Asymptotic behavior for the heat equation with magnetic field and Hardy potential

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The Hamiltonian

Consider the magnetic Schrödinger operator

$$H_B = \big(- i\nabla_x - A(x)\big)^2 - \frac{c_d}{|x|^2} \qquad \text{in} \qquad L^2(\mathbb{R}^d), \quad d \geq 2. \tag{1}$$

- $A : \mathbb{R}^d \to \mathbb{R}^d$ is a *magnetic potential* (1-form)
- B is the *magnetic tensor* (2-form).
- The relationship between A and B: $B = \mathrm{d}A$.
- The Maxwell equation: $\mathrm{d}B = 0$, i.e. B is a closed form.
- The dimensional quantity c_d in (1) is the best constant in the *classical Hardy inequality*

$$\forall \psi \in C_0^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \,\mathrm{d}x \geq c_d \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} \,\mathrm{d}x \,. \tag{2}$$

Explicitly:

$$c_d = \left(\frac{d-2}{2}\right)^2, \quad d \geq 3.$$

Conventionally put $c_d := 0$ for $d = 2$.

The corresponding heat equation

Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + H_B\, u = 0\,, \\ \qquad u(x,0) = u_0(x)\,, \end{cases} \tag{3}$$

where $(x,t) \in \mathbb{R}^d \times (0,\infty)$ and $u_0 \in L^2(\mathbb{R}^d)$. Formally

$$u(x,t) := \mathrm{e}^{-tH_B} u_0(x).$$

GOAL: Determine the large time behavior of the heat semigroup e^{-tH_B}.

General facts:

• It is well known that the large-time behaviour of a heat semigroup is determined by spectral-threshold properties of its generator.
• An important characterisation of this threshold behaviour is given by the existence/non-existence of Hardy-type inequalities.

- DIAMAGNETIC INEQUALITY: For any smooth A

$$\big|(\nabla - iA)\psi(x)\big| \geq \big|\nabla|\psi|(x)\big| \tag{4}$$

holds for a. e. $x \in \mathbb{R}^d$ and any $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$.

DIAMAGNETIC INEQUALITY+HARDY INEQUALITY $\Longrightarrow$ H_B can be defined as a self-adjoint operator in $L^2(\mathbb{R}^d)$ $\Longrightarrow$ the Cauchy problem is well-posed.

- More specifically, H_B is defined as the self-adjoint operator in $L^2(\mathbb{R}^d)$ associated with the quadratic form

$$h_B[\psi] := \int_{\mathbb{R}^d} \big|(\nabla - iA)\psi(x)\big|^2 \,\mathrm{d}x - c_d \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x \qquad \mathrm{D}(h_B) := \overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_{h_B}}. \tag{5}$$

Here the norm with respect to which the closure is taken is defined by

$$\|\psi\|_{h_B} := \sqrt{h_B[\psi] + \|\psi\|^2_{L^2(\mathbb{R}^d)}}\,. \tag{6}$$

• $\inf \sigma(H_B) \geq 0 = \inf \sigma(H_0)$.

Then $\|e^{-tH_B}\|_{L^2 \to L^2} \leq 1 = \|e^{-tH_0}\|_{L^2 \to L^2}$.

• The decay of the heat semigroup in the presence of magnetic field can be only better with respect to $B = 0$.

• If B is compactly supported then

$$\sigma(H_B) = \sigma(H_0) = [0, \infty].$$

• Then $\|e^{-tH_B}\|_{L^2 \to L^2} = 1$ and no extra decay of the heat semigroup is seen at this level.

