Undecidability of the Spectral Gap

David Pérez–García

Joint work with Toby Cubitt (UCL) and Michael Wolf (TUM)

Short version: **Nature 528, 207-211 (10 December 2015)**

Long version (146 pages!): **arXiv:1502.04573**

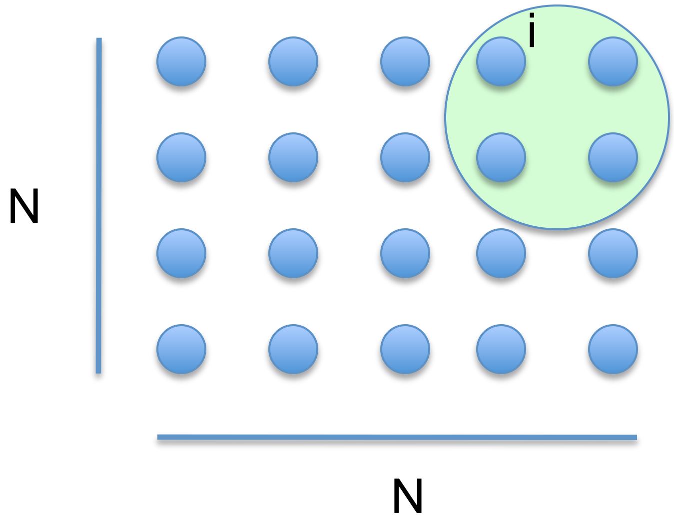
Outlook

- Basic definitions
- Our result
- Main ingredients in the proof
- Some consequences

BASIC DEFINITIONS

Spectral gap problem

Setup



Particles in a lattice

d-dimensional Hilbert space associated to each site.

Finite range translational invariant Hamiltonian

$$H = \sum_i h_i \otimes 1_{rest}$$

Spectral Gap: $\Delta_N = \lambda_1(N) - \lambda_0(N)$

Spectral Gap Problem:

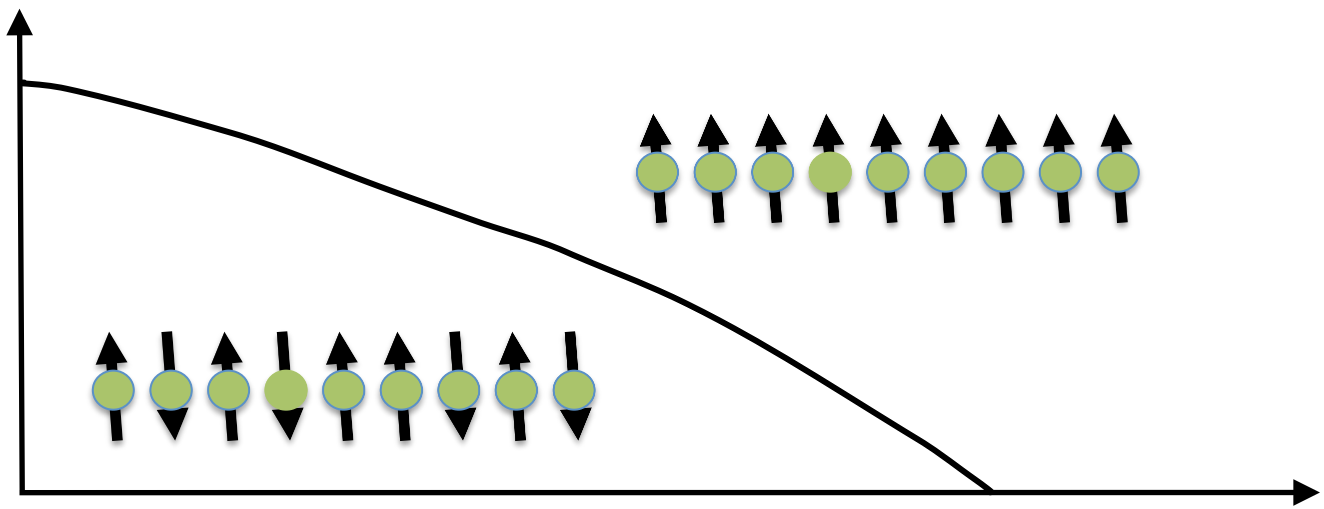
How does the spectral gap behave as N goes to infinity?

Does the system have gap? i.e is there a c>0 such that $\Delta_N > c$ for all N?

Where does it appear?

Spectral Gap in **condensed matter physics**:

- It defines the concept of quantum ***phase***, *phase transition, phase diagram, …*



Spectral Gap in **quantum information and computation**:

- It measures the efficiency in adiabatic quantum computation and quantum state engineering

