A. Santi, Marie-Curie of INdAM at the University of Edinburgh

Killing superalgebras and filtered deformations

(joint works with P. de Medeiros and J. Figueroa-O'Farrill)

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Plan of the talk:

– Killing superalgebras

– Filtered deformations

– Spencer cohomology and Killing spinors

– Classifications (so far): $d = 11$ and $d = 4$

– Outlook

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Plan of the talk:

- Killing superalgebras
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Second part (time-dependent):

- Manifolds M with a Poincaré superstructure $\mathcal{D}$ (nonintegr. distrib.)
- Tanaka's approach
- Classification of maximally homogeneous $(M, \mathcal{D})$

Prelude

to *"Killing superalgebras and filtered deformations"*

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Let (V, η) be a real d-dimensional Lorentzian vector space. In this talk $d = 11$ or $d = 4$. The double-cover $\mathrm{Spin}(V)$ of the connected component $\mathrm{SO}^o(V) \subset \mathrm{SO}(V)$ is identifiable with a group of invertible elements in the so-called Clifford algebra $\mathcal{C}\ell(V)$. Such algebra is the vector space $\mathcal{C}\ell(V) = \Lambda^\bullet V = \mathbb{R} \oplus V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^d V$ with a special product.

$\mathcal{C}\ell(V)$ always admits a representation as a suitable matrix algebra, for example $\mathcal{C}\ell(V) \simeq \mathbb{R}(32) \oplus \mathbb{R}(32)$ if $d = 11$ and $\mathcal{C}\ell(V) \simeq \mathbb{R}(4)$ if $d = 4$. Since $\mathrm{Spin}(V) \subset \mathcal{C}\ell(V)$, we have a representation of $\mathrm{Spin}(V)$ (and hence of $\mathfrak{so}(V) = Lie(\mathrm{Spin}(V))$) by means of matrices acting on $S = \mathbb{R}^{32}$ if $d = 11$ and $S = \mathbb{R}^4$ if $d = 4$, called *spinor representation*.

Since $\mathcal{C}\ell(V) = \Lambda^\bullet V$, we also have that polyvectors and forms on V correspond to matrices acting on S. The actions of such matrices are called *Clifford products between elements of $\Lambda^\bullet V$ and elements of S.*

Prelude

to *"Killing superalgebras and filtered deformations"*

Def. A *Lie superalgebra* is a real vector space of the form

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

endowed with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

- $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subset \mathfrak{g}_{\bar{0}}$, $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{1}}$, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{0}}$;
- for any homogeneous X, Y (i.e. with $X \in \mathfrak{g}_{\bar{i}}$, $Y \in \mathfrak{g}_{\bar{j}}$)

$$[X, Y] = -(-1)^{|X||Y|}[Y, X] \qquad \left(|X| = \text{ parity of } X = \begin{cases} 0 \\ 1 \end{cases} \right)$$

- for any homogeneous X, Y, Z

$$(-1)^{|Z||X|}[X, [Y, Z]] + (-1)^{|Y||Z|}[Z, [X, Y]] + (-1)^{|X||Y|}[Y, [Z, X]] = 0$$

The Poincaré superalgebra

Let (V, η) be fixed Lorentzian vector space and S the associated spinor $\mathfrak{so}(V)$-module. On S there is $\mathfrak{so}(V)$-invariant symplectic form $\langle -, - \rangle$ such that $\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle$ for all $v \in V$, $s_1, s_2 \in S$. The transpose of Clifford action $V \otimes S \to S$ gives a way to square spinors: a map $k : \odot^2 S \to V$ known as Dirac current:

$$\eta(k(s,s), v) = \langle v \cdot s, s \rangle \qquad v \in V\,,\ s \in S$$

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Def. The Poincaré superalgebra is the Lie superalgebra $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$ where

(i) $\mathfrak{p}_{\bar{0}} = \mathfrak{so}(V) \oplus V$;

(ii) $\mathfrak{p}_{\bar{1}} = S$;

(iii) the nonzero Lie brackets are:

$$[A, B] = AB - BA\,, \quad [A, s] = As\,, \quad [A, v] = Av\,, \quad [s, s] = k(s, s)\,,$$

for all $A, B \in \mathfrak{so}(V)$, $s \in S$, $v \in V$.

