Size Estimates for the Inverse Stokes Problem

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Stokes System

In this work we are interested in the study of size estimates for an obstacle $D \subset \mathbb{R}^d$ inside a cavity $\Omega \subset \mathbb{R}^d$, which is fulfill for a fluid, governed for the Stokes system. In particular we consider the following problem.

$$(1) \qquad \left\{ \begin{array}{rcccl} -\mathrm{div}(\sigma(u,p)) & = & 0 & , & \text{in } \Omega \setminus \overline{D}, \\ \mathrm{div} u & = & 0 & , & \text{in } \Omega \setminus \overline{D}, \end{array} \right.$$

where

- $\sigma(u,p) = 2\mu e(\nabla u) - pI$, $e(A) = \frac{A+A^T}{2}$.
- $u(x)$ velocity field.
- $p(x)$ presure, scalar function.
- $\mu(x)$ viscosity, scalar positive function.

Boundary Value Problem

$$(2) \qquad \begin{cases} -\mathrm{div}(\sigma(u,p)) &= 0 &, & \text{in } \Omega \setminus \overline{D}, \\ \mathrm{div}u &= 0 &, & \text{in } \Omega \setminus \overline{D}, \\ u &= g &, & \text{on } \partial\Omega, \\ u &= 0 &, & \text{on } \partial D, \end{cases}$$

- $g \in H^{1/2}(\partial\Omega)$ satisfies the compatibility condition $\displaystyle\int_{\partial\Omega} g \cdot n = 0$.
- The condition $u|_{\partial D} = 0$ is the no-slip condition.
- Existence: There exists $u \in (H^1(\Omega \setminus D))^d$ and $p \in L^2(\Omega \setminus D)$ satisfying (2).

Inverse Problem

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- Inverse Problem: size estimates of D from the boundary data (g, ψ) known on the boundary $\partial\Omega$.

Previous Result

- Álvarez, Conca, Fritz, Kavian, Ortega (2005). Identification of immersed obstacles via boundary measurements.
- Heck, Uhlmann, Wang (2007). Reconstruction of obstacles inside a bounded domain filled with an incompressible fluid.
- Lin, Uhlmann, Wang (2010). Optimal three-ball inequality for Stokes System.
- Ballerini (2011). Stable determination of an immersed body in a stationary Stokes fluid.

- Alessandrini, Rosset, Seo (1999). Optimal size estimates for the inverse conductivity problem with one measurements.
- Alessandrini, Morassi, Rosset (2002). Detecting an inclusion in an elastic body by boundary measurements.
- Nguyen, Wang (2014). Estimate of an inclusion in a body with discontinuous conductivity.

Inverse Problem

- When D is present, let us denote by u the solution of the following problem

$$
(3) \qquad \begin{cases} -\mathrm{div}(\sigma(u,p)) &= 0 \quad , \quad \text{in } \Omega \setminus \overline{D}, \\ \mathrm{div}\ u &= 0 \quad , \quad \text{in } \Omega \setminus \overline{D}, \\ u &= g \quad , \quad \text{on } \partial\Omega, \\ u &= 0 \quad , \quad \text{on } \partial D, \end{cases}
$$

Inverse Problem

- Let us denote by u_0 the corresponding velocity of the fluid in the case without obstacle D, namely u_0 is the solution of the Dirichlet problem

$$(4) \qquad \left\{ \begin{array}{rcccl} -\mathrm{div}(\sigma(u_0, p_0)) & = & 0 & , & \text{in } \Omega, \\ \mathrm{div}\ u_0 & = & 0 & , & \text{in } \Omega, \\ u_0 & = & g & , & \text{on } \partial\Omega. \end{array} \right.$$

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- Considere the numbers

$$W_0 = \int_{\partial\Omega} g\psi_0 \qquad \text{and} \qquad W = \int_{\partial\Omega} g\psi$$

where $\psi = \sigma(u, p) \cdot n|_{\partial\Omega}$ and $\psi_0 = \sigma(u_0, p_0) \cdot n|_{\partial\Omega}$

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- Inverse Problem: consist of estimating, from above and below, the measure $|D|$ of the obstacle D, in terms of the ratio

$$
\frac{W - W_0}{W_0}.
$$

Mathematical Setup

Definition

Let $\Omega \subset \mathbb{R}^n$ a bounded domain. We say $\partial\Omega$ is of a class $C^{k,\alpha}$, with constants $\rho_0,\ M_0 > 0$, where k is a nonnegative integer and $\alpha \in [0,1[$ if, for any $x_0 \in \partial\Omega$ there exists a rigid transformation of coordinates, in which $x_0 = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) :\ x_n > \varphi(x')\},$$

where φ is a function of class $C^{k,\alpha}(B'_\rho(0))$ such that

$$\begin{aligned} \varphi_i &= 0, \\ \nabla\varphi(0) &= 0,\ k \geq 1, \\ \|\varphi\|_{C^{k,\alpha}(B'_{\rho_0}(0))} &\leq M_0\rho_0. \end{aligned}$$