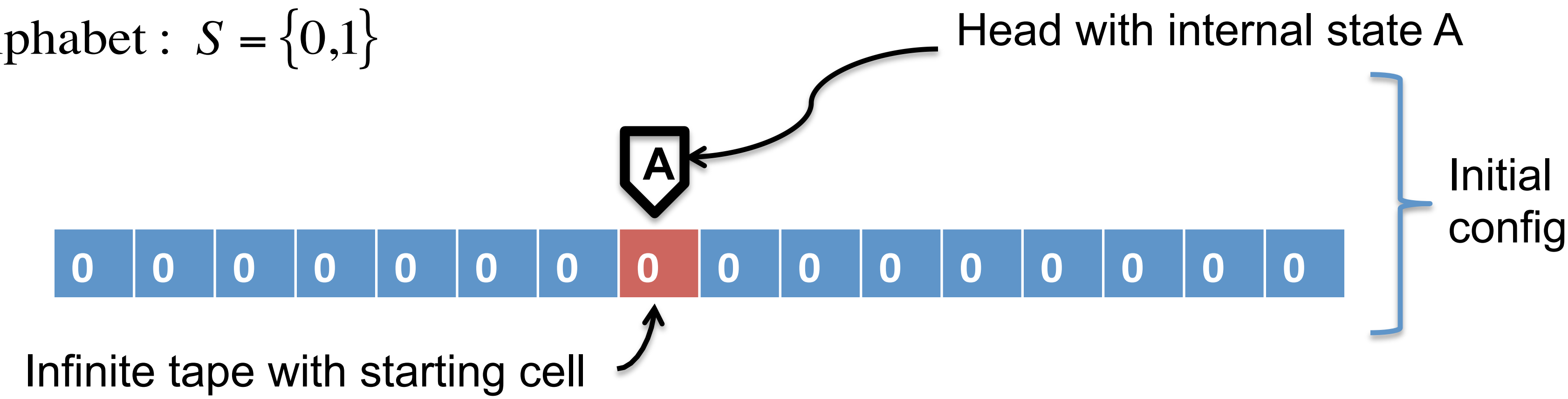
BASIC DEFINITIONS

Undecidability

Turing Machines

Finite number of internal states: $Q = \{A, B, C, \ldots\} \cup \{\text{halting state H}\}$

Finite alphabet: $S = \{0,1\}$



Instructions: $\delta : Q \times S \rightarrow Q \times \{L, R\} \times S$

Turing Machines $\Leftrightarrow$ Natural numbers

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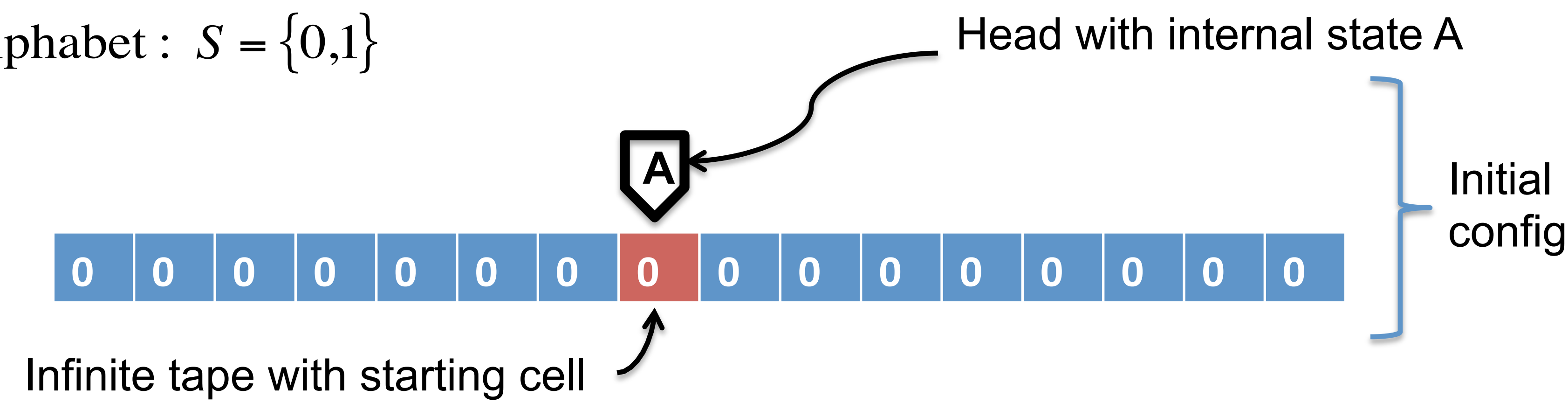
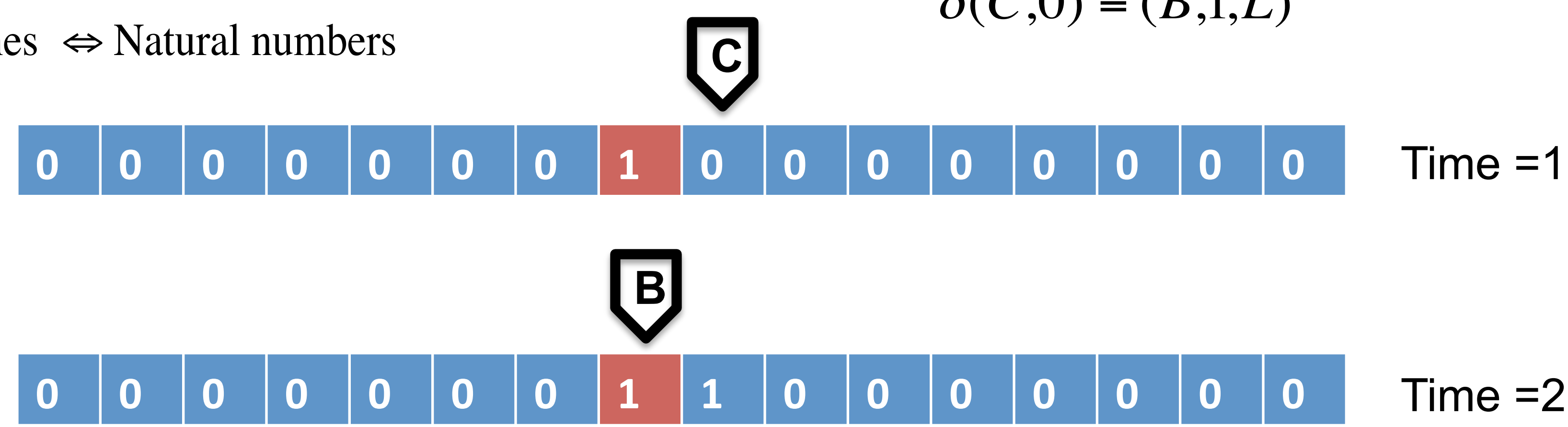


