Killing Spinors in Generalized Geometry

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Superstring Solutions, Supersymmetry and Geometry
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Based on joint work with R. Rubio and C. Tipler (arXiv:1503.07562), and C. Shahbazi (in progress).

Outline

- This talk is about the internal geometry of 4-dimensional $N = 1$ supersymmetric compactifications of heterotic supergravity.

- Classical ingredients of this geometry are complex manifolds (trivial canonical bundle), Kähler-Ricci-flat metrics ($\sim SU(3)$-holonomy), (stable) holomorphic bundles (prescribed Chern classes), ... all shaken, not stirred.

- In the presence of fluxes, the $SU(3)$-holonomy condition for the metric is replaced by: 1. a *balanced condition* for a hermitian metric **and** 2. the Bianchi identity, coupling the metric with the gauge field

$$dd^c\omega = \alpha'(\operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F).$$

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- Recent developments in the study of moduli of heterotic flux compactifications shows that generalized geometry is an essential ingredient of this new geometry (Melnikov-Sharpe '11, de la Ossa-Svanes '14, Anderson-Gray-Sharpe '14, GF-Rubio-Tipler '15).

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Part I: **Heterotic flux compactifications**

Heterotic supergravity

The (bosonic) field content of heterotic supergravity is a Lorentz metric $\hat{g}$ on ten-dimensional space-time M^{10}, dilaton $\phi \in C^\infty(M^{10})$, 3-form flux H and gauge field A with field strength F

$$\mathrm{Ric}^{\hat{g}}_{ij} + 2\nabla^{\hat{g}}_i \nabla^{\hat{g}}_j \phi - \frac{1}{4} H_{ikl} H_j^{kl} - \alpha' \operatorname{tr} F_{ik} F_j^k + \alpha' \operatorname{tr} R_{ik} R_j^k + O(\alpha'^2) = 0, \qquad \textbf{EM}$$

$$d^*(e^{-2\phi} H) + O(\alpha'^2) = 0,$$

$$d_A^*(e^{-2\phi} F) + \frac{e^{-2\phi}}{2} * (F \wedge * H) + O(\alpha'^2) = 0,$$

$$\nabla^- \epsilon + O(\alpha'^2) = 0, \qquad \textbf{SUSY}$$

$$(2d\phi - H) \cdot \epsilon + O(\alpha'^2) = 0,$$

$$F \cdot \epsilon + O(\alpha'^2) = 0$$

$$dH - \alpha' (\operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F) + O(\alpha'^2) = 0, \qquad \textbf{Bianchi}$$

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$$Hol(g) \subset SU(3)$$

- Combined with Yau's solution of the Calabi Conjecture in 1976 this led to important advances in heterotic model building and moduli space.
- Yau's result relies in an important separation of parameters in killing spinor equations (complex vs metric parameters): reduces the problem to the complex Monge-Ampere equation for the Kähler potential:

$$\log \det \partial_i \overline{\partial}_j \varphi = f$$

- This 'separation of variables' produces complex and metric moduli splitting in moduli space $\Rightarrow$ algebraic methods.

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Hull-Strominger Geometry

In 1986 Hull and Strominger characterized warped 4d compactifications of the heterotic string, with $N = 1$ supersymmetry and nonzero flux $H \neq 0$

$$M^{10} = \mathbb{R}^{1,3} \times M^6 \qquad \hat{g} = e^{2f} \cdot (\eta \oplus g) \qquad f \in C^\infty(M^{10})$$

where (M^6, g) compact Riemannian

$$P_G \to M^{10} \qquad G \subset SO(32), E_8 \times E_8$$

Imposing $N = 1$ supersymmetry, on M^6 we obtain:

- $SU(3)$-structure (ψ, ω) with metric g and almost complex structure $J\colon TM^6 \to TM^6$, $\psi \in \Lambda^{3,0}$, $\omega \in \Lambda^{1,1}$,
- $\phi = f \in C^\infty(M^6)$,
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d\Omega &= 0, \\
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d^*\omega - i(\overline{\partial} - \partial)\log\|\Omega\| &= 0, \\
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where $\Omega = e^{2\phi}\psi$, $H = i(\overline{\partial} - \partial)\omega$, $\phi = \frac{1}{8}\log\|\Omega\|$

Remark: The system is obtained taking Susy + Bianchi.

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Theorem (Fernandez-Ivanov-Ugarte-Villacampa '08-'10)

$$\mathrm{EM} + \mathrm{SUSY} + \mathrm{Bianchi} \Leftrightarrow \qquad (\Uparrow) \quad \textit{and} \quad g^{i\bar{j}} R_{i\bar{j}} = 0, \; R_{\bar{i}\bar{j}} = 0.$$

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Strominger System (ST)

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- First non-Kähler solutions of the Strominger system were found by Li and Yau in 2005, and in non-Kählerian complex manifolds by Fu and Yau in 2008.
- Since then, a long list of people has been studying the existence problem for the Strominger system ... still no analogue of Yau's Thm.

Remark: Crucial symmetry between curvature 2-form R of ∇ and F!