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Rem. The Poincaré superalgebra $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$ has also a $\mathbb{Z}$-grading of the form $\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$ where $\mathfrak{p}_{-2} = V$, $\mathfrak{p}_{-1} = S$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$. The grading is compatible with the Lie superalgebra structure in the sense that

(i) $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_{i+j}$ for all $i, j \in \mathbb{Z}$;

(ii) $\mathfrak{p}_{\bar{0}} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_0$ and $\mathfrak{p}_{\bar{1}} = \mathfrak{p}_{-1}$.

Killing superalgebras

Let (M, g, F) be Lorentzian mnfd (M, g), $\dim M = 11$, with closed $F \in \Omega^4(M)$ and endowed with a spin bundle $S(M) \longrightarrow M$ (the fiber $S(M)_x \simeq S = \mathbb{R}^{32}$). The bosonic equations of supergravity are two coupled PDE [Cremmer-Julia-Scherk '78]:

$$\left.\begin{aligned} \mathrm{Ric}(X,Y) &= -\tfrac{1}{2} g(i_X F, i_Y F) + \tfrac{1}{6} g(X,Y)|F|^2 \\ d * F &= \tfrac{1}{2} F \wedge F \end{aligned}\right\} \quad (*)$$

Supersymmetry transf. $\delta_\varepsilon \Psi = D\varepsilon + O(\Psi)$ of the gravitino Ψ gives the so-called superconnection on $S(M)$:

$$D_X \varepsilon = \nabla_X \varepsilon - \frac{1}{24}[X \cdot F - 3F \cdot X] \cdot \varepsilon \ ,$$

for all v.f. X and sections ε of $S(M)$.

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for all v.f. X and sections ε of $S(M)$.

Def. A supersymmetry of a solution of $(*)$ is a pair (X, ε) given by

(i) *a Killing vector field* for (g, F), *i.e.* a v.f. X s.t $\mathcal{L}_X g = \mathcal{L}_X F = 0$;

(ii) *a (generalized) Killing spinor, i.e.* a section ε of $S(M)$ s.t. $D\varepsilon = 0$.

Killing superalgebras

Thm[Figueroa-O'Farrill, Meessen, Philip '05] The $\mathbb{Z}_2$-graded v.s. $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ of supersymmetries of (M, g, F) has a natural structure of a Lie superalgebra, called the Killing superalgebra.

Ex. (M, g) Minkowski, $F = 0$, $D = \nabla$ then $\mathfrak{g}_{\bar{1}} \simeq S$, $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{so}(V) \oplus V$ and $\mathfrak{g} = \mathfrak{p}$.

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The Killing superalgebra is useful algebraic invariant of a supergravity bkgd:

- late '90s: first general check of AdS/CFT correspondence;
- early 2000s: it contracts under Penrose limit;
- mid 2000s: homogeneity conjecture by Meessen, i.e. if $\dim(\mathfrak{g}_{\bar{1}}) > \frac{1}{2}\dim S = 16$ then bkgd is locally homogeneous;
- more generally it is useful in construction of bkgds with prescribed infinitesimal automorphisms.

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Thm[Figueroa-O'Farrill, Hustler '12] If $\dim(\mathfrak{g}_{\bar{1}}) > 16$ then bkgd is loc. hom.

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Thm[Figueroa-O'Farrill, Hustler '12] If $\dim(\mathfrak{g}_{\bar{1}}) > 16$ then bkgd is loc. hom.

Thm[Figueroa-O'Farrill, —] Any Killing superalgebra is a filtered deformation of a subalgebra of $\mathfrak{p}$.

Filtered deformations are studied in mathematics and are related e.g. to homogeneous spaces, G-structures, the classification of the simple infinite dimensional Lie (super)algebras and also parabolic geometries, etc. What does this mean practically in our case?

Filtered deformations

Let's go back to the Poincaré superalgebra $\mathfrak{p}$: it has a compatible $\mathbb{Z}$-grading $\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0 = V \oplus S \oplus \mathfrak{so}(V)$ and we shall be interested in graded subalgebras $\mathfrak{a} \subset \mathfrak{p}$, i.e.

$$\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 = V' \oplus S' \oplus \mathfrak{h} \ ,$$

where $V' \subset V$, $S' \subset S$ and $\mathfrak{h} \subset \mathfrak{so}(V)$. If $\dim S' > 16$ then $V' = V$ (this is the algebraic fact underlying homogeneity thm). The Lie brackets of $\mathfrak{a}$ are:

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$$\begin{aligned} [A,B] &= AB - BA \\ [A,v] &= Av \\ [A,s] &= As \\ [s,s] &= \kappa(s,s) \\ [v,s] &= 0 \\ [v,w] &= 0 \end{aligned}$$

$A, B \in \mathfrak{h}$, $s \in S'$, $v, w \in V'$.

