Stability of difference equations and applications to wave propagation on networks

Guilherme Mazanti
joint work with Yacine Chitour and Mario Sigalotti

Partial differential equations, optimal design and numerics
Benasque – August 31st, 2015

CMAP, École Polytechnique
Team GECO, Inria Saclay
France

Outline

Introduction

Linear difference equations

$$\Sigma_\delta(L, A) : \qquad u(t) = \sum_{j=1}^{N} A_j(t) u(t - L_j), \qquad t \geq 0$$

- $L_1, \dots, L_N$: *(rationally independent)* positive delays.
- $A_1(t), \dots, A_N(t)$: time-dependent $d \times d$ matrices.
- $u(t) \in \mathbb{C}^d$.
- Notation: $L_{\min} = \min_j L_j$, $L_{\max} = \max_j L_j$.

Introduction
Linear difference equations

$$\Sigma_\delta(L, A): \qquad u(t) = \sum_{j=1}^{N} A_j(t) u(t - L_j), \qquad t \geq 0$$

- $L_1, \ldots, L_N$: *(rationally independent)* positive delays.
- $A_1(t), \ldots, A_N(t)$: time-dependent $d \times d$ matrices.
- $u(t) \in \mathbb{C}^d$.
- Notation: $L_{\min} = \min_j L_j$, $L_{\max} = \max_j L_j$.

Motivation:

- Applications to transport and wave propagation.
- Generalization of simpler cases: $N = 1$, autonomous.

Introduction

Motivation: transport systems

Hyperbolic PDEs $\rightarrow$ difference equations: [Cooke, Krumme, 1968], [Slemrod, 1971], [Greenberg, Li, 1984], [Coron, Bastin, d'Andréa Novel, 2008], [Fridman, Mondié, Saldivar, 2010], [Gugat, Sigalotti, 2010]...

Introduction

Motivation: transport systems

Hyperbolic PDEs $\rightarrow$ difference equations: [Cooke, Krumme, 1968], [Slemrod, 1971], [Greenberg, Li, 1984], [Coron, Bastin, d'Andréa Novel, 2008], [Fridman, Mondié, Saldivar, 2010], [Gugat, Sigalotti, 2010]...

$$\begin{cases} \partial_t u_i(t, x) + \partial_x u_i(t, x) \\ \qquad + \alpha_i(t, x) u_i(t, x) = 0, & t \in \mathbb{R}_+,\; x \in [0, L_i],\; i \in [\![1, N]\!], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^N m_{ij}(t) u_j(t, L_j), & t \in \mathbb{R}_+,\; i \in [\![1, N]\!]. \end{cases}$$

Introduction

Motivation: transport systems

Hyperbolic PDEs $\rightarrow$ difference equations: [Cooke, Krumme, 1968], [Slemrod, 1971], [Greenberg, Li, 1984], [Coron, Bastin, d'Andréa Novel, 2008], [Fridman, Mondié, Saldivar, 2010], [Gugat, Sigalotti, 2010]...

$$\begin{cases} \partial_t u_i(t, x) + \partial_x u_i(t, x) \\ \qquad + \alpha_i(t, x) u_i(t, x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_i],\ i \in [\![1, N]\!], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^N m_{ij}(t) u_j(t, L_j), & t \in \mathbb{R}_+,\ i \in [\![1, N]\!]. \end{cases}$$

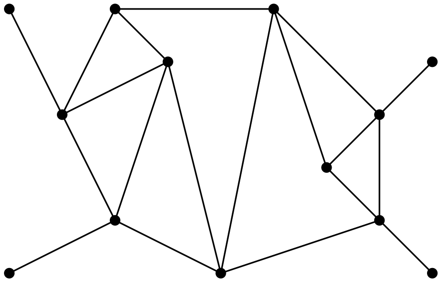
Method of characteristics: for $t \geq \max_i L_i$,

$$u_i(t, 0) = \sum_{j=1}^N m_{ij}(t) u_j(t, L_j) = \sum_{j=1}^N m_{ij}(t) e^{-\int_0^{L_j} \alpha_j(t-s, L_j-s) ds} u_j(t - L_j, 0).$$

Set $v(t) = (u_i(t, 0))_{i \in [\![1, N]\!]}$. Then v satisfies a difference equation.