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Part II: **Moduli**

Moduli

- In recent work (arXiv:1503.07562, GF-Rubio-Tipler) it is proved that the Strominger system is an elliptic system of equations: an elliptic complex S^* of (multidegree, real) differential operators is constructed, so that $H^1(S^*)$ is the infinitesimal moduli.

- Previous work by de la Ossa-Svanes '14, Anderson-Gray-Sharpe '14 propose a (complex) vector space as infinitesimal moduli: the first Dolbeault cohomology $H^1(\mathcal{Q})$ of a holomorphic double extension $\mathcal{Q}$.

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The flux map

In an ideal situation (obstructions!), given a solution of the Strominger system we obtain an open patch in the moduli space

$$H^1(S^*) \supset U \subset \mathcal{M}_{ST}$$

Using the transgression formula for the Chern-Simons three-form: well defined map given by *flux charge*

$$\mathcal{M}_{ST} \supset U \to H^3(M, \mathbb{R})$$

Idea: (neglect ∇, take A abelian) given by ($A' = A + a$)

$$flux : (\Omega', A', \omega') \mapsto [d^c_{J'}\omega' - d^c\omega + 2\alpha' a \wedge F_{A'}]$$

Flux quantization: restricts to $flux^{-1}(H^3(M, \mathbb{Z}))$ (string theory)

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$$\mathcal{M}_{ST} \supset U \to H^3(M, \mathbb{R})$$

Idea: (neglect ∇, take A abelian) given by $(A' = A + a)$

$$flux : (\Omega', A', \omega') \mapsto [d^c_{J'}\omega' - d^c\omega + 2\alpha' a \wedge F_{A'}]$$

Flux quantization: restricts to $flux^{-1}(H^3(M, \mathbb{Z}))$ (string theory)

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Idea: $\mathit{flux} : (\Omega', A', \omega') = 0$ implies

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Moduli of Killing spinors

The local moduli space $\mathring{\mathcal{M}}_{ST}$ of solutions of the killing spinor equations is constructed using *generalized diffeomorphisms*. Restricting to the *inner generalized diffeomorphisms*, obtain and $H^2(X,\mathbb{R})$-bundle $\widehat{\mathcal{M}}_{ST}$ over $\mathring{\mathcal{M}}_{ST}$

$$\begin{tikzcd}
H^2(X,\mathbb{R}) \arrow[r] & \widehat{\mathcal{M}}_{ST} \arrow[d] & & \\
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Key: $\widehat{\mathcal{M}}_{ST}$ is an even dimensional manifold.

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The moduli $\widehat{\mathcal{M}}_{ST}$ carries a natural Kähler structure.

Evidence: there is a natural map $T_p\widehat{\mathcal{M}}_{ST} \to H^1(\mathcal{Q})$ (complex).

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Part III: **Generalized geometry**

Basics on generalized geometry

Given a smooth manifold M, $T \oplus T^*$ has canonical pairing and bracket

$$\langle X+\xi, X+\xi\rangle = X_i \xi_i, \qquad [X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X \eta - Y_i d\xi_{[ij]}$$

It has structure group $\mathrm{O}(n,n)$, and symmetries $\Omega^2_{cl} \rtimes \mathrm{Diff}(M)$, with B-fields acting by

$$X+\xi \mapsto X+\xi+X_i B_{[ij]}$$

Twisted version: an **exact** Courant algebroid

$$0 \to T^* \to E \to T \to 0.$$

is isomorphic to

$$(T+T^*, \langle,\rangle, [,]_H = [,] + X_i Y_j H_{[ijk]})$$

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A metric is a reduction of the frame bundle from $\mathrm{GL}(n)$ to $\mathrm{O}(n)$.

A generalized metric is a reduction from $\mathrm{O}(n,n)$ to $\mathrm{O}(n) \times \mathrm{O}(n) \cong$ a rank n positive-definite subbundle $V_+ \subset E$.

A generalized metric on an exact Courant algebroid is actually equivalent to a usual metric g together with two-form b-field,

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A connection on E is a differential operator

$$D : \Gamma(E) \to \Gamma(T^* \otimes E),$$

satisfying the Leibniz rule $\left(D_e f e' = \pi(e)(f)e' + f D_e e\right)$ and compatible with the metric $\left(\pi(e)\langle e', e''\rangle = \langle D_e e', e''\rangle + \langle e', D_e e''\rangle\right)$.

The space of connections is affine, modelled on $\Gamma(E^* \otimes \mathfrak{o}(E))$.

There is a well-defined torsion $T_D \in \Lambda^3 E$

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An example: the Gualtieri-Bismut connection

Let V_+ be a generalized metric (recall $V_\pm \cong T$). Define, by projecting, a map $C\colon E \to E$, $C(V_+) = V_-$, $C(V_-) = V_+$ and

$$D^B_e e' := [e_-, e'_+]_+ + [e_+, e'_-]_- + [Ce_-, e'_-]_- + [Ce_+, e'_+]_+,$$

The connection D_B preserves $V_\pm$ and has totally skew torsion

$$T_{D^B} = \pi_+^* H + \pi_-^* H.$$

Projecting to T, D^B encodes two metric connections with totally skew symmetric torsion

$$\nabla^\pm = \nabla^g \pm \frac{1}{2} g^{-1} H$$

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The canonical Levi-Civita connection

Given a metric V_+, we can define the Levi-Citiva connection

$$D^{LC} = D_B - \frac{1}{3} T_{D_B}.$$

which encodes **four different metric connections** on M:

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Remark: not unique torsion-free connection compatible with V_+! ($\sim$ conformal generalized geometry).