Instructions: $\delta : Q \times S \to Q \times \{L, R\} \times S$

E.g. $\delta(A,0) = (C,1,R)$

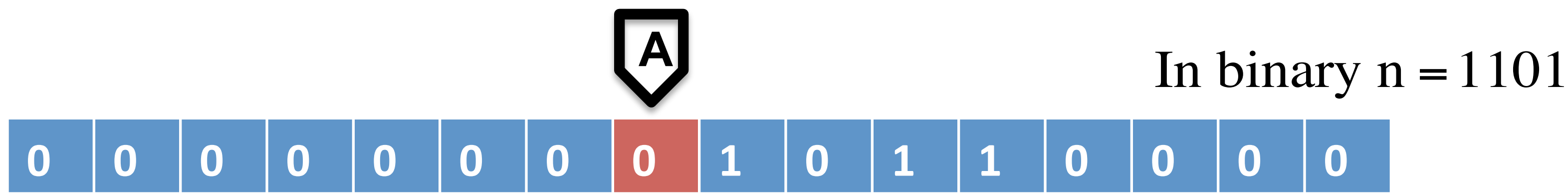
$\delta(C,0) = (B,1,L)$

Turing Machines $\Leftrightarrow$ Natural numbers



The halting problem of a TM

A TM **halts on input n** if it eventually enters the halting state
when the TM starts with the head in the starting cell and starting internal state, and with the tape initialized in n, written in binary just at the right of the starting cell.



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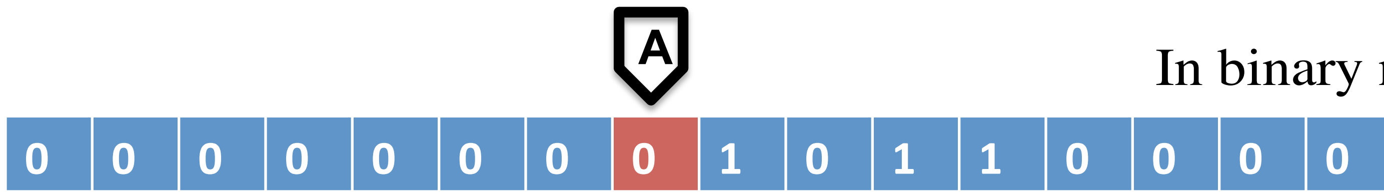

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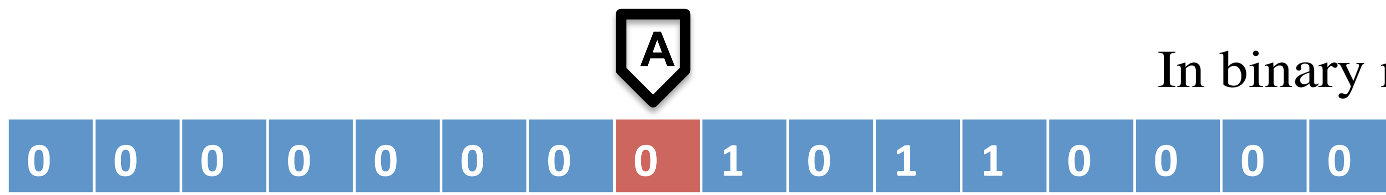

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Theorem (1936, Turing): The halting problem is *undecidable*. That is, there is no algorithm (= TM) that on input another TM (=n), decides whether it halts or not.

Theorem (1936, Turing): There exists a TM M, called *universal* (UTM), so that it halts on input n iff the TM=n halts on input 0.

Corollary: There is no algorithm that on input a natural number n, decides whether the UTM halts or not on input n.

Gödel axiomatic independence

There is a close connection between (Turing) undecidability and (Gödel) axiomatic independence.

Fix a decision problem and an axiom system A such that

(a) there is an algorithm that generates exactly the axioms of A
(b) there is an algorithm that, when fed an instance n of the decision problem, outputs a statement Y_n in the language of A such that
- if Y_n is provable in A, then the answer to n is YES, and
- if $\neg Y_n$ is provable in A, then the answer to n is NO.

Under these assumptions**, if the decision problem is (Turing) undecidable, then at least one of its instance statements Y_n is independent of A.**

We will state and prove (Turing) undecidability of the Spectral Gap Problem. There is a corresponding statement for (Gödel) independence.

OUR RESULT

Our result (informal statement)

Problem (Spectral Gap):

Input: nearest-neighbor interaction h

Output: decide if H has a gap or not.