The repulsive effect of the magnetic field A

In the absence of magnetic field, $H_0 := -\Delta_x - c_d/|x|^2$ is critical in the sense that c_d is optimal in (2) and there is no weight $\omega > 0$ such that

$$\forall \psi \in C_0^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \, \mathrm{d}x - c_d \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} \, \mathrm{d}x \geq \int_{\mathbb{R}^d} \omega(x) |\psi|^2 \mathrm{d}x.$$

On the contrary, whenever B is non-trivial it holds

Theorem (CC, D. Krejcirik '14)

Let $d \geq 2$. Suppose that B is smooth and closed. If $B \neq 0$, then there exists a positive constant $c_{d,B}$ such that for any smooth A satisfying $\mathrm{d}A = B$, the following inequality holds for all $\psi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |(\nabla - iA)\psi(x)|^2 \, \mathrm{d}x - c_d \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} \, \mathrm{d}x \geq c_{d,B} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{1 + |x|^2 \log^2(|x|)} \, \mathrm{d}x \,. \tag{7}$$

See also [Laptev and Weidl '99], [Weidl '99], [Alziary, Fleckinger-Pelle and Takac '03], [Balinsky, Laptev, and Sobolev '04], [Ekholm and Portmann '14], etc. for particular/weaker related results.

To exploit this subtle repulsive property of the magnetic field, we introduce a weighted space

$$L^2_w(\mathbb{R}^d) := L^2(\mathbb{R}^d, w(x)\,\mathrm{d}x)\,, \qquad \text{where} \qquad w(x) := \mathrm{e}^{|x|^2/4}\,, \tag{8}$$

and reconsider H_B as an operator from $L^2_w(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. That is, we restrict the initial data u_0 in (3) to lie in $L^2_w(\mathbb{R}^d)$.

We then consider the polynomial *decay rate*

$$\gamma_B := \sup\left\{\gamma \;\middle|\; \exists C_\gamma > 0,\, \forall t \geq 0,\; \left\|\mathrm{e}^{-tH_B}\right\|_{L^2_w(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq C_\gamma\,(1+t)^{-\gamma}\right\}. \tag{9}$$

where

$$\left\|\mathrm{e}^{-tH_B}\right\|_{L^2_w(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = \inf_{u_0\in L^2_\omega(\mathbb{R}^d)} \frac{\left\|\mathrm{e}^{-tH_B}u_0\right\|_{L^2(\mathbb{R}^d)}}{\left\|u_0\right\|_{L^2_\omega(\mathbb{R}^d)}}.$$

MAIN GOAL: FIND γ_B.

What was done for γ_B, $H_B = (-i\nabla - A(x))^2 - c_d/|x|^2$

- No magnetic field:

$$H_0 = -\Delta - c_d/|x|^2, \quad d \geq 3.$$

Zuazua-Vazquez JFA '00:

$$\gamma_0 = \frac{1}{2}.$$

- No singular potential (2-d case):

$$H_B = (-i\nabla - A(x))^2.$$

Krejcirik PDE Calc Var '13: $\gamma_0 = \frac{1}{2}$ if $B = 0$. Otherwise,

$$\gamma_B \geq \frac{1+\beta}{2},$$

where

$$\beta := \operatorname{dist}(\Phi_B, \mathbb{Z}), \qquad \Phi_B := \frac{1}{2\pi} \int_{\mathbb{R}^2} {}^*B(x)\, dx., \tag{10}$$

where ${}^*B = \operatorname{rot} A = \partial_2 A_1 - \partial_1 A_2$.

The gauge invariance

Why writing B in the subscript of H_B rather than A ?

• Let $\mathrm{d}\tilde{A} = B$. Then $\mathrm{d}(\tilde{A} - A) = 0$, *i.e.* $\tilde{A} - A$ is closed. By the Poincaré lemma, $\tilde{A} - A$ is exact, *i.e.* there exists a smooth function f such that $\tilde{A} = A + \mathrm{d}f$.

• Let $\tilde{h}_B$ and $\tilde{H}_B$ be respectively the form and operator generated by $\tilde{A}$. Then

$$\phi := \mathrm{e}^{if}\psi, \quad \psi \in C_0^\infty(\mathbb{R}^d) \Longrightarrow h_B[\psi] = \tilde{h}_B[\phi].$$

It follows that

$$\mathsf{D}(h_B) = \mathsf{D}(\tilde{h}_B), \quad \tilde{H}_B = \mathrm{e}^{if} H_B \,\mathrm{e}^{-if}. \tag{11}$$

$\Longrightarrow$ The operators corresponding to different gauges are unitarily equivalent and isospectral.