Filtered deformations

There is also a natural filtration $\mathfrak{a}^\bullet$ on $\mathfrak{a}$, i.e.

$$\mathfrak{a} = \mathfrak{a}^{-2} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^{-1} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^{0} = \mathfrak{a}_0 \supset \mathfrak{a}^{1} = 0 \ .$$

Def. A filtered deformation of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ with same underlying vector space as $\mathfrak{a}$ and a new Lie bracket $[-,-]$ which satisfies:

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Def. A filtered deformation of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ with same underlying vector space as $\mathfrak{a}$ and a new Lie bracket $[-,-]$ which satisfies:

(i) $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j} \oplus \mathfrak{a}_{i+j+1} \oplus \cdots$,

(ii) components of $[-,-]$ of zero degree coincide with original bracket.

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Def. A filtered deformation of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ with same underlying vector space as $\mathfrak{a}$ and a new Lie bracket $[-,-]$ which satisfies:

$$\begin{aligned}
[A,B] &= AB - BA \\
[A,v] &= Av + t\tau(A,v) \\
[A,s] &= As \\
[s,s] &= \kappa(s,s) + t\gamma(s,s) \\
[v,s] &= t\beta(v,s) \\
[v,w] &= t\alpha(v,w) + t^2\delta(v,w)
\end{aligned}$$

for some maps $\tau : \mathfrak{h} \otimes V' \to \mathfrak{h}$, $\gamma : \odot^2 S' \to \mathfrak{h}$, $\beta : V' \otimes S' \to S'$, $\alpha : \Lambda^2 V' \to V'$ and $\delta : \Lambda^2 V' \to \mathfrak{h}$ subject to the Jacobi identities for all values of the parameter t, which has been introduced to keep track of the filtration degree.

Killing superalgebras are filtered deformations

Thm[Figueroa-O'Farrill, —] Any Killing superalgebra $\mathfrak{g}$ is a filtered deformation of a graded subalgebra $\mathfrak{a}$ of the Poincaré superalgebra $\mathfrak{p}$.

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The proof is simple. First let's start with a classical analogy [Kostant, '55]. Let (M,g) be a pseudo-Riemannian mnfd with Lie algebra $\mathfrak{g}_{\bar{0}}$ of Killing vectors. On the bundle $\mathscr{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM)$ we have a connection D defined by

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A - R(X,\xi) \end{pmatrix}$$

for $(\xi, A) \in \Gamma(\mathscr{E}_{\bar{0}})$, where $R : \Lambda^2 TM \to \mathfrak{so}(TM)$ is Riemann curvature. A section (ξ, A) is parallel precisely if ξ is Killing vector. This allows to localise at any $x \in M$. We introduce the notation $(V,\eta) = (T_x M, g_x)$, $\mathfrak{so}(V) = \mathfrak{so}(T_x M)$ so that the fiber of $\mathscr{E}_{\bar{0}}$ at x is $\mathfrak{p}_{\bar{0}} = V \oplus \mathfrak{so}(V)$. In particular $\mathfrak{g}_{\bar{0}}$ is identified with a (not yet $\mathbb{Z}$-graded!) subspace of $\mathfrak{p}_{\bar{0}}$ and for all $(\xi, A_\xi), (\zeta, A_\zeta) \in \mathfrak{g}_{\bar{0}}$ we have the bracket:

$$[(\xi, A_\xi), (\zeta, A_\zeta)] = (A_\xi \zeta - A_\zeta \xi, [A_\xi, A_\zeta] + R(\xi,\zeta))$$

Killing superalgebras are filtered deformations

Let $\mathrm{ev}_x^{\bar 0} : \mathfrak{g}_{\bar 0} \longrightarrow V$ be the composition of evaluation at x with projection onto $V = T_x M$ and set $\mathrm{Im}(\mathrm{ev}_x^{\bar 0}) = V' \subseteq V$. Let also $\mathfrak{h} = \mathrm{Ker}(\mathrm{ev}_x^{\bar 0})$: it consists of elements of $\mathfrak{g}_{\bar 0}$ of the form $(0, A) \in V \oplus \mathfrak{so}(V)$ at x and defines a subspace of $\mathfrak{so}(V)$. There is a short exact sequence of v.s.