Theorem:

The Spectral Gap problem is undecidable.

⇕

There is no algorithm that on input h decides it

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Theorem:

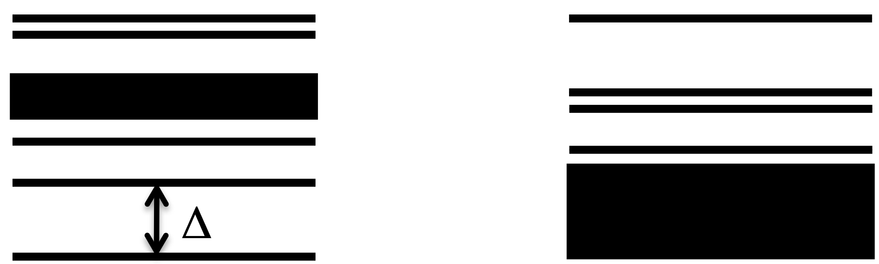
The Spectral Gap problem is undecidable.

There is no algorithm that on input h decides it

Even with the **promise** that

1.In the gapped case, the gap is larger than the norm of h for all system sizes and the ground state is unique and product ("all spins up").

2.In the gapless case, eigenvalues become dense in a region (of diverging size) just above the ground energy.



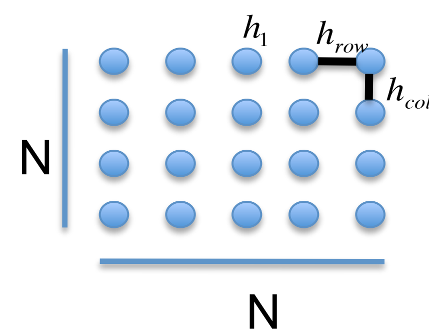
Gap   Gapless

Our result (formal statement)

Theorem:

the Hamiltonian given by

$$H(n)=\sum_{rows}\sum_{c}h_{row}^{(c,c+1)}(n)+\sum_{columns}\sum_{r}h_{row}^{(r,r+1)}(n)+\sum_{i}h_1^i(n)$$



Have the following properties

1. All terms $h_{col}(n), h_{row}(n), h_1(n)$ have operator norm bounded by 1.
2. If the UTM halts on input n, then, for all system sizes N, $0\in\mathrm{spec}(H)\subset\{0\}\cup[1,\infty)$ (that is, the gap is ≥ 1) and the unique eigenstate with eigenvalue 0 is $|\uparrow\uparrow\cdots\uparrow\uparrow\rangle$
3. If the UTM does not halt on input n, then the spectrum becomes dense in the real line.

Our result (formal statement)

Given the binary rep. of a natural number $n = n_1 n_2 \ldots n_{|n|}$, we call $\varphi = 0, n_{|n|} \ldots n_2 n_1 \in Q$

Theorem:

$$\begin{cases} h_{col}(n) = A \\ h_{row}(n) = B + \beta \left(e^{i\pi\varphi} C + e^{i\pi 2^{-|\varphi|}} D + h.c. \right) \\ h_1(n) = \alpha(n) P \end{cases}$$

Only dependency on n

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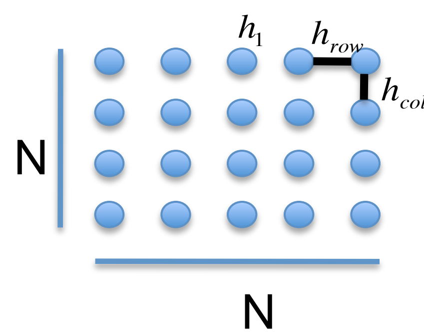

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Given the binary rep. of a natural number $n = n_1 n_2 \ldots n_{|n|}$, we call $\varphi = 0, n_{|n|} \ldots n_2 n_1 \in Q$

Theorem: We give explicitly a dimension d, matrices A,B,C,D, P and a rational number β as small as desired so that

- A, B are hermitian and with coefficients in $Z + \beta Z + \frac{\beta}{\sqrt{2}} Z$
- C, D, P have integer coefficients

And if we define, for each natural number n, $\begin{cases} h_{col}(n) = A \\ h_{row}(n) = B + \beta \left(e^{i\pi\varphi} C + e^{i\pi 2^{-|\varphi|}} D + h.c. \right) \\ h_1(n) = \alpha(n) P \end{cases}$

($\alpha(n) < \beta$ an algebraic computable number).

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Then the Hamiltonian given by

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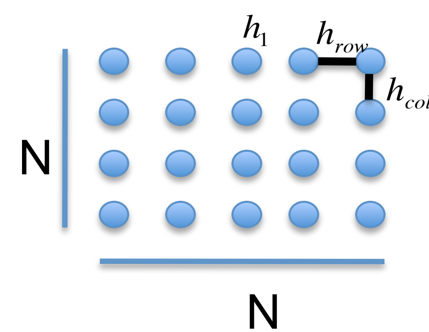