• For our purpose it is enough to work in one gauge. We chose the Poincare gauge since this choice is convenient because we wish to work in the spherical coordinates.

The *Poincaré* (or *transverse*) gauge:

$$x \cdot A(x) = 0, \quad \forall x \in \mathbb{R}^d. \tag{12}$$

• We can assume (12) without loss of any generality, because of the gauge invariance of the physical theory.

EXISTENCE OF THE GAUGE (12): For a given smooth tensor field B, define the vector potential

$$A(x) := \int_0^1 x \cdot B(xu)\, u\, \mathrm{d}u \tag{13}$$

Proposition

Let $d \geq 2$. Suppose that B is smooth and closed. Then the vector potential A defined by (13) satisfies (12) and $\mathrm{d}A = B$.

Spherical coordinates

For this reason it will be convenient to introduce spherical coordinates

$$\mathcal{L} : S^{d-1} \times (0, \infty) \to \mathbb{R}^d : \{(\sigma, r) \mapsto \sigma r\}\,. \tag{14}$$

• We use the letters q and x to denote points in $S^{d-1} \times (0, \infty)$ and $\mathbb{R}^d$, respectively.

• Writing $q = (q', q^d)$, where $q' \in S^{d-1}$ and $q^d \in (0, \infty)$, we obviously have $q^d = r = |x|$ and $q' = \sigma = x/|x| = \partial/\partial_r$, with $x \in \mathbb{R}^d$. In these coordinates, the metric acquires the block-diagonal form

$$g = r^2\, \mathrm{d}\sigma^2 + \mathrm{d}r^2 = \begin{pmatrix} r^2\,\gamma & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad |g| = r^{2(d-1)}\,|\gamma|\,, \tag{15}$$

where $\mathrm{d}\sigma^2 = \gamma_{\mu\nu}(\theta)\,\mathrm{d}\theta^\mu \otimes \mathrm{d}\theta^\nu$ is the metric of S^{d-1}, using local coordinates $\theta^1, \dots, \theta^{d-1}$ on the sphere S^{d-1}.

The transfer matrix reads

$$\nabla\mathcal{L} = \begin{pmatrix} r\,\nabla'\sigma \\ \sigma \end{pmatrix}, \qquad (16)$$

in a concise notation where $\nabla'\sigma$ is the gradient with respect to the coordinates $\theta^1, \dots, \theta^{d-1}$.

• The transformed magnetic potential and magnetic tensor are respectively given by

$$\mathsf{A} := \nabla\mathcal{L}\cdot(A\circ\mathcal{L}) \qquad \text{and} \qquad \mathsf{B} := \nabla\mathcal{L}\cdot(B\circ\mathcal{L})\cdot(\nabla\mathcal{L})^T.$$

• In addition we have

$$|\mathsf{A}|^2 := \mathsf{A}_j g^{jk}\mathsf{A}_k \text{ and } |\mathsf{A}|^2_{S^{d-1}} := \mathsf{A}_\mu\gamma^{\mu\nu}\mathsf{A}_\nu.$$

• The radial component in the spherical coordinates basis vanishes: $\mathsf{A}_d = 0$.

We introduce the quantity

$$\nu_B(r) := \inf_{\varphi \in H^1(S^{d-1}), \varphi \neq 0} \frac{\displaystyle\int_{S^{d-1}} \left| \left(\mathrm{d}' - i\mathsf{A}(\sigma, r) \right) \varphi(\sigma) \right|^2_{S^{d-1}} \mathrm{d}\sigma}{\displaystyle\int_{S^{d-1}} |\varphi(\sigma)|^2 \, \mathrm{d}\sigma}, \qquad (17)$$