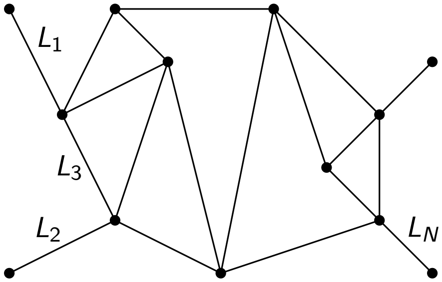
Introduction

Motivation: wave propagation on networks



Introduction

Motivation: wave propagation on networks



Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$

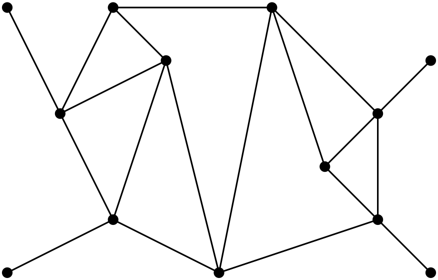
$$\partial_{tt}^2 u_i(t, x) = \partial_{xx}^2 u_i(t, x)$$

$$u_i(t, q) = u_j(t, q), \qquad \forall q \in \mathcal{V},\ \forall i, j \in \mathcal{E}_q$$

$+$ conditions on vertices.

Introduction

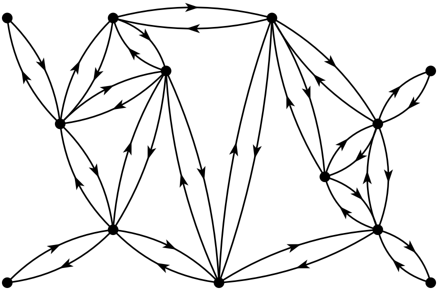
Motivation: wave propagation on networks



Introduction

Motivation: wave propagation on networks

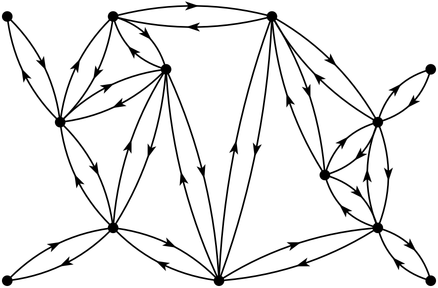
D'Alembert decomposition on travelling waves:



Introduction

Motivation: wave propagation on networks

D'Alembert decomposition on travelling waves:



System of $2N$ transport equations.

Can be reduced to a system of difference equations.

Introduction

Motivation: case $N = 1$

- When $N = 1$: $u(t) = A(t)u(t - L)$.
- Can be reduced to $u_n = A_n u_{n-1}$.

Introduction

Motivation: case $N = 1$

- When $N = 1$: $u(t) = A(t)u(t - L)$.
- Can be reduced to $u_n = A_n u_{n-1}$.

Autonomous system

$$u_n = A u_{n-1}$$

$$A \in \mathcal{M}_d(\mathbb{C})$$

Exponential stability
$\iff \rho(A) < 1$

$$\rho(A) = \lim_{n \to +\infty} |A^n|^{\frac{1}{n}}$$
$$= \max_{\lambda \in \sigma(A)} |\lambda|$$

Introduction

Motivation: case $N = 1$

- When $N = 1$: $u(t) = A(t)u(t - L)$.
- Can be reduced to $u_n = A_n u_{n-1}$.

Autonomous system

$$u_n = A u_{n-1}$$

$$A \in \mathcal{M}_d(\mathbb{C})$$

Exponential stability $\iff \rho(A) < 1$

$$\rho(A) = \lim_{n \to +\infty} \left|A^n\right|^{\frac{1}{n}} = \max_{\lambda \in \sigma(A)} \left|\lambda\right|$$

Arbitrary switching

$$u_n = A_n u_{n-1}$$

$$A_n \in \mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})$$

Uniform exponential stability $\iff \rho_{\mathrm{J}}(\mathfrak{B}) < 1$

$$\rho_{\mathrm{J}}(\mathfrak{B}) = \lim_{n \to +\infty} \sup_{A_1, \dots, A_n \in \mathfrak{B}} \left|A_1 A_2 \cdots A_n\right|^{\frac{1}{n}}$$

(cf. [Jungers, 2009])

Introduction

Motivation: autonomous [Cruz, Hale, 1970], [Henry, 1974], [Michiels et al., 2009]

$$\Sigma_\delta^{\mathrm{aut}}(L, A): \qquad u(t) = \sum_{j=1}^{N} A_j u(t - L_j), \qquad t \geq 0$$

- Stability for rationally independent $L_1, \dots, L_N$ characterized by $\rho_{\mathrm{HS}}(A) = \max_{(\theta_1, \dots, \theta_N) \in [0, 2\pi]^N} \rho\left(\sum_{j=1}^N A_j e^{i\theta_j}\right)$.

