Evolution of spoon-shaped networks

Alessandra Pluda

Department of Mathematics, University of Pisa

Partial differential equations, optimal design and numerics

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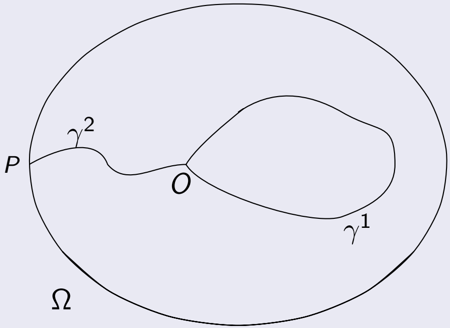
Definition



$$\Gamma_0 = \gamma^1([0,1]) \cup \gamma^2([0,1])\,,$$

$\gamma^1 : [0,1] \to \Omega\,,$
$\gamma^2 : [0,1] \to \overline{\Omega}\,.$
$\gamma^1(0) = \gamma^1(1) = \gamma^2(0) = O.$
$\gamma^2(1) = P \in \partial\Omega.$

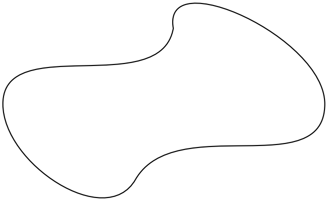
The evolution by curvature of a spoon-shaped network is the *geometric gradient flow* of the *Length functional*, that is, the sum of the lengths of all the curves of the network:

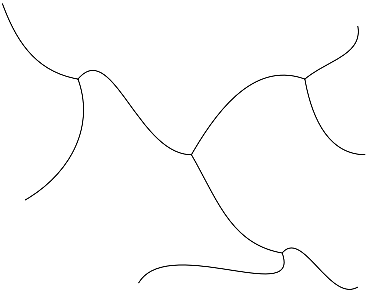
$$L(\Gamma) = L_1 + L_2 = \sum_{i=1}^{2} \int_0^1 |\gamma^i_x(\xi)| \, d\xi \, .$$

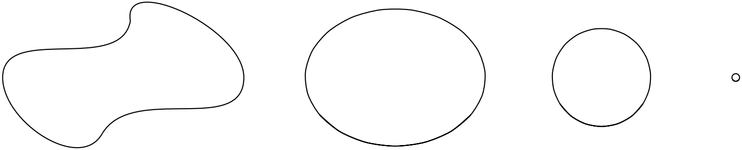
The equation that describes the motion by curvature is

$$\gamma^i_t(x,t) = \frac{\gamma^i_{xx}(x,t)}{|\gamma^i_x(x,t)|^2} \, .$$

Closed curve and tree-like network



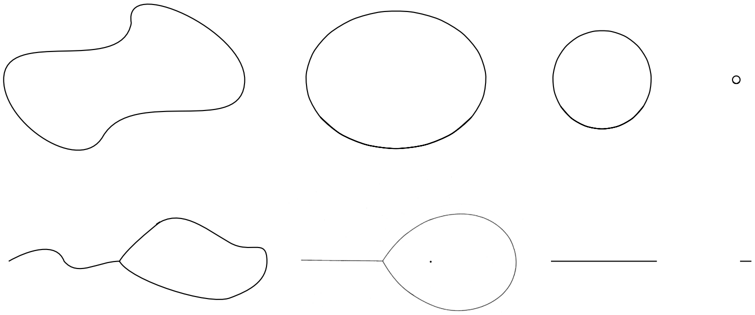




Theorem (Gage-Hamilton-Grayson)

A simple closed curve evolving by curvature becomes eventually convex and then shrinks to a point in a finite time.

Comparison between closed curve and spoon-shaped network



Theorem

For any initial smooth, spoon-shaped network Γ_0 in a smooth, convex, open set $\Omega \subset \mathbb{R}^2$ there exists a unique smooth solution of the problem in a maximal time interval $[0, T)$.

Theorem

If $[0, T)$, with $T < \infty$, is the maximal time interval of existence of a smooth solution Γ_t of the problem, then at least one of the following possibilities holds:

- $\liminf_{t \to T} L_2(t) = 0\,,$
- $\limsup_{t \to T} \int_{\Gamma_t} k^2 ds = +\infty\,.$

Rescaled networks

We rescale the flow in its maximal time interval $[0, T)$.
Fixed $x_0 \in \mathbb{R}^2$, let $\widetilde{F}_{x_0} : \Gamma \times [-1/2 \log T, +\infty) \to \mathbb{R}^2$ be the map

$$\widetilde{F}_{x_0}(p, \mathfrak{t}) = \frac{F(p,t) - x_0}{\sqrt{2(T-t)}} \qquad \mathfrak{t}(t) = -\frac{1}{2}\log{(T-t)} \tag{1}$$

then, the rescaled networks are given by

$$\widetilde{\Gamma}_{x_0,\mathfrak{t}} = \frac{\Gamma_t - x_0}{\sqrt{2(T-t)}} \tag{2}$$

and they evolve according to the equation

$$\frac{\partial}{\partial \mathfrak{t}} \widetilde{F}_{x_0}(p, \mathfrak{t}) = \widetilde{\underline{v}}(p, \mathfrak{t}) + \widetilde{F}_{x_0}(p, \mathfrak{t}) \tag{3}$$

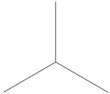
where

$$\widetilde{\underline{v}}(p, \mathfrak{t}) = \frac{\underline{v}(p, t(\mathfrak{t}))}{\sqrt{2(T - t(\mathfrak{t}))}} = \widetilde{\underline{k}} + \widetilde{\underline{\lambda}} = \widetilde{k}\nu + \widetilde{\lambda}\tau \qquad \text{and} \qquad t(\mathfrak{t}) = T - e^{-2\mathfrak{t}}. \tag{4}$$

We obtained a smooth flow of spoon-shaped networks $\tilde{\Gamma}_{\mathfrak{t}}$ defined for $\mathfrak{t} \in [-\frac{1}{2}\log T, +\infty)$.

Assume that the length of the curve $\gamma^2(x,t)$ is uniformly bounded away from zero for $t \in [0,T)$. Then, for every $x_0 \in \mathbb{R}^2$ a sequence of rescaled networks $\widetilde{\Gamma}_{x_0,t}$ converges in the C^1_{loc} topology to a limit set
which is one of the following:

- a halfline from the origin,
- a straight line through the origin,
- an infinite flat triod centered at the origin,



- a Brakke spoon.