- $\nu_B(r)$ is the lowest eigenvalue of a magnetic Laplace-Beltrami operator in $L^2(S^{d-1})$.
- Assuming that B is smooth and compactly supported, the limit

$$\mathsf{A}_\infty(\sigma) := \lim_{r\to\infty} \mathsf{A}(\sigma, r) \qquad (18)$$

exists as a smooth vector field from the unit sphere S^{d-1} to $\mathbb{R}^d$ and we may also define the corresponding number

$$\nu_B(\infty) := \lim_{r\to\infty} \nu_B(r)\,. \qquad (19)$$

Theorem

Let $d \geq 2$. Suppose that B is smooth, closed and compactly supported. Then

$$\gamma_B = \frac{1 + \sqrt{\nu_B(\infty)}}{2}\,. \tag{20}$$

We find

$$\nu_B(\infty) = \mathrm{dist}(\Phi_B, \mathbb{Z})^2 \qquad \text{if} \qquad d = 2\,. \tag{21}$$

and

$$\nu_B(\infty) = 0 \qquad \text{if} \qquad d \geq 3\,. \tag{22}$$

Theorem (main result- C.C, D. Krejcirik '14)

Let $d \geq 2$. Suppose that B is smooth, closed and compactly supported. Then

$$\gamma_B = \begin{cases} \dfrac{1+\beta}{2} & \textit{if} \quad d = 2\,, \\ \dfrac{1}{2} & \textit{if} \quad d \geq 3\,, \end{cases}$$

where

$$\beta := \mathrm{dist}(\Phi_B, \mathbb{Z})\,, \qquad \Phi_B := \frac{1}{2\pi}\int_{\mathbb{R}^2} {}^*B(x)\,\mathrm{d}x\,. \tag{23}$$

The self-similarity variables

Define the *self-similar variables* $(y, s) \in \mathbb{R}^d \times (0, \infty)$ by

$$y := (t+1)^{-1/2}\, x\,, \qquad s := \log(t+1)\,. \tag{24}$$

If u is a solution of (3), we then define a new function

$$\tilde{u}(y, s) := \mathrm{e}^{sd/4}\, u\big(\mathrm{e}^{s/2} y, \mathrm{e}^{s} - 1\big)\,. \tag{25}$$

The inverse transform is given by

$$u(x, t) = (t+1)^{-d/4}\, \tilde{u}\big((t+1)^{-1/2} x, \log(t+1)\big)\,. \tag{26}$$

It is straightforward to check that $\tilde{u}$ satisfies a weak formulation of the Cauchy problem

$$\begin{cases} \tilde{u}_{,s} + \big(-i\nabla_y - A_s(y) \big)^2 \tilde{u} - \dfrac{c_d}{|y|^2} \tilde{u} - \dfrac{1}{2} y \cdot \nabla_y \tilde{u} - \dfrac{d}{4} \tilde{u} = 0 \,, & (y,s) \in \mathbb{R}^d \times (0,\infty \\ \tilde{u}(y,0) = u_0(y) \,, & y \in \mathbb{R}^d \,, \end{cases} \tag{27}$$

with the new, s-dependent magnetic potential

$$A_s(y) := \mathrm{e}^{s/2} A(\mathrm{e}^{s/2} y) \,. \tag{28}$$

The self-similarity transform $u \mapsto \tilde{u}$ acts as a unitary transform in $L^2(\mathbb{R}^d)$:

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = \|\tilde{u}(s)\|_{L^2(\mathbb{R}^d)} \tag{29}$$

The natural space to study the evolution is not $L^2(\mathbb{R}^d)$ but rather the weighted space $L^2_w(\mathbb{R}^d)$
We thus define an additional transform

$$\tilde{v}(y,s) := w(y)^{1/2} \, \tilde{u}(y,s) \tag{30}$$

that casts (27) *formally* to

$$\begin{cases} \tilde{v}_{,s} + \big(-i\nabla_y - A_s(y)\big)^2\tilde{v} - \dfrac{c_d}{|y|^2}\tilde{v} + \dfrac{|y|^2}{16}\tilde{v} - \dfrac{1}{2}iy\cdot A_s(y)\,\tilde{v} = 0(y,s)\in\mathbb{R}^d\times(0,\infty) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tilde{v}(y,0) = v_0(y), y\in\mathbb{R}^d \end{cases} \tag{31}$$

where $v_0 := w^{1/2}u_0$.