Theorem (Hale, 1975; Silkowski, 1976)

The following are equivalent:

- $\rho_{\mathrm{HS}}(A) < 1$;
- $\Sigma_\delta^{\mathrm{aut}}(L, A)$ *is exponentially stable for some* $L \in (0, +\infty)^N$ *with rationally independent components;*
- $\Sigma_\delta^{\mathrm{aut}}(L, A)$ *is exponentially stable for every* $L \in (0, +\infty)^N$.

Main result and applications
Main result

$$\Sigma_\delta(L, A) : \qquad u(t) = \sum_{j=1}^{N} A_j(t) u(t - L_j), \qquad t \geq 0$$

- $\mathrm{X}_p^\delta = L^p([-L_{\max}, 0], \mathbb{C}^d)$, $p \in [1, +\infty]$.
- Exponential stability of $\Sigma_\delta(L, A)$ uniformly with respect to a given set $\mathcal{A}$ of functions $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$.
- $\Sigma_\delta(L, \mathcal{A})$: family of systems $\Sigma_\delta(L, A)$ for $A \in \mathcal{A}$.
- RI: set of all $L = (L_1, \dots, L_N) \in (0, +\infty)^N$ with rationally independent components.

Main result and applications

Main result

Theorem (Chitour, M., Sigalotti)

Let $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$ bounded and $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$. The following statements are equivalent:

- $\mu(L, \mathfrak{B}) < 1$;
- $\Sigma_\delta(L, \mathcal{A})$ *is exponentially stable in* X_p^δ *for some* $p \in [1, +\infty]$ *and* $L \in \mathrm{RI}$;
- $\Sigma_\delta(L, \mathcal{A})$ *is exponentially stable in* X_p^δ *for every* $p \in [1, +\infty]$ *and* $L \in (0, +\infty)^N$.

Main result and applications

Main result

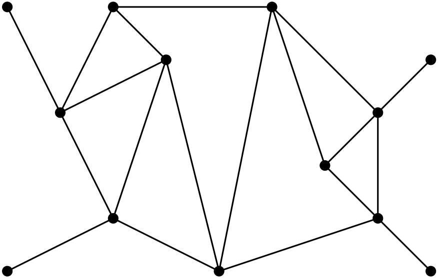
Theorem (Chitour, M., Sigalotti)

Let $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$ bounded and $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$. The following statements are equivalent:

- $\mu(L, \mathfrak{B}) < 1$;
- $\Sigma_\delta(L, \mathcal{A})$ *is exponentially stable in* X_p^δ *for some* $p \in [1, +\infty]$ *and* $L \in$ RI;
- $\Sigma_\delta(L, \mathcal{A})$ *is exponentially stable in* X_p^δ *for every* $p \in [1, +\infty]$ *and* $L \in (0, +\infty)^N$.

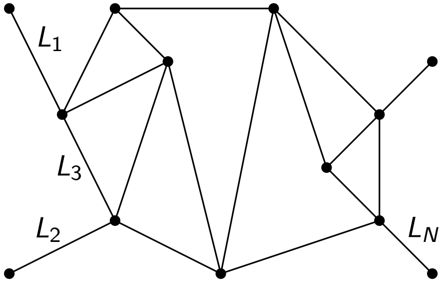
Main result and applications

Application to wave propagation on networks



Main result and applications

Application to wave propagation on networks

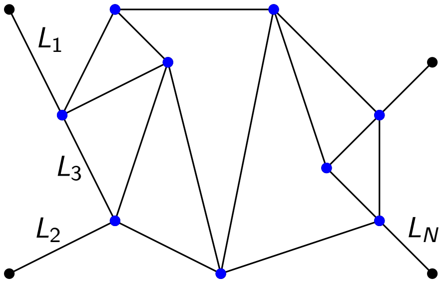


Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$

$$\begin{aligned}
&\partial_{tt}^2 u_i(t, x) = \partial_{xx}^2 u_i(t, x),\\
&u_i(t, q) = u_j(t, q),\\
&\qquad\qquad \forall q \in \mathcal{V},\ \forall i, j \in \mathcal{E}_q,
\end{aligned}$$