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INGREDIENTS OF THE PROOF

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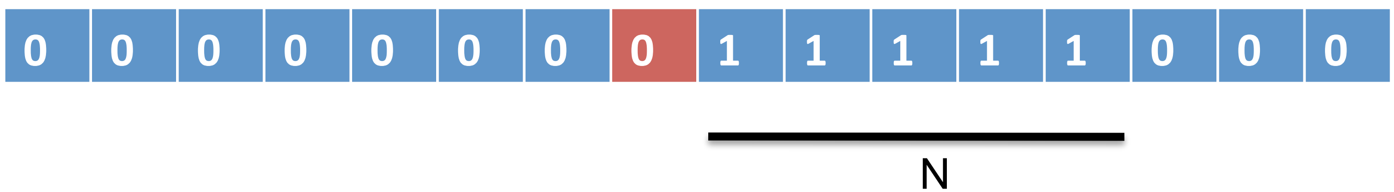

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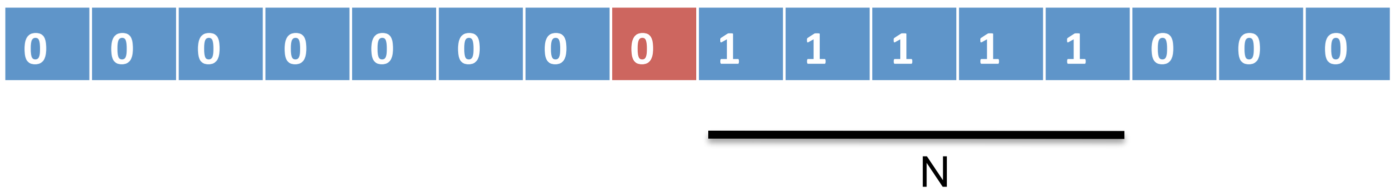

Outputs always n (in binary) and moreover it

1. Uses N+3 space
2. Takes time $O(\text{poly}(N)2^N)$
3. Never moves the head to the left of the starting cell.
4. Has deterministic head movement.

Observation: there cannot exist a similar CLASSICAL Turing Machine.

It is “simply” the Quantum Turing Machine associated to the quantum phase estimation algorithm.

Ingredients of the proof

Ingredient 2. (a Hamiltonian whose spectrum depends on whether the UTM on input n halts or not)

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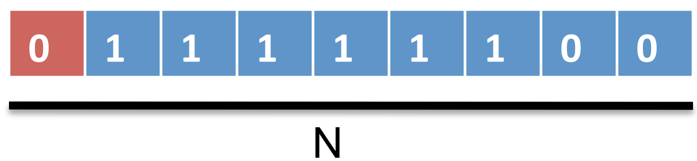
Construction, for each classical or quantum Turing Machine, of a 1D nearest neighbor interaction so that on a system of size N, it has a unique ground state which encodes (in superposition) the evolution of the TM for a time ξ^N on a tape of size N initialized in N-3 written in unary.



By adding a penalty term to the halting state, one gets that the Hamiltonian has energy 0 if the TM does not halt and energy ξ^{-N} if it halts.

By choosing the QTM of Ingredient 1 followed by the UTM we get a Hamiltonian which has different ground energies depending on the behavior of the UTM on input n.

The idea of a state that encodes the evolution of a Turing Machine goes back to Feynmann and Kitaev. Here we exploit recent constructions of Gottesman and Irani.

Ingredients of the proof

Ingredient 3. (Amplifying the spectral difference between halting and not-halting)

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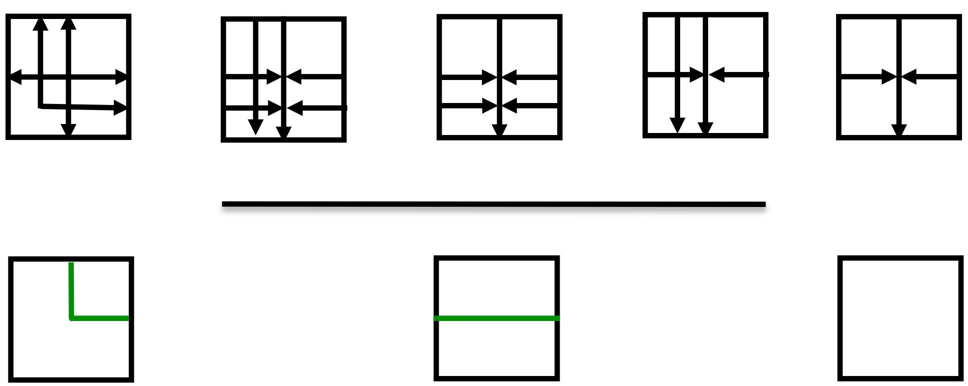
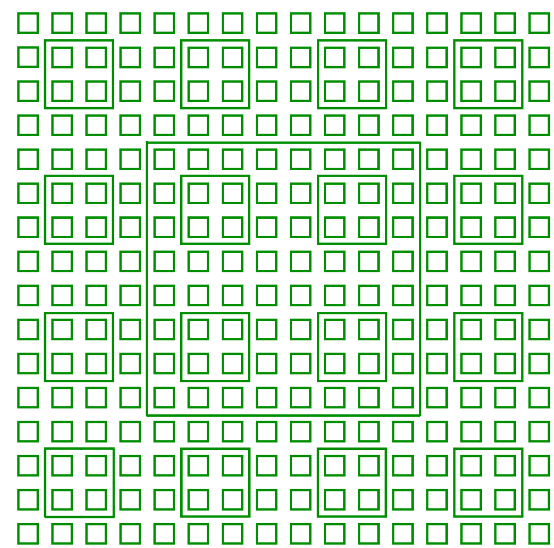
For that we rely on Robinson's aperiodic tiling which allows us to have infinitely many copies of the previous Hamiltonian in all possible system sizes.