• Looking for solutions of (27) with an initial datum $u_0 \in L^2_w(\mathbb{R}^d)$ is the same as looking for solutions of (31) with the initial datum $v_0 \in L^2(\mathbb{R}^d)$.

• choice of the gauge $\Rightarrow$ the non-symmetric term vanishes.

• The harmonic-oscillator potential in (31) ensures the compactness of the resolvent of

$$L_s := \big(-i\nabla_y - A_s(y)\big)^2 - \frac{c_d}{|y|^2} + \frac{|y|^2}{16}.$$

• Consequently, L_s has a purely discrete spectrum.

• Let $\lambda_B(s)$ denote the first eigenvalue of L_s

$$\lambda_B(s) = \min_{\psi\in \mathsf{D}(l),\psi\neq 0} \frac{l_s[\psi]}{\|\psi\|^2_{L^2(\mathbb{R}^d)}}, \tag{32}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\tilde{v}(s)\|^2_{L^2(\mathbb{R}^d)} = -l_s[\tilde{v}(s)] \tag{33}$$

for every $s \geq 0$. Now, using the spectral bound

$$l_s[\tilde{v}(s)] \geq \lambda_B(s)\,\|\tilde{v}(s)\|^2_{L^2(\mathbb{R}^d)}\,, \tag{34}$$

from Gronwall's inequality we get

$$\|\tilde{v}(s)\|_{L^2(\mathbb{R}^d)} \leq \|v_0\|_{L^2(\mathbb{R}^d)}\,\mathrm{e}^{-\int_0^s \lambda_B(\tau)\,\mathrm{d}\tau}, \quad \forall s \geq 0. \tag{35}$$

Undoing the variables we get

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2_\omega(\mathbb{R}^d)}\,\mathrm{e}^{-\int_0^{\log(t+1)} \lambda_B(\tau)\,\mathrm{d}\tau}, \quad \forall s \geq 0. \tag{36}$$

• The problem of large-time behaviour of (1) is reduced to a spectral analysis of the family of operators $\{L_s\}_{s\geq 0}$.

The following theorem is probably the most important auxiliary result of this paper.

Theorem

Let $d \geq 2$. Suppose that B is smooth, closed and compactly supported. , then the operator L_s converges to L_∞ in the norm-resolvent sense as $s \to \infty$, i.e.,

$$\lim_{s \to \infty} \left\| \mathsf{L}_s^{-1} - \mathsf{L}_\infty^{-1} \right\|_{\mathcal{H} \to \mathcal{H}} = 0 \,. \tag{37}$$

CONSEQUENCE: The spectrum of L_s converges to the spectrum of L_∞ as $s \to \infty$, where L_∞ is an Ahoronov-Bohn type operator given by

$$L_\infty := \big(- i\nabla_y - A_\infty(y) \big)^2 - \frac{c_d}{|y|^2} + \frac{|y|^2}{16}.$$

In particular,

$$\lambda_B(s) \to \lambda_B(\infty), \ \text{as } s \to \infty,$$

where $\lambda_B(\infty)$ is the first eigenvalue of L_∞.

Proposition

Let $d \geq 2$. Suppose that B is smooth, closed and compactly supported. We have

$$\sigma(L_\infty) = \left\{ n + \frac{1 + \sqrt{\nu_{B,\ell}(\infty)}}{2} \right\}_{n,\ell \in \mathbb{N}},$$

where $\{\nu_{B,\ell}(\infty)\}_{\ell \in \mathbb{N}}$ is the set of eigenvalues of the operator $\big(-i\nabla_\sigma - \mathsf{A}_\infty(\sigma)\big)^2$ in $L^2(S^{d-1})$.

Once we get the spectrum of L_∞ we can handle to prove the main result...

C. C. and D. Krejčiřík, *The Hardy inequality and the heat equation with magnetic field in any dimension*, submitted (available at arxiv.org).

Thank you for your attention !