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g}_{\bar 0} \xrightarrow{\mathrm{ev}_x^{\bar 0}} V' \longrightarrow 0$$

and hence a (non-canonical) v.s. isomorphism $\mathfrak{g}_{\bar 0} \cong \mathfrak{a}_{\bar 0} := V' \oplus \mathfrak{h}$. Geometrically this amounts to choosing a splitting i.e. for every $v \in V'$ a Killing v.f. $\xi \in \mathfrak{g}_{\bar 0}$ with $\xi(x) = v$. Translating the Lie bracket of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{a}_{\bar 0}$ shows that $\mathfrak{g}_{\bar 0}$ is a filt. def. of $\mathfrak{a}_{\bar 0} = V' \oplus \mathfrak{h} \subset \mathfrak{p}_{\bar 0} = V \oplus \mathfrak{so}(V)$. Clearly Riemann curvature always contributes to components of the Lie bracket of positive filtration degree, independently of the choice of the splitting.

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We now superize this construction.

Killing superalgebras are filtered deformations

Let $\mathscr{E} = \mathscr{E}_{\bar{0}} \oplus \mathscr{E}_{\bar{1}}$ be the vector bundle where $\mathscr{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM)$ and $\mathscr{E}_{\bar{1}} = S(M)$. On $\mathscr{E}$ we have a connection D defined on $\mathscr{E}_{\bar{0}}$ as before and by the superconnection on $\mathscr{E}_{\bar{1}}$. Then the Killing superalgebra $\mathfrak{g}$ is given by

$$\mathfrak{g}_{\bar{1}} = \{\varepsilon \in \Gamma(\mathscr{E}_{\bar{1}}) | D\varepsilon = 0\} \ ,$$

$$\mathfrak{g}_{\bar{0}} = \{(\xi, A) \in \Gamma(\mathscr{E}_{\bar{0}}) | D(\xi, A) = 0 \quad \text{and} \quad \nabla_\xi F + A \cdot F = 0\} \ ,$$

where $A \cdot F$ is the natural action of $\mathfrak{so}(TM)$ on 4-forms. Again an element of $\mathfrak{g}$ is determined by the value at a point $x \in M$ of the corresponding parallel section of $\mathscr{E}$ and $\mathfrak{g}$ defines a subspace of $T_xM \oplus S(M)_x \oplus \mathfrak{so}(T_xM)$. We introduce the notation

$$(V, \eta) = (T_xM, g_x) \ , \qquad S = S(M)_x \ , \qquad \mathfrak{so}(V) = \mathfrak{so}(T_xM) \ ,$$

and identify $\mathfrak{g}$ with a (not yet $\mathbb{Z}$-graded) subspace of $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$. Again $\mathfrak{g}$ is not, in general, a Lie subalgebra of the Poincaré superalgebra $\mathfrak{p}$.

Killing superalgebras are filtered deformations

The Lie brackets of $\mathfrak{g}$ are given as follows. If $(\xi, A_\xi), (\zeta, A_\zeta) \in \mathfrak{g}_{\bar{0}}$ then as before

$$[(\xi, A_\xi), (\zeta, A_\zeta)] = (A_\xi \zeta - A_\zeta \xi, [A_\xi, A_\zeta] + R(\xi, \zeta)) \tag{1}$$

whereas the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is given by the so-called spinorial Lie derivative of Lichnerowicz and Kosmann

$$\begin{aligned} [(\xi, A_\xi), \varepsilon] &= \nabla_\xi \varepsilon + A_\xi \cdot \varepsilon \\ &= \frac{1}{24}(\xi \cdot F - 3F \cdot \xi) \cdot \varepsilon + A_\xi \cdot \varepsilon \end{aligned} \tag{2}$$

where $A_\xi \cdot \varepsilon$ is spinor representation of $\mathfrak{so}(TM)$ on $S(M)$ and we used $D\varepsilon = 0$. Lastly the Lie square of a Killing spinor is its Dirac current, which belongs to $\mathfrak{g}_{\bar{0}}$:

$$\begin{aligned} [\varepsilon, \varepsilon] &= (\kappa(\varepsilon, \varepsilon), -\nabla \kappa(\varepsilon, \varepsilon)) \\ &= (\kappa(\varepsilon, \varepsilon), \gamma^F(\varepsilon, \varepsilon)) \end{aligned} \tag{3}$$

where last identity follows from a simple computation and $\gamma^F(\varepsilon, \varepsilon) \in \mathfrak{so}(TM)$ is defined such that for all $X \in TM$,

$$\gamma^F(\varepsilon, \varepsilon)(X) = -\frac{1}{12}\kappa\left((X \cdot F - 3F \cdot X) \cdot \varepsilon, \varepsilon\right) .$$