Main result and applications

Application to wave propagation on networks



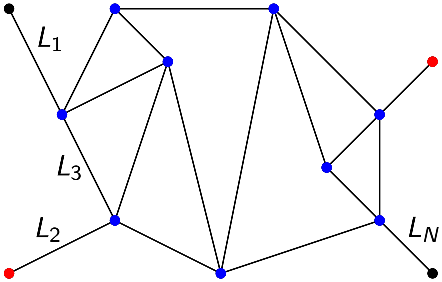
Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$

Interior vertices: $\mathcal{V}_{\text{int}}$

$$
\begin{aligned}
&\partial_{tt}^2 u_i(t,x) = \partial_{xx}^2 u_i(t,x),\\
&u_i(t,q) = u_j(t,q),\\
&\qquad\qquad \forall q \in \mathcal{V},\ \forall i,j \in \mathcal{E}_q,\\
&\textstyle\sum_{i \in \mathcal{E}_q} \partial_n u_i(t,q) = 0,\\
&\qquad\qquad\qquad \forall q \in \mathcal{V}_{\text{int}},
\end{aligned}
$$

Main result and applications

Application to wave propagation on networks



Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$

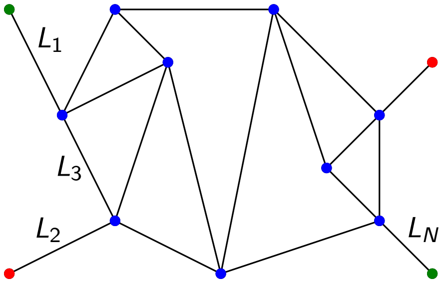
Interior vertices: $\mathcal{V}_{\text{int}}$
Damped vertices: $\mathcal{V}_{\text{d}}$

$$\partial_{tt}^2 u_i(t,x) = \partial_{xx}^2 u_i(t,x),$$
$$u_i(t,q) = u_j(t,q), \qquad \forall q \in \mathcal{V},\ \forall i,j \in \mathcal{E}_q,$$
$$\sum_{i\in\mathcal{E}_q} \partial_n u_i(t,q) = 0, \qquad \forall q \in \mathcal{V}_{\text{int}},$$

$$\partial_t u_i(t,q) = -\eta_q(t)\partial_n u_i(t,q), \qquad \forall q \in \mathcal{V}_{\text{d}},$$

Main result and applications

Application to wave propagation on networks



Edges: $\mathcal{E}$

Vertices: $\mathcal{V}$

$\mathcal{V} = \mathcal{V}_{\mathrm{int}} \cup \mathcal{V}_{\mathrm{d}} \cup \mathcal{V}_{\mathrm{u}}$

Interior vertices: $\mathcal{V}_{\mathrm{int}}$

Damped vertices: $\mathcal{V}_{\mathrm{d}}$

Undamped vertices: $\mathcal{V}_{\mathrm{u}}$

$$\partial_{tt}^2 u_i(t, x) = \partial_{xx}^2 u_i(t, x),$$

$$u_i(t, q) = u_j(t, q), \quad \forall q \in \mathcal{V},\ \forall i, j \in \mathcal{E}_q,$$

$$\sum_{i \in \mathcal{E}_q} \partial_n u_i(t, q) = 0, \quad \forall q \in \mathcal{V}_{\mathrm{int}},$$

$$\partial_t u_i(t, q) = -\eta_q(t) \partial_n u_i(t, q), \quad \forall q \in \mathcal{V}_{\mathrm{d}},$$

$$u_i(t, q) = 0,\ \forall q \in \mathcal{V}_{\mathrm{u}}.$$

Main result and applications

Application to wave propagation on networks

- System: $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$, where $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, $(\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}} \in \mathcal{D}$.
- $((u_i)_{i \in \mathcal{E}}, (\partial_t u_i)_{i \in \mathcal{E}}) \in \mathsf{X}_p^\omega = W_0^{1,p}(\mathcal{G}, L) \times L^p(\mathcal{G}, L)$, where $W_0^{1,p}(\mathcal{G}, L)$: $W^{1,p}$ on each edge, continuous at all vertices, equal to 0 on $q \in \mathcal{V}_{\mathrm{u}}$.