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Killing spinors and Calabi-Yau metrics

If M is spin and $\dim M = 2n$, by $V_+ \cong T$, we can talk about the spinor bundle $S_\pm(V_+)$, so that the restrictions $D_\pm^{LC} : V_+ \to V_+ \otimes (V_\pm)^*$, extend to a differential operator on spinors

$$D_\pm^{LC} : S_+(V_+) \to S_+(V_+) \otimes (V_\pm)^*,$$

with associated Dirac operator

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The **Killing spinor equations** for a spinor $\eta \in S_+(V_+)$ are

$$D_+^{LC}\eta = 0, \qquad \not{D}_-^{LC}\eta = 0.$$

Proposition (______,Rubio,Tipler)

If (V_+, η) is a solution to the Killing spinor eq. with $\eta \neq 0$ pure, then $H = 0$ and g is a metric with holonomy contained in $SU(n)$.

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If $\dim M = 2n$ and (V_+, η) is a solution to the Killing spinor eq. with $\eta \neq 0$ pure, then $H = 0$ and g is a metric with holonomy contained in $SU(n)$.

Proof: Reduces to Ivanov-Papadopoulos No-Go Theorem.

Remark: embedded in generalized geometry, deformations of Calabi-Yau metrics encode closed B-fields providing natural complexification Kähler cone.

$$V_+ = \{X + X_i g_{ij} + X_i b_{ij}\},$$

Killing spinors and Calabi-Yau metrics

The **Killing spinor equations** for a spinor $\eta \in S_+(V_+)$ are

$$D_+^{LC}\eta = 0, \qquad \not{D}_-^{LC}\eta = 0.$$

Proposition (_____,Rubio,Tipler)

If $\dim M = 2n$ and (V_+, η) is a solution to the Killing spinor eq. with $\eta \neq 0$ pure, then $H = 0$ and g is a metric with holonomy contained in $SU(n)$.

Proof: Reduces to Ivanov-Papadopoulos No-Go Theorem.

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Killing spinors and the Strominger system

Using the bundle of frames of the internal manifold M^6, jointly with gauge bundle, construct a principal G-bundle P over the internal manifold with vanishing first Pontryagin class.

$$p_1(P) = 0.$$

Choice of invariant class

$$[\hat{H}] \in H^3(P, \mathbb{R})^G.$$

determines an equivariant (twisted) exact Courant algebroid

$$0 \to T^*P \to \hat{E} \to TP \to 0,$$

that can be reduced to a non-exact Courant algebroid $E \to M$. As a vector bundle, $E \cong T + \operatorname{ad} P + T^*$, but not canonically.

Theorem (_____,Rubio,Tipler)

A solution to the Killing spinor eq. on E with $\eta \neq 0$ is equivalent to a solution of the Strominger system.

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The Courant algebroid

More explicitely: assume for simplicity that gauge bundle is $SU(r)$-bundle, with associated hermitian vector bundle V. Then

$$E = TM \oplus \operatorname{End} TM \oplus \operatorname{End}_{skw} V \oplus T^*M$$

Pairing: for $e = X + s + t + \xi$

$$\langle e, e \rangle = X_i \xi_i - \alpha' \operatorname{tr} ss + \alpha' \operatorname{tr} tt.$$

Recall: B-field transformations act on E by

$$Y \to Y + Y_i B_{[ij]}.$$

Bracket: the B-field part of $[e, \cdot]$ is

$$B_{ij} = -\partial_{[i}\xi_{j]} + Y_k (d^c\omega)_{[kij]} - 2\alpha' \operatorname{tr}(R_{ij}, s) + 2\alpha' \operatorname{tr}(F_{ij}, t)$$

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A unifying framework

Generalized geometry is a unifying framework for the theory of the Strominger system and the well-stablished theory for metrics with $SU(n)$-holonomy.

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\begin{aligned}
d\Omega &= 0,\\
F_{\bar{i}\bar{j}} = 0, \qquad R_{\bar{i}\bar{j}} &= 0,\\
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d^*\omega - i(\overline{\partial} - \partial)\log\|\Omega\| &= 0,\\
2i\partial\overline{\partial}\omega - \alpha'(\operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F) &= 0,
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Remark: for simplicity, I have assumed that $\|\Omega\| = 1$ (constant dilaton). General case requires *conformal generalized geometry*.

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In the next episode ...

Twisted heterotic supergravity

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- Present very interesting features:
  - compactifications to $6d$ with non-zero flux at zero order in α' ($\sim$ Maldacena-Nuñez)
  - very small moduli space
  - explicit examples in Hopf surfaces ($\sim$ WZW model).
  - toy model for analysis of standard compactifications.

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Thank you!