin{cases} f^*(\overline{\sigma}^x(x, \overline{u}(x)) = \overline{\sigma}^t(x, \overline{u}(x)) + g(\overline{u}(x)) & \text{a.e. } x \in \Omega \\ \forall s \in D_g, \ -f(0) = \overline{\sigma}^t(\cdot, s) + g(s) & \text{a.e. } x \in \{\overline{u} = s\} \end{cases} \qquad (**)$$

Remark: this determines $\overline{\sigma}$ on $G_{\overline{u}} \setminus D_g \times \mathbb{R}$

Explicit optimal $\overline{\sigma}$ if g, γ are convex continuous

Let $\overline{u}$ a minimizer for $J(\Omega)$. Then there exists $\psi(x)$ such that (Euler equation)

$$\psi(x) \in \partial f(\nabla \overline{u}) \ , \ \operatorname{div} \psi \ \in \partial g(\overline{u}) \text{ in } \Omega \ , \ -\psi \cdot \nu_\Omega \in \partial \gamma(u(x)) \text{ in } \Gamma_1.$$

With ψ , we may associate the field

$$\overline{\sigma} \ = \ \begin{pmatrix} \psi(x) \\ -t \operatorname{div}\psi(x) + \eta(x) \end{pmatrix} \quad , \quad \eta := f^*(\psi) + g^*(\operatorname{div}\psi) \ .$$

It is divergence free (s1) and satisfies boundary condition (s4). As $g(t) \geq g(\overline{u}) + \operatorname{div}\psi(t - \overline{u})$, we deduce inequality (s2) from the Euler Eq combined with Fenchel inequality.

Thus $\overline{\sigma} \in \mathcal{B}$ and optimality condition $(**)$ holds by exploiting Fenchel equality

$$g(\overline{u}(x)) = \overline{u}(x) \operatorname{div}\psi(x) + g^*(\operatorname{div}\psi)$$

Proof of inequality $J(\Omega) \leq J^*(\Omega)$

Assume that $\Gamma_0 = \partial\Omega, \Gamma_1 = \emptyset$,

Step 1. We use $u \in W_0^{1,2}(\Omega) \rightsquigarrow v = 1_u(x,t) \in \mathcal{A}_0$ where

$$\mathbf{1}_u(x,t) := \begin{cases} 1 & \text{if } t \leq u(x) \\ 0 & \text{if } t > u(x) \end{cases} \quad , \quad \mathcal{A}_0 := v_0 + BV(\Omega \times \mathbb{R})$$

Let $F : \mathcal{A}_0 \to (-\infty, +\infty]$ be the **convex** functional defined by

$$F(v) \;:=\; \int_{\Omega\times\mathbb{R}} h(t, Dv)\;.$$

Then $F(\mathbf{1}_u) = \int_\Omega (f(\nabla u) + g(u))$. By classical convex duality arguments:

$$J^*(\Omega) \;=\; \inf\left\{F(v) \;:\; v \in \mathcal{A}_0\right\}\;.$$

Q: is previous infimum unchanged if we restrict to functions of the form $v = \mathbf{1}_u$??

Generalized coarea formula

Proposition Let $v \in \mathcal{A}_0$ such that $F(v) < +\infty$. Then

(i) For a.e; $x \in \Omega$, $v(x, \cdot)$ is non increasing with $v(x, -\infty) = 1$, $v(x, +\infty) = 0$.

(ii) Let $u_s(x) := \inf \left\{ \tau \in \mathbb{R} : v(x, \tau) \leq s \right\}$. Then $u_s \in W_0^{1,2}(\Omega)$ for a.e $s \in (0, 1)$ and

$$F(v) = \int_0^1 \left(\int_\Omega (f(\nabla u_s) + g(u_s)) \right) ds.$$

Corollary $J(\Omega) = \min \{ F(v) : v \in \mathcal{A}_0 \}$ $(= J^*(\Omega))$.
If $v \in \mathcal{A}_0$ is optimal, then u_s is solution to $J(\Omega)$ for a.e $s \in (0, 1)$

In particular, if $J(\Omega)$ has finitely many solutions $\overline{u}_j$, v must be piece-wise constant jumping across the graphs of the $\overline{u}_j$'s

Let $d = 1$, $\Omega = (0, h)$, $\Gamma_0 = \{0, h\}$, $u_0 = 0$.