Theorem

Suppose that $\mathcal{D} = L^\infty(\mathbb{R}, \mathfrak{D})$ for some $\mathfrak{D} \subset \mathbb{R}_+^d$, $d = \#\mathcal{V}_{\mathrm{d}}$. The following statements are equivalent:

- $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ *is exponentially stable in* X_p^ω *for some* $p \in [1, +\infty]$ *and* $L \in \mathrm{RI}$;
- $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ *is exponentially stable in* X_p^ω *for every* $p \in [1, +\infty]$ *and* $L \in (0, +\infty)^N$.

Main result and applications

Application to wave propagation on networks

Topological characterization of exponential stability:

Theorem

Let $\mathfrak{D} \subset \mathbb{R}_+^d$ be bounded, $\mathcal{D} = L^\infty(\mathbb{R}, \mathfrak{D})$. Then $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ is exponentially stable in X_p^ω for some p if and only if $\mathcal{G}$ is a tree, $\mathcal{V}_{\mathrm{u}}$ contains only one point, and $\overline{\mathfrak{D}} \subset (0, +\infty)^d$.

Main result and applications

Application to wave propagation on networks

Topological characterization of exponential stability:

Theorem

Let $\mathfrak{D} \subset \mathbb{R}_+^d$ be bounded, $\mathcal{D} = L^\infty(\mathbb{R}, \mathfrak{D})$. Then $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ is exponentially stable in X_p^ω for some p if and only if $\mathcal{G}$ is a tree, $\mathcal{V}_{\mathrm{u}}$ contains only one point, and $\overline{\mathfrak{D}} \subset (0, +\infty)^d$.

$\Longleftarrow$: classical methods (see e.g. [Dáger, Zuazua, 2006]). One has that, for every $t, s \geq 0$,

$$\|u(t+s)\|_{\mathsf{X}_2^\omega}^2 = \|u(t)\|_{\mathsf{X}_2^\omega}^2 - \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+s} 2\eta_q(\tau) \left|\frac{\partial u_i}{\partial x}(\tau, 0)\right|^2 d\tau$$

and, since $\eta_q(\tau) \geq \eta_{\min}$, it suffices to show that $\exists c, \ell > 0$ s.t.

$$\sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+\ell} \left|\frac{\partial u_i}{\partial x}(\tau, 0)\right|^2 d\tau \geq c \|u(t)\|_{\mathsf{X}_2^\omega}^2 .$$

Main result and applications

Application to wave propagation on networks

$\Longrightarrow$: (only for the case $L \in \mathrm{RI}$)

- Exponential stability for $L \in \mathrm{RI}$ $\iff$ exponential stability for every L.

Main result and applications

Application to wave propagation on networks

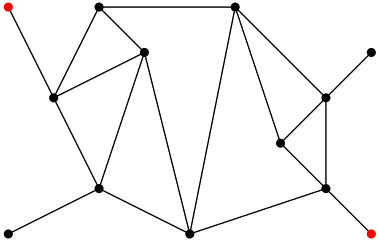
$\Longrightarrow$: (only for the case $L \in \mathrm{RI}$)

- Exponential stability for $L \in \mathrm{RI}$ $\iff$ exponential stability for every L.
- Take $L' = (1, 1, \dots, 1)$.

Main result and applications
Application to wave propagation on networks

$\Longrightarrow$: (only for the case $L \in \mathrm{RI}$)

- Exponential stability for $L \in \mathrm{RI} \iff$ exponential stability for every L.
- Take $L' = (1, 1, \ldots, 1)$.
- If the graph is not a tree, or if $\mathcal{V}_{\mathrm{u}}$ contains two or more points, or if $\overline{\mathfrak{D}}$ has a point with one coordinate zero:



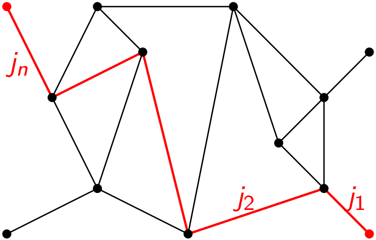
Two vertices in $\mathcal{V}_{\mathrm{u}}$.