We will sometimes use the following notation, for $h > 0$:

$$\Omega_h = \{x \in \Omega :\ d(x, \partial\Omega) > h\}.$$

A Priori Information

We assume that:

- $\Omega \subset \mathbb{R}^d$ to be a bounded domain such that $\partial\Omega$ is connected, and it has smooth boundary, i.e., $\partial\Omega$ is of a class $C^{2,\alpha}$ with constants ρ_0, M_0. Further, $|\Omega| \leq M_1 \rho_0^n$, where $M_1 > 0$.
- We consider $D \subset \Omega$, which represents the obstacle we want to size estimates from the boundary measurements, on which we require that $\Omega \setminus \overline{D}$ is connected, ∂D connected and is of class $C^{2,\alpha}$ with constants ρ, L.
- Additionally, we suppose that the obstacle is well contained in Ω, this meaning $d(D, \partial\Omega) \geq d_0$.
- We assume that D satisfies the scale-invariant fatness condition with constant $Q > 0$, i.e., $diam(D) \leq Q\rho$.

Main Theorems

Theorem 1

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain such that $\partial\Omega \in C^{2,\alpha}$, with constants ρ_0, M_0, $\alpha \in [0,1]$ and let $h_1 > 0$ be such that $|D_{h_1}| \geq \frac{1}{2}|D|$. Moreover, we assume that

$$g \in H^{3/2}(\partial\Omega),\; g \not\equiv 0, \quad \int_{\partial\Omega} g \cdot n \, ds = 0, \quad \frac{\|g\|_{H^{1/2}(\partial\Omega)}}{\|g\|_{L^2(\partial\Omega)}} \leq c_0,$$

for a given constant $c_0 > 0$. Also suppose that there exists a point $p \in \partial\Omega$, such that,

$$g = 0 \text{ on } \partial\Omega \cap B_{\rho_0}(p).$$

Then, we have

$$|D| \leq K \left(\frac{\int_{\partial\Omega} (\psi - \psi_0) g}{\int_{\partial\Omega} \psi_0 g} \right). \tag{5}$$

The constant K only depending on $|\Omega|, \rho_0, M_0, \rho, \|g\|_{H^{1/2}(\partial\Omega)} / \|g\|_{L^2(\partial\Omega)}$.

Theorem 2

Under the hypotheses of Theorem 1 and let D satisfying the scale-invariant fatness condition with constant Q. Then

$$C\frac{W - W_0}{\|g\|_{H^{3/2}(\partial\Omega)} W_0^{1/2}} \leq |D|^{1/2}, \tag{6}$$

where C only depends on $|\Omega|, \rho, L, Q$.

Lemma

Let $u_0 \in H^1(\Omega)$ be the solution to problem (4) and $u \in H^1(\Omega \setminus \overline{D})$ be the solution to problem (3). Then

$$\int_D |\nabla u_0|^2 \leq \int_{\partial\Omega} (\psi - \psi_0) g = \int_{\partial D} u_0 \cdot \sigma(u,p) n, \tag{7}$$

where n denotes the exterior unit normal to ∂D.

Proposition (A. Ballerini)

Under the hypothesis of Theorem 1, there exists a constant $s > 1$, depending only on d and M_0, such that for every $r > 0$ and for every $\bar{x} \in \Omega_{sr}$, we have

$$\int_{B_r(x)} |\nabla u_0|^2 dx \geq C_\rho \int_\Omega |\nabla u_0|^2 dx,$$

where C_ρ is a constant depending only on d, M_0, M_1, c_0, r.