Write divergence free σ as $\sigma = (\partial_t w, -\partial_x w)$ for some scalar potential $w : [0, a] \times \mathbb{R} \to \mathbb{R}$. Then $J^*(\Omega)$ reads

$$\sup \left\{ w(h,0) - w(0,0) \ : \ f^*(\partial_t w) + \partial_x w \leq g(t) \right\}$$

Remark: it looks like Monge-Kantorovich problem (in dual form)

- $\delta_{(0,0)}$ is transported to $\delta_{(h,0)}$
- constraint $|\nabla w| \leq 1$ is substituted with the one above

By using value-function

Let us perturbe the infimum problem $J(\Omega)$ by taking

$$\Omega = (0, x) \quad , \quad u_0(0) = 0 \quad , \quad u_0(x) = t \ .$$

Then, we introduce the value function

$$V(x,t) \ := \ \inf \left\{ \int_0^x (f(u') + g(u)) \, ds \ , \ u(0) = 0 \ , \ u(x) = t \right\}$$

LEMMA $\quad \overline{\sigma} = (\partial_t V, -\partial_x V)$ solves $J^*(\Omega)$.

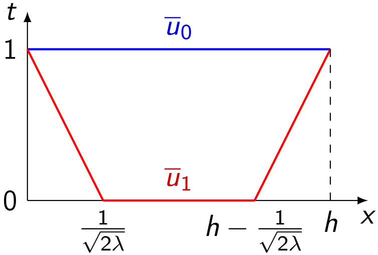
Proof: By Bellman's dynamic optimization principle, $V(x,t)$ solves in the viscosity sense $\partial_x V + f^*(\partial_t V) = g(t)$ (needs only $\leq$)
Besides: $J^*(\Omega) \geq - \int_0^h \overline{\sigma}^t(s,0) \, ds = V(h,0) = J(\Omega)$ $\square$

Remark: This solution $\overline{\sigma}(x,t)$ is singular closed to $\{x = 0\}$

Free boundary example

Let $\Omega_h = (0,h)$, $f = \frac{|z|^2}{2}$, $g_\lambda = \lambda\ \mathbf{1}_{t>0}$ $(D_g = \{0\})$, $u_0 = 1$.

$$\begin{aligned} J_\lambda(\Omega_h) &:= \inf\left\{ \int_0^h \frac{u'^2}{2}\, ds + \lambda|u>0| \ :\ u(0)=u(h)=1 \right\} \\ &= \min\{\lambda h, 2\sqrt{2}\sqrt{\lambda}\} \quad \text{(non differentiable in } h\text{)} \end{aligned}$$



$\overline{u}_0, \overline{u}_1$ **are local minimizers**

$$\mathrm{Argmin}(J_\lambda(\Omega_h)) = \begin{cases} \{\overline{u}_0\} & \text{if } \lambda h^2 < 8 \\ \{\overline{u}_1\} & \text{if } \lambda h^2 > 8 \\ \{\overline{u}_0, \overline{u}_1\} & \text{if } \lambda h^2 = 8 \end{cases}$$

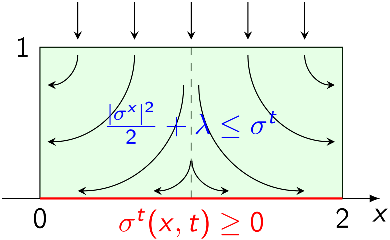
Numerical computation of optimal flow

We treat a case with two solutions $\overline{u}_0, \overline{u}_1$.

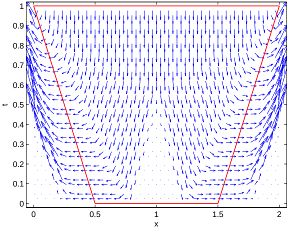
$$\Omega = [0,2], \qquad g(t) = \begin{cases} 2 & \text{if} t > 0 \\ 0 & \text{if} t \leq 0, \end{cases} \qquad (\lambda = 2),$$

$$\mathcal{S}_\epsilon(\Omega) := \sup_{\sigma \in \mathcal{B}} \left\{ -\int_\Omega \sigma^t(x,1)dx - \epsilon \int_{\Omega \times [0,1]} |\sigma|^2 \right\} \quad (\epsilon \geq 0)$$