Main result and applications

Application to wave propagation on networks

$\Longrightarrow$: (only for the case $L \in \mathrm{RI}$)

- Exponential stability for $L \in \mathrm{RI}$ $\iff$ exponential stability for every L.
- Take $L' = (1, 1, \ldots, 1)$.
- If the graph is not a tree, or if $\mathcal{V}_{\mathrm{u}}$ contains two or more points, or if $\overline{\mathfrak{D}}$ has a point with one coordinate zero:



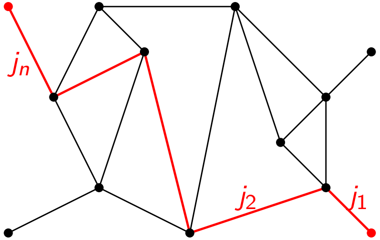
Two vertices in $\mathcal{V}_{\mathrm{u}}$.
$(j_1, j_2, \ldots, j_n)$: path

Main result and applications

Application to wave propagation on networks

$\Longrightarrow$: (only for the case $L \in \mathrm{RI}$)

- Exponential stability for $L \in \mathrm{RI}$ $\iff$ exponential stability for every L.
- Take $L' = (1, 1, \ldots, 1)$.
- If the graph is not a tree, or if $\mathcal{V}_{\mathrm{u}}$ contains two or more points, or if $\overline{\mathfrak{D}}$ has a point with one coordinate zero:



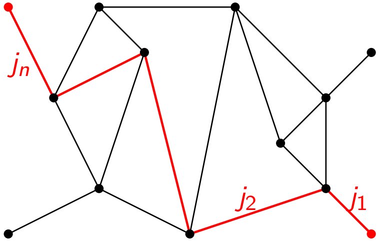
Two vertices in $\mathcal{V}_{\mathrm{u}}$.
$(j_1, j_2, \ldots, j_n)$: path
$u_{j_i}(t, x) = \pm \sin(2\pi t) \sin(2\pi x)$:
periodic solution

Main result and applications

Application to wave propagation on networks

$\Longrightarrow$: (only for the case $L \in \mathrm{RI}$)

- Exponential stability for $L \in \mathrm{RI} \iff$ exponential stability for every L.
- Take $L' = (1, 1, \ldots, 1)$.
- If the graph is not a tree, or if $\mathcal{V}_{\mathrm{u}}$ contains two or more points, or if $\overline{\mathfrak{D}}$ has a point with one coordinate zero:



Two vertices in $\mathcal{V}_{\mathrm{u}}$.
$(j_1, j_2, \ldots, j_n)$: path
$u_{j_i}(t, x) = \pm \sin(2\pi t) \sin(2\pi x)$: periodic solution
No exponential stability for L', then no exponential stability for L.

Proof of the main result

Explicit solution

$$\Sigma_\delta(L, A): \qquad u(t) = \sum_{j=1}^N A_j(t) u(t - L_j), \qquad t \geq 0$$

Lemma

Let $L \in (0, +\infty)^N$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, and $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$. The solution $u : [-L_{\max}, +\infty) \to \mathbb{C}^d$ of $\Sigma_\delta(L, A)$ is, for $t \geq 0$,

$$u(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \sum_{\substack{j \in [\![1, N]\!] \\ L \cdot \mathbf{n} - L_j \leq t}} \Xi_{\mathbf{n} - e_j, t}^{L, A} A_j(t - L \cdot \mathbf{n} + L_j) u_0(t - L \cdot \mathbf{n}),$$

where the matrices $\Xi_{\mathbf{n}, t}^{L, A}$ are defined inductively by

$$\Xi_{\mathbf{n}, t}^{L, A} = \sum_{\substack{k=1 \\ n_k \geq 1}}^N A_k(t) \Xi_{\mathbf{n} - e_k, t - L_k}^{L, A}, \qquad \Xi_{0, t}^{L, A} = \mathrm{Id}_d\,.$$

Proof of the main result
Explicit solution

$$\Sigma_\delta(L, A): \qquad u(t) = \sum_{j=1}^{N} A_j(t) u(t - L_j), \qquad t \geq 0$$