Sketch of the Proof: Theorem 1

- We consider a intermediate domain $\Omega_{d_0/2}$.
- Let us cover D_{h_1} with cubes Q_l of side ϵ.
- Then, we have

$$\int_D |\nabla u_0|^2 \geq \int_{\cup_{l=1}^{l} Q_l} |\nabla u_0|^2 \geq \frac{|D_{h_1}|}{\epsilon^d} \int_{Q_{\bar{l}}} |\nabla u_0|^2,$$

where $\bar{l}$ is chosen in such way that

$$\int_{Q_{\bar{l}}} |\nabla u_0|^2 = \min_{l} \int_{Q_l} |\nabla u_0|^2.$$

- From the Proposition of Ballerini we obtain that

$$\int_D |\nabla u_0|^2 \geq \frac{|D_{h_1}|}{\epsilon^d} \int_{Q_{\bar{l}}} |\nabla u_0|^2 \geq \frac{\frac{1}{2}|D|}{\epsilon^d} \int_{Q_{\bar{l}}} |\nabla u_0|^2$$
$$\geq |D| K \int_\Omega |\nabla u_0|^2 = \left(K \int_{\partial\Omega} \psi_0 \, g \right) |D|.$$

- From the Lemma, we have that

$$\int_{\partial\Omega} (\psi - \psi_0) g \geq \int_D |\nabla u_0|^2 \geq \left(K \int_{\partial\Omega} \psi_0 \, g \right) |D|.$$

- Therefore,

$$|D| \leq K \frac{\int_{\partial\Omega} (\psi - \psi_0) g}{\int_{\partial\Omega} \psi_0 \, g}.$$

Sketch of the Proof: Theorem 2

- Applying Holder's inequality to the estimate in the Lemma, and using a appropriate Poincare inequality, we obtain

$$W - W_0 \leq \overline{C}\left(\int_D |\nabla u_0|^2\right)^{1/2}\left(\int_{\partial D} |\sigma(u,p)\cdot n|^2\right)^{1/2}$$

- The first integral in the right hand side can be estimates as follows

$$\begin{aligned}\int_D |\nabla u_0|^2 &\leq C|D|^{1/2}\sup_D |\nabla u_0| \leq C|D|^{1/2}\sup_{\Omega_{d_0/2}} |u_0| \\ &\leq C|D|^{1/2}\|\nabla u_0\|_{L^2(\Omega)} = C|D|^{1/2}W_0^{1/2}.\end{aligned}$$

- For the second integral, we can deduce that

$$\|\sigma(u,p)\cdot n\|_{L^2(\partial D)} \leq C\|g\|_{H^{3/2}(\partial\Omega)}.$$

- Therefore,

$$C\frac{W-W_0}{\|g\|_{H^{3/2}(\partial\Omega)}W_0^{1/2}} \leq |D|^{1/2}.$$

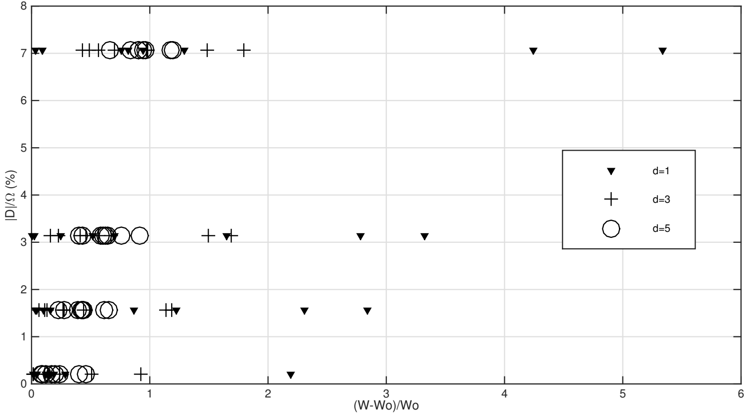

Figure: Influence of d_0 for circle inclusion.

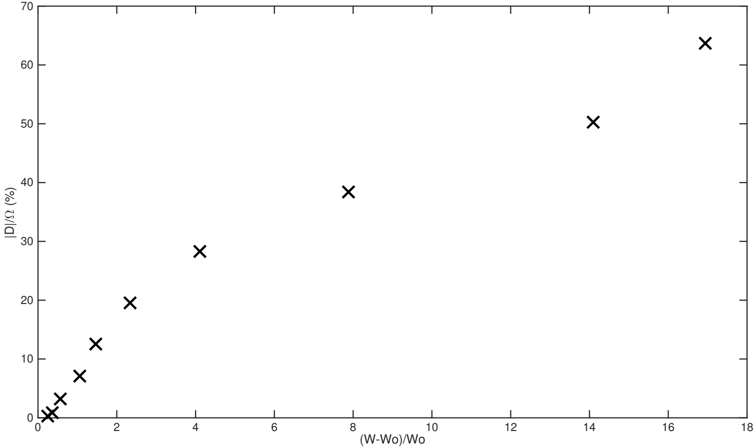

Figure: Influence of the size of the circle.

Thank you for your attention