We also use the trivial fact that we can encode any valid tiling in the ground state of a nearest neighbor Hamiltonian defined in terms of the tiles.

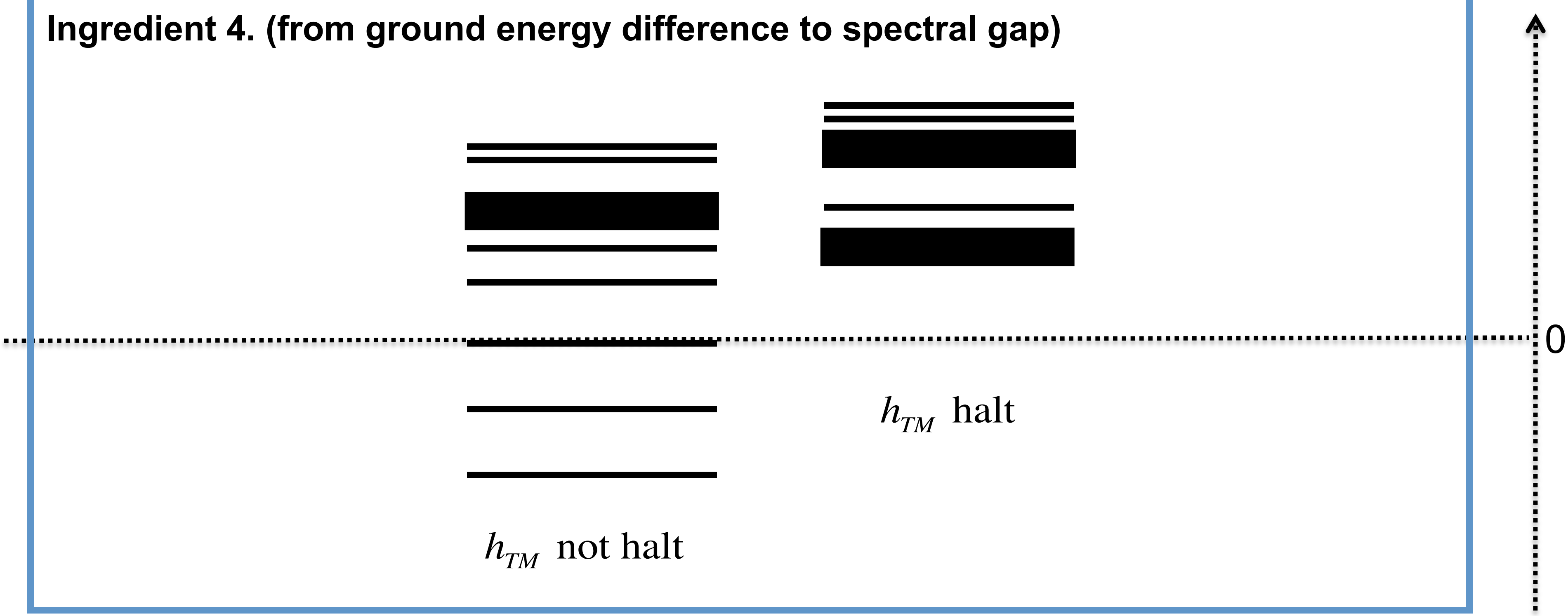




Ingredients of the proof

After Ingredient 3 we have a nearest neighbor interaction so that in a square region of size L we have a difference in energy of order L^2 depending on whether the UTM on input n halts or not. By adding a term $h = \alpha Id$ we can make one energy positive and the other negative.

Ingredient 4. (from ground energy difference to spectral gap)