Lemma

Let $L \in (0, +\infty)^N$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, and $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$. The solution $u : [-L_{\max}, +\infty) \to \mathbb{C}^d$ of $\Sigma_\delta(L, A)$ is, for $t \geq 0$,

$$u(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \sum_{\substack{j \in [\![1, N]\!] \\ L \cdot \mathbf{n} - L_j \leq t}} \Xi_{\mathbf{n} - e_j, t}^{L, A} A_j(t - L \cdot \mathbf{n} + L_j) u_0(t - L \cdot \mathbf{n}),$$

where the matrices $\Xi_{\mathbf{n},t}^{L,A}$ are defined inductively by

$$\Xi_{\mathbf{n},t}^{L,A} = \sum_{\substack{k=1 \\ n_k \geq 1}}^{N} A_k(t) \Xi_{\mathbf{n} - e_k, t - L_k}^{L,A}, \qquad \Xi_{0,t}^{L,A} = \mathrm{Id}_d .$$

Proof of the main result

Explicit solution

$$\Sigma_\delta(L, A): \qquad u(t) = \sum_{j=1}^N A_j(t) u(t - L_j), \qquad t \geq 0$$

Lemma

Let $L \in (0, +\infty)^N$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, and $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$. The solution $u : [-L_{\max}, +\infty) \to \mathbb{C}^d$ of $\Sigma_\delta(L, A)$ is, for $t \geq 0$,

$$u(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \Theta_{\mathbf{n}, t}^{L, A} u_0(t - L \cdot \mathbf{n}),$$

where the matrices $\Xi_{\mathbf{n}, t}^{L, A}$ are defined inductively by

$$\Xi_{\mathbf{n}, t}^{L, A} = \sum_{\substack{k=1 \\ n_k \geq 1}}^N A_k(t) \Xi_{\mathbf{n} - e_k, t - L_k}^{L, A}, \qquad \Xi_{0, t}^{L, A} = \mathrm{Id}_d.$$

Proof of the main result
Exponential type

$\mathcal{A}$: set of uniformly locally bounded functions taking values in $\mathcal{M}_d(\mathbb{C})^N$.

Definition

$\Sigma_\delta(L, \mathcal{A})$ is of:

- *exponential type γ* in X_p^δ if $\forall \varepsilon > 0\ \exists K > 0$ s.t. $\forall A \in \mathcal{A}$, $\forall u_0 \in \mathsf{X}_p^\delta$, the solution u satisfies $\left\lVert u_t \right\rVert_{\mathsf{X}_p^\delta} \leq K e^{(\gamma+\varepsilon)t} \left\lVert u_0 \right\rVert_{\mathsf{X}_p^\delta}$;
- *Θ-exponential type γ* if $\forall \varepsilon > 0\ \exists K > 0$ s.t. $\forall A \in \mathcal{A}$, $\forall \mathbf{n} \in \mathbb{N}^N$, a.e. $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, one has $\left|\Theta_{\mathbf{n},t}^{L,A}\right| \leq K e^{(\gamma+\varepsilon)t}$;

Exponential stability: exponential type $\gamma < 0$.

Proof of the main result

Exponential type

Definition

The *maximal Lyapunov exponent of $\Sigma_\delta(L, \mathcal{A})$* in X_p^δ is

$$\lambda_p(L, \mathcal{A}) = \limsup_{t \to +\infty} \sup_{A \in \mathcal{A}} \sup_{\substack{u_0 \in \mathsf{X}_p^\delta \\ \left\lVert u_0 \right\rVert_{\mathsf{X}_p^\delta} = 1}} \frac{\ln \left\lVert u_t \right\rVert_{\mathsf{X}_p^\delta}}{t}.$$

Proof of the main result

Exponential type

Definition

The *maximal Lyapunov exponent of* $\Sigma_\delta(L, \mathcal{A})$ in X_p^δ is

$$\lambda_p(L, \mathcal{A}) = \limsup_{t \to +\infty} \sup_{A \in \mathcal{A}} \sup_{\substack{u_0 \in \mathsf{X}_p^\delta \\ \left\lVert u_0 \right\rVert_{\mathsf{X}_p^\delta} = 1}} \frac{\ln \left\lVert u_t \right\rVert_{\mathsf{X}_p^\delta}}{t}.$$

Proposition

$\lambda_p(L, \mathcal{A}) = \inf\{\gamma \in \mathbb{R} \mid \Sigma_\delta(L, \mathcal{A})$ *is of exponential type* γ *in* $\mathsf{X}_p^\delta\}$.
In particular,

$$\Sigma_\delta(L, \mathcal{A}) \text{ exponentially stable } \iff \lambda_p(L, \mathcal{A}) < 0.$$

