Potential theory of the porous medium equation, obstacle problems, and all that

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Introduction

- The aim is to give an overview of the potential theory of the porous medium equation.
- Superharmonic functions, Perron solutions, Riesz measures, etc. have their counterparts.
- The obstacle problem is a crucial tool.

Porous medium equation

- The porous medium equation (PME for short) is

$$\partial_t u - \Delta u^m = 0$$

  with $m > 1$.
- Weak solutions: $u^m \in L^2(0, T; H^1(\Omega))$ such that

$$\int_{\Omega_T} -u\partial_t\varphi + \nabla u^m \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = 0$$

  for all smooth φ.
- Weak supersolutions: require "$\geq$" for positive φ.

p-parabolic equation

- The other prototype is the p-parabolic equation

$$\partial_t u - \operatorname{div} |\nabla u|^{p-2}\nabla u = 0$$

  with $p > 2$.

- Weak solutions: $u \in L^p(0, T; W^{1,p}(\Omega))$ such that

$$\int_{\Omega_T} -u\partial_t\varphi + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = 0$$

  for all smooth φ.

Nonlinear analogue of superharmonic functions

A function u is a semicontinuous supersolution, if it is

1. lower semicontinuous, and
2. satisfies the comparison principle with respect to continuous weak solutions:
   For $U_{t_1,t_2} \Subset \Omega_T$, if $v \in C(\overline{U}_{t_1,t_2})$ is a weak solution in U_{t_1,t_2} with $v \leq u$ on $\partial_p U_{t_1,t_2}$, then $v \leq u$ in U_{t_1,t_2}.

Definition due to F. Riesz.

Barenblatt solution I

- The Barenblatt solution is given by

$$\mathcal{B}_m(x,t) = \begin{cases} t^{-\lambda}\left(C - \frac{\lambda(m-1)}{2mn}\frac{|x|^2}{t^{2\lambda/n}}\right)_+^{1/(m-1)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

  where $\lambda = n/(n(m-1)+2)$, and $C > 0$ can be chosen freely.
- Barenblatt, Zeldovich-Kompaneets

Barenblatt solution II

- Nonlinear counterpart of the fundamental solution.
- Found by looking for solutions in the form

$$u(x,t) = t^{-\alpha} F(|x| t^{\beta}).$$

- $\mathcal{B}_m$ is a semicontinuous supersolution in $\mathbf{R}^{n+1}$, but not a weak supersolution in any open set containing the origin.

The friendly giant I

- The friendly giant is given by

$$G_m(x,t) = \begin{cases} t^{-\frac{1}{m-1}} F(x), & t > 0, \\ 0, & t \leq 0, \end{cases}$$

  where $F \geq 0$ solves the boundary value problem

$$\begin{cases} \Delta F^m + \frac{1}{m-1} F = 0, & \text{in } \Omega, \\ F = 0, & \text{on } \partial\Omega \end{cases}$$

  in a bounded domain Ω.
- Constructed by Dahlberg-Kenig.

The friendly giant II

- Since $F > 0$ in Ω,
$$\lim_{t \to 0} G_m(x,t) = \infty$$
for all $x \in \Omega$.
- G_m is a semicontinuous supersolution in $\Omega \times \mathbf{R}$ but not a weak supersolution.
- In fact G_m^m is not integrable in any cylinder intersecting the line $t = 0$.
- No counterpart in linear theory!

What about obstacle problems?

- By lower semicontinuity, one may approximate a semicontinuous supersolution by an increasing sequence of smooth functions.
- Solve the obstacle problem with these smooth functions as obstacles.
- Outcome: any semicontinuous supersolution is an increasing limit of weak supersolutions. (Lindqvist)

Measure data problems

- A consequence of the approximation is the existence of Riesz measures of semicontinuous supersolutions.
- For a class of semicontinuous supersolutions u, there is a measure μ such that

$$\partial_t u - \Delta u^m = \mu.$$

  (Kinnunen-Lindqvist).
- Conversely, one may start from the measure, and construct a semicontinuous supersolution so that the above equation holds.

Why obstacle problems?

- Heuristic principle: solutions to obstacle problems are as regular as weak solutions, as long as the obstacle allows it.
- The point of using obstacle problems is to have the ability to construct supersolutions with favorable regularity properties.
- Regularity in time is a particularly delicate issue.