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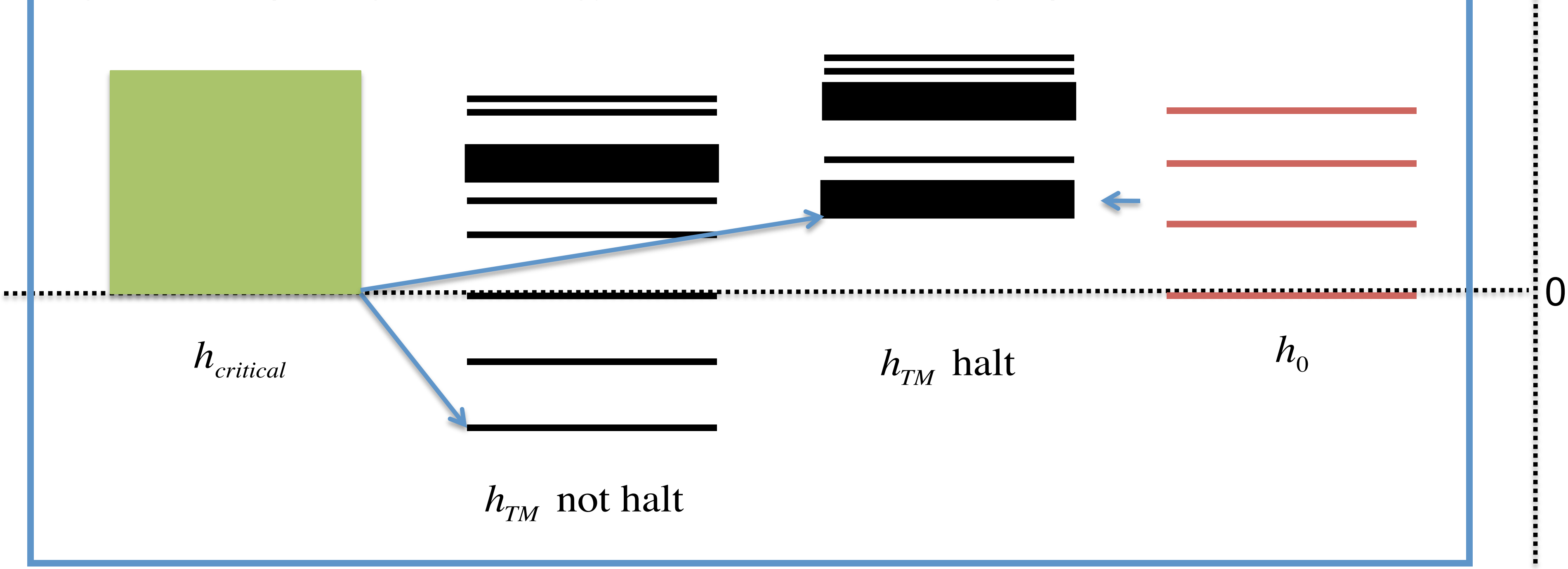

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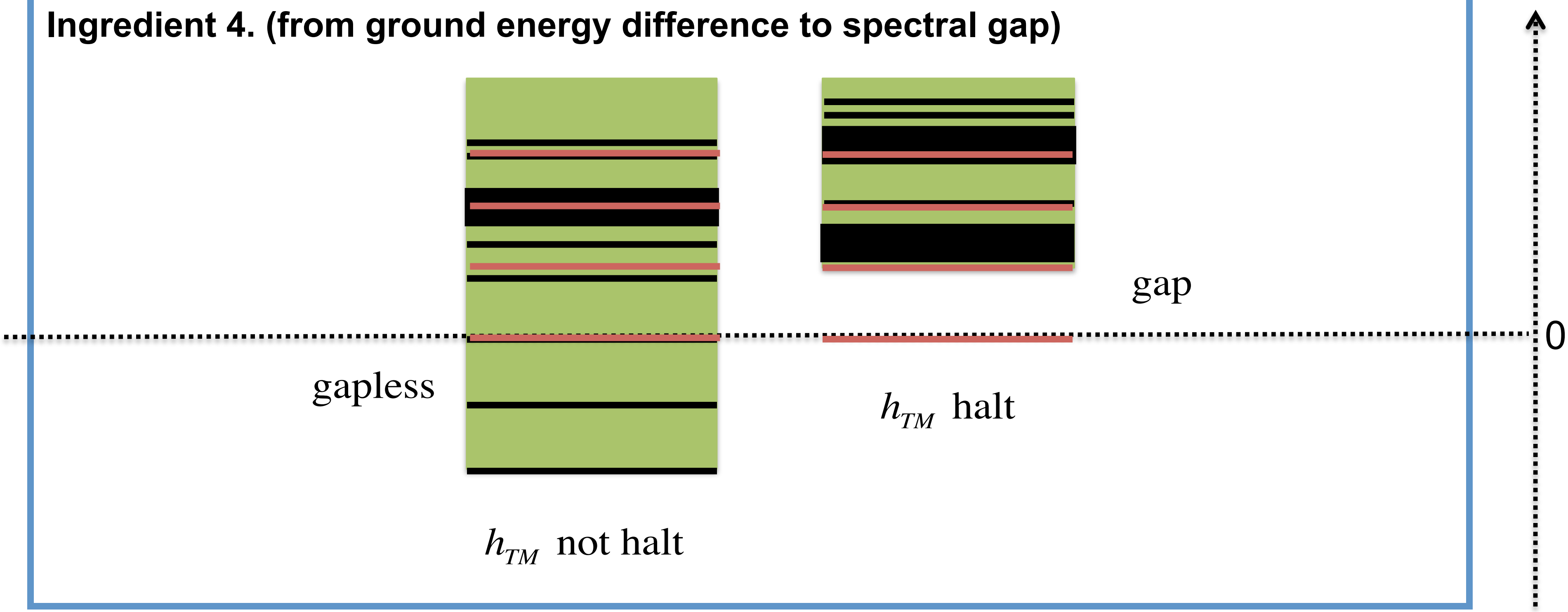
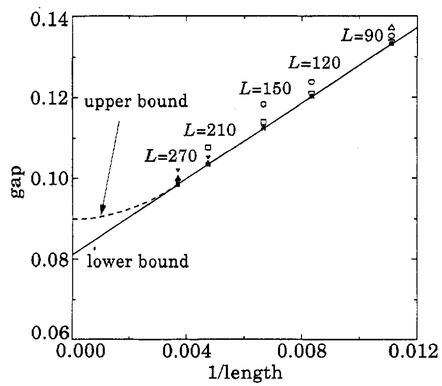


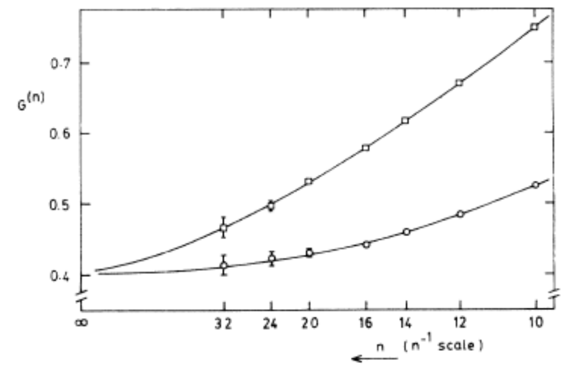
SOME CONSEQUENCES

Implications of the result

In condensed matter physics, most knowledge is numerical. Got by increasing the system size and extrapolating the result. Example: Haldane's conjecture.