Proof of the main result

Arbitrary time-dependence

Let $\mathcal{A}$ be uniformly locally bounded. Recall that

$$u(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \Theta_{\mathbf{n}, t}^{L, A} u_0(t - L \cdot \mathbf{n}), \qquad t \geq 0.$$

Proof of the main result

Arbitrary time-dependence

Let $\mathcal{A}$ be uniformly locally bounded. Recall that

$$u(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \Theta_{\mathbf{n},t}^{L,A} u_0(t - L \cdot \mathbf{n}), \qquad t \geq 0.$$

Theorem

- *Let $L \in (0, +\infty)^N$. If $\Sigma_\delta(L, \mathcal{A})$ is of Θ-exponential type γ then $\forall p \in [1, +\infty]$ it is of exponential type γ in X_p^δ.*
- *Let $L \in \mathrm{RI}$. If $\exists p \in [1, +\infty]$ s.t. $\Sigma_\delta(L, \mathcal{A})$ is of exponential type γ in X_p^δ, then it is of Θ-exponential type γ.*

Proof of the main result

Arbitrary switching

Case of $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$ with $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$ bounded.
An explicit formula for the coefficients $\Theta_{\mathbf{n},t}^{L,A}$ and $\Xi_{\mathbf{n},t}^{L,A}$ motivates the following definition.

Definition

$$\mu(L, \mathfrak{B}) = \limsup_{\left|\mathbf{n}\right|_1 \to +\infty} \sup_{\substack{B^r \in \mathfrak{B} \\ \text{for } r \in \mathcal{L}_{\mathbf{n}}(L)}} \left| \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{\left|\mathbf{n}\right|_1} B_{v_k}^{L_{v_1} + \dots + L_{v_{k-1}}} \right|^{\frac{1}{x}},$$

where $\mathcal{L}_{\mathbf{n}}(L) = \{L \cdot \mathbf{k} \mid \mathbf{k} \in \mathbb{N}^N,\ L \cdot \mathbf{k} < L \cdot \mathbf{n}\}$.

$V_{\mathbf{n}}$: set of all permutations of $(\underbrace{1, \dots, 1}_{n_1 \text{ times}}, \underbrace{2, \dots, 2}_{n_2 \text{ times}}, \dots, \underbrace{N, \dots, N}_{n_N \text{ times}})$.

Proof of the main result
Arbitrary switching

Theorem

- $\lambda_p(L, \mathcal{A}) = \ln \mu(L, \mathfrak{B})$;

Proof of the main result
Arbitrary switching

Theorem

- $\lambda_p(L, \mathcal{A}) = \ln \mu(L, \mathfrak{B})$;
- *for every* $L \in \mathrm{RI}$ *and* $L' \in (0, +\infty)^N$, $\lambda_p(L', \mathcal{A}) \leq m_1 \ln \mu(L, \mathfrak{B})$;
- *for every* $L, L' \in \mathrm{RI}$, $m_2 \lambda_p(L, \mathcal{A}) \leq \lambda_p(L', \mathcal{A}) \leq m_1 \lambda_p(L, \mathcal{A})$.

Here, $\{m_1, m_2\} = \left\{\min_{j \in [\![1, N]\!]} \frac{L_j}{L'_j}, \max_{j \in [\![1, N]\!]} \frac{L_j}{L'_j}\right\}$.

Proof of the main result

Arbitrary switching

Theorem

- $\lambda_p(L, \mathcal{A}) = \ln \mu(L, \mathfrak{B})$;
- *for every* $L \in \mathrm{RI}$ *and* $L' \in (0, +\infty)^N$, $\lambda_p(L', \mathcal{A}) \leq m_1 \ln \mu(L, \mathfrak{B})$;
- *for every* $L, L' \in \mathrm{RI}$, $m_2 \lambda_p(L, \mathcal{A}) \leq \lambda_p(L', \mathcal{A}) \leq m_1 \lambda_p(L, \mathcal{A})$.

Here, $\{m_1, m_2\} = \left\{\min_{j \in [\![1, N]\!]} \frac{L_j}{L_j'}, \max_{j \in [\![1, N]\!]} \frac{L_j}{L_j'}\right\}$.

This theorem implies immediately our main result.