Perron's method

- We want to solve the Dirichlet problem with continuous boundary values $g : \partial_p \Omega_T \to [0, \infty)$.
- The *upper class* $\mathfrak{U}_g$: semicontinuous supersolutions v which satisfy

$$\liminf_{z \to \xi} v(z) \geq g(\xi)$$

  for all $\xi \in \partial_p \Omega_T$. The *upper Perron solution* is then

$$\overline{H}_g(z) = \inf_{v \in \mathfrak{U}_g} v(z).$$

- Lower Perron solution $\underline{H}_g$ has a similar definition in terms of semicontinuous subsolutions below g on the boundary.

Resolutivity I

- Immediate from definitions: $\underline{H}_g \leq \overline{H}_g$.
- Resolutivity: for which functions g it holds $\underline{H}_g = \overline{H}_g$?
- The common function is the Perron solution H_g.

Resolutivity II

- Wiener's resolutivity theorem for the PME:

$$\underline{H}_g = \overline{H}_g$$

  holds for all continuous g in any space-time cylinder Ω_T. (Kinnunen-Lindqvist-L.)
- Crucial in the proof: find supersolutions v such that $\partial_t v^m$ is an L^2 function.
- Such supersolutions come from the obstacle problem.

Turning the tables

- One can also use semicontinuous supersolutions to construct solutions to the obstacle problem.
- The procedure is analogous to the balayage concept from classical potential theory.

Smallest solution

- Consider an obstacle ψ, positive, continuous and compactly supported in Ω_T.
- Define

$$\mathcal{U}_\psi = \{v \text{ semicontinuous supersolution} : v \geq \psi \text{ a.e. in } \Omega_T\}$$

  and

$$\tilde{u}_\psi(x,t) = \inf\{v(x,t) : v \in \mathcal{U}_\psi\}.$$

- The smallest solution to the obstacle problem u_ψ is then

$$u_\psi(x,t) = \operatorname*{ess\,liminf}_{(y,s)\to(x,t)} \tilde{u}_\psi(y,s),$$

  the lower semicontinuous regularization of $\tilde{u}_\psi$.
- Parviainen-Lindqvist (p-parabolic equation) , Avelin-L. (PME).

Variational solutions to the PME obstacle problem

- A function u is a variational solution to the PME obstacle problem if

$$\int_{\Omega_T} \partial_t u(v^m - u^m) + \nabla u^m \cdot (\nabla v^m - \nabla u^m) \,\mathrm{d}x \,\mathrm{d}t \geq 0$$

  for comparison functions $v \geq \psi$.

- Bögelein-L.-Scheven.

The connection between variational solutions and the smallest solution

- The smallest solution is a pointwise limit of variational solutions. (Avelin-L.)
- By stability of variational solutions, the smallest solution is a variational solution for sufficiently regular obstacles.

Open problems

- Are variational solutions to the PME obstacle problem unique?
- Are semicontinuous supersolutions to the PME the same as viscosity supersolutions?
- For the p-parabolic equation, the answer to both questions is yes.

Uniqueness I

- One way to solve the uniqueness: show that a variational solution is the smallest supersolution.
- A variational solution u is a weak solution to the PME in $\{u > \psi\}$.
- Since $u = \psi$ on $\partial\{u > \psi\}$, by comparison $u \leq v$ whenever v is a supersolution with $\psi \leq v$. Thus u must be the smallest solution.

Uniqueness II

- The gap in the above argument is that $\{u > \psi\}$ is a general open set in R^{n+1}, not a space-time cylinder.
- A sufficiently strong comparison principle in general open sets for the PME is not known.

Viscosity solutions I

- One would like to prove a comparison principle for viscosity solutions.
- This turns out to be a bit complicated even for simple equations like

$$-\operatorname{div}(a(x)\nabla u) = 0,$$

where the coefficient a is a smooth, strictly positive.

Viscosity solutions II

- Formally one may write the PME as

$$\partial_t - \mathrm{div}(mu^{m-1}\nabla u) = 0.$$

- The coefficient mu^{m-1} can be rough and vanish on a large set...

Thank you for your attention.