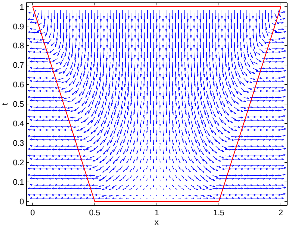
- $\epsilon = 0$ unperturbed dual Pb $J^*(\Omega)$
- $\epsilon > 0$ viscosity term ($\rightsquigarrow$ select solution of minimal L^2-norm)



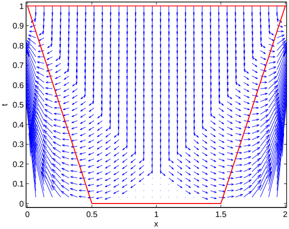
Numerical solution by Matlab + 2d-finite elements



(a) $\epsilon = 0$



(b) $\epsilon > 0$



(c) Singular solution

- Singular solution (c) constructed by symmetrization of gradient rotated of $\overline{\sigma} = (\partial_t V, -\partial_x V)$ (value function)
- Time of computation is very high (Matlab optimization toolbox).

3- Min-Max formulation

• We observe that $F(v) = \sup_{\sigma\in\mathcal{B}} L(v,\sigma)$ where $\mathcal{B}$ is the class of fields $\sigma \in L^\infty(\Omega\times\mathbb{R}, \mathrm{div})$ which satisfy (s1)(s2)(s3) and

$$L(v,\sigma) := \int_{\Omega\times\mathbb{R}} \sigma \cdot Dv$$

Thus according to our duality result

$$J(\Omega) \;=\; \min\{F(v) : v \in \mathcal{A}_0\} \;=\; \min_{v\in\mathcal{A}_0} \sup_{\sigma\in\mathcal{B}} \; L(v,\sigma)\;.$$

• Similarly we have for every $\sigma \in \mathcal{B}$:
$\inf_{v\in\mathcal{A}_0} L(v,\sigma) = -\int_\Omega \sigma^t(x,0)$ if $\mathrm{div}\,\sigma = 0$ ($+\infty$ otherwise).
Thus

$$J^*(\Omega) \;=\; \sup_{\sigma\in\mathcal{B}} \inf_{v\in\mathcal{A}_0} \; L(v,\sigma)\;.$$

Remark: Divergence free condition on σ is treated by duality (v represents a pressure).

Saddle point

Let $(\overline{v}, \overline{\sigma}) \in \mathcal{A}_0 \times \mathcal{B}$. Then $(\overline{v}, \overline{\sigma})$ is an optimal pair iff

$$L(\overline{v}, \sigma) \;\leq\; L(\overline{v}, \overline{\sigma}) \;\leq\; L(v, \overline{\sigma}) \quad , \quad \text{for all } (v, \sigma) \in \mathcal{A}_0 \times \mathcal{B}$$

We can use then an approximation scheme

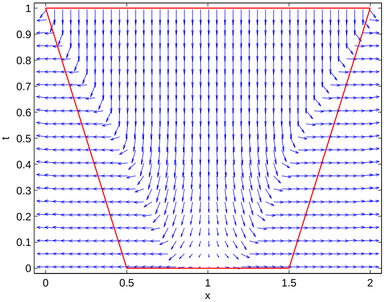
$$\begin{cases} \sigma^h_{n+1} = \mathrm{Proj}_{B^h}(\sigma^h_n + \alpha \nabla^h \overline{v}^h_n) \\ v^h_{n+1} = v^h_n - \beta(\mathrm{div}^h\, \sigma^h_{n+1}) \\ \overline{v}^h_{n+1} = 2v^h_{n+1} - v^h_n \end{cases}$$

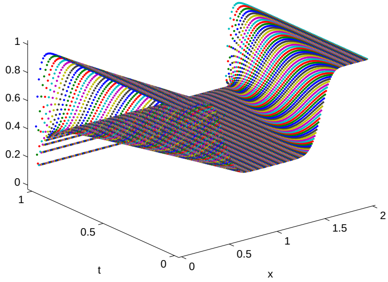
where

- h_x, h_t are the size parameters of a $d+1$-cartesian grid G_h
- div^h is adjoint to ∇^h, $Proj_{B^h}$ is a suitable non linear projector (discretization of constraint $\mathcal{B}$)
- $\alpha\beta c_h^2 < 1$ with $\quad c_h = \sup_{\|v^h\| \neq 0} \frac{\|\nabla^h v^h\|}{\|v^h\|} = \frac{2}{\|h\|}$