Schollwöck and co. d=5, DMRG (1995)



Nichtingale and co. d=3, MC (1986)

Implications of the result

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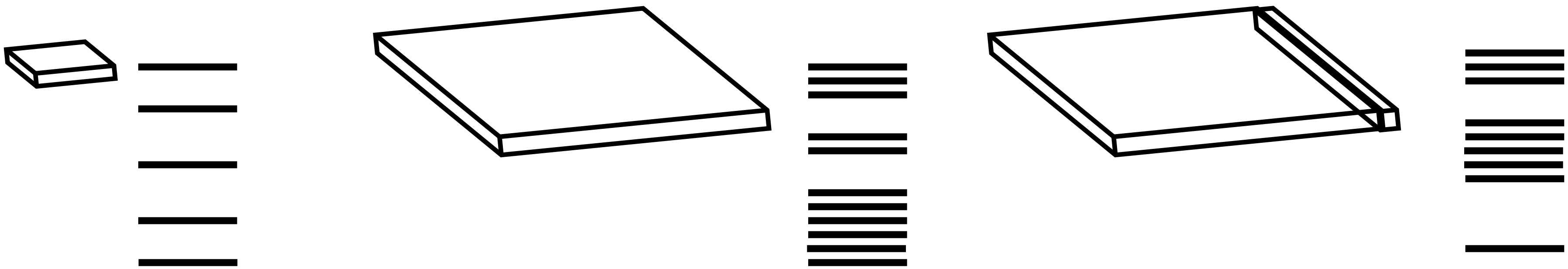
Our result implies that there exist systems that look gapless for all systems sizes $< L_c$
But a gap opens from L_c on. Moreover, this (uncomputable) critical size can be arbitrarily large.

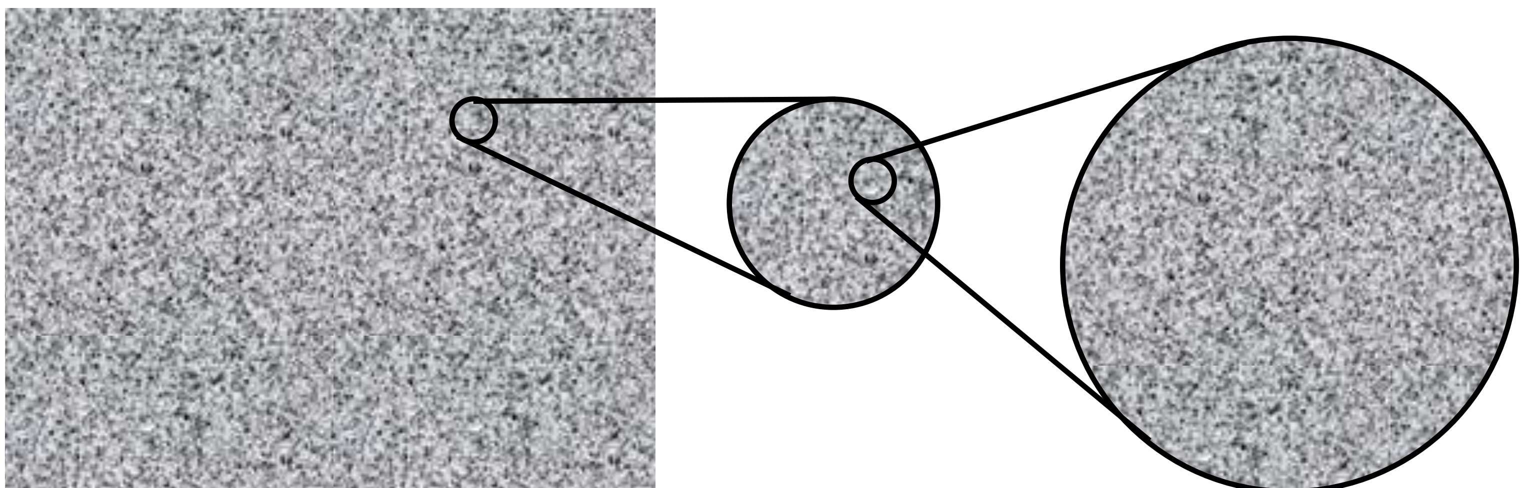


Local dimension	5	6	7	8	9	10
Critical size L (lattice $L \times L$)	15	84	420	2310	10^7	10^{35000}

(arXiv:1512.05687)

Implications of the result

There exist nearest neighbor interactions with uncomputable “fractal” phase diagrams.



THANKS FOR YOUR ATTENTION