Scheme MAC + Orthogonal projections

Scheme MAC seems well adapted. Here are some results by Minh Phan





Optimal v exhibits two plateaus corresponding to solutions $\overline{u}_0, \overline{u}_1$

4- Case $d > 1$. A case of Existence

Theorem Assume that

- $\alpha|z| - \delta \leq f(z) \leq \beta(1 + |z|)$
- f^* is bounded on its domain
- $g = g(t)$ is a bounded nondecreasing function.
- $\Gamma_0 = \partial\Omega$

Then there exists a calibration, i.e. the supremum $J^*(\Omega)$ is attained.

Example. The next free boundary pb falls into this framework.

$$\inf\left\{ \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \left|\{u > 0\}\right| \;:\; u = 1 \text{ on } \partial\Omega \right\}$$

Proof strategy

Let for instance $u_0 = 1, g(t) = 1_{(0,+\infty)}, f(z) = |z|$.

We consider the modified dual problem

$$\widetilde{J}^*(\Omega) := \sup_{\sigma \in \widetilde{K}(\Omega)} \left\{ -\int_\Omega \sigma^t(x,1)\,dx \ , \quad \mathrm{div}\sigma = 0 \ \text{ on } \Omega \times \mathbb{R} \ \right\}$$

which resembles $J^*(\Omega)$, BUT now competitors σ belong to

$$\widetilde{K}(\Omega) = \left\{ \sigma \in L^\infty(\Omega \times \mathbb{R}; \mathbb{R}^{d+1}) \ : \ |\sigma^t| + |\sigma^x| \leq g \ \text{ a.e. in } \Omega \times \mathbb{R} \right\}.$$

It turns out that $\widetilde{J}^*(\Omega)$ is attained and agrees with $J^*(\Omega)$

[cf. rearrangement results for functionals with non constant density, Landes 2008].

5- About Munford Shah problem

A celebrated example of non convex variational Pb:

$$J(\Omega) := \inf_{u \in SBV(\Omega)} \left\{ \int_{\Omega \setminus S_u} \frac{1}{2} |\nabla u|^2 \, dx + H^{d-1}(S_u) + \frac{1}{2} \int_{\Omega} |u - h(x)|^2 \, dx \right\}$$

- Existence due to Ambrosio in 1990
- Sufficient conditions for global minimizers (calibrations) , G.Alberti, GB, G. Dal Maso (2001)

A possible dual Pb

$$J^*(\Omega) := \sup_{\sigma \in K(\Omega)} \left\{ \begin{array}{ll} -\int_\Omega \sigma^t(x,0)\,dx \ , & \operatorname{div}\sigma = 0 \ \text{ on } \Omega \times \mathbb{R} \\ \sigma^x \cdot \nu_\Omega = 0 & \text{a.e. on } \partial\Omega \times \mathbb{R} \end{array} \right\}$$

BUT here the convex constraint is **non local**. In order to account the jump energy, $\sigma \in K(\Omega)$ requires two conditions:

- $\frac{1}{2}|\sigma^x|^2 \leq \sigma^t + \frac{1}{2}|t - h(x)|^2$
- $\left| \int_{t_1}^{t_2} \sigma^x(x,s)\,ds \right| \leq 1$, for every t_1, t_2

THEOREM $J(\Omega) \geq J^*(\Omega)$ with equality if, for an admissible (u, σ), one has

$$\sigma(x, u(x)) = (\nabla u(x), \tfrac{1}{2}(|\nabla u|^2 - |u - h|^2)) \quad \text{a.e. } x \in \Omega$$

$$\int_{u_-(x)}^{u^+(x)} \sigma^x(x,t) \cdot \nu_u = 1 \quad H^{d-1} \text{ a.e. } x \in S_u$$

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Open problem

Do we have the equality $J(\Omega) = J^*(\Omega)$?

Difficulty: no coarea formula for Munford Shah functional !