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Let $\mathrm{ev}_x^{\bar{0}} : \mathfrak{g}_{\bar{0}} \longrightarrow V$ be as before and $\mathrm{ev}_x^{\bar{1}} : \mathfrak{g}_{\bar{1}} \longrightarrow S$ the evaluation at x in $\mathfrak{g}_{\bar{1}}$. Set $\mathrm{Im}(\mathrm{ev}_x^{\bar{0}}) = V' \subseteq V$, $\mathfrak{h} = \mathrm{Ker}(\mathrm{ev}_x^{\bar{0}})$ and $\mathrm{Im}(\mathrm{ev}_x^{\bar{1}}) = S' \subseteq S$. Choosing a splitting of

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g}_{\bar{0}} \xrightarrow{\mathrm{ev}_x^{\bar{0}}} V' \longrightarrow 0$$

we see that $\mathfrak{g} \cong \mathfrak{a} := V' \oplus S' \oplus \mathfrak{h} \subset \mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ as a v.s. The result follows then from translating brackets (1), (2) and (3) of $\mathfrak{g}$ on $\mathfrak{a}$ under this identification.

Killing superalgebras are filtered deformations

Let $\mathrm{ev}_x^{\bar 0} : \mathfrak{g}_{\bar 0} \longrightarrow V$ be as before and $\mathrm{ev}_x^{\bar 1} : \mathfrak{g}_{\bar 1} \longrightarrow S$ the evaluation at x in $\mathfrak{g}_{\bar 1}$. Set $\mathrm{Im}(\mathrm{ev}_x^{\bar 0}) = V' \subseteq V$, $\mathfrak{h} = \mathrm{Ker}(\mathrm{ev}_x^{\bar 0})$ and $\mathrm{Im}(\mathrm{ev}_x^{\bar 1}) = S' \subseteq S$. Choosing a splitting of

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Rem. There are terms in (1), (2) and (3) which always contribute to components of Lie bracket of positive filtration degree, independently of the choice of splitting:

$$[(\xi, A_\xi), (\zeta, A_\zeta)] = (A_\xi \zeta - A_\zeta \xi, [A_\xi, A_\zeta] + R(\xi, \zeta)) \tag{1}$$

$$[(\xi, A_\xi), \varepsilon] = \frac{1}{24}(\xi \cdot F - 3F \cdot \xi) \cdot \varepsilon + A_\xi \cdot \varepsilon \tag{2}$$

$$[\varepsilon, \varepsilon] = (\kappa(\varepsilon, \varepsilon), \gamma^F(\varepsilon, \varepsilon)) \tag{3}$$

where $\gamma^F(\varepsilon, \varepsilon)(X) = -\frac{1}{12}\kappa((X \cdot F - 3F \cdot X) \cdot \varepsilon, \varepsilon)$.

Main Motivations and Questions

Motivations.

- The classification of highly supersymmetric bkgds of $d = 11$ supergravity is still largely open. There is classification of maximally supersymmetric ones [Figueroa-O'Farrill, Papadopoulos '03] and non-existence results for 31 and 30 supersymmetries [Gran, Gutowski, Papadopoulos, Roest '07&'10]. There are examples with 26 [Michelson '02], $24, 22, 20, 18$ [Gauntlett, Hull '02].
- Idea: instead of studying directly bkgds, we set to study filt. def.
- The problem of classifying filt. def. of $\mathbb{Z}$-graded Lie (super)algebras is well-defined mathematically [Sternberg, Guillemin mid '60s] and also subject of recent investigations [Kac, Cantarini, Cheng 2000s] and [Kruglikov, The '14].

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Observations/Questions.

– Is every filt. def. a Killing superalgebra? There might be counterexample..

– In supergravity, filtered deformations are further constrained.

– On the other hand, as we will shortly see at least in the case $d = 4$, filt. def. are interesting also in relation with the problem of determining the geometries which admit rigidly supersymmetric field theories.

Spencer cohomology

Deformations of an algebraic structure are governed (at least to first order) by a cohomology theory. In this case, this is Spencer cohomology, a refinement by degree of the usual Chevalley-Eilenberg cohomology of a Lie (super)algebra and its adjoint representation to the case of $\mathbb{Z}$-graded Lie (super)algebras $\mathfrak{a} = \oplus \mathfrak{a}_j$.

Let us start with the Poincaré superalgebra $\mathfrak{a} = \mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$ in $d = 4$ or $d = 11$, where $\mathfrak{p}_{-2} = V$, $\mathfrak{p}_{-1} = S$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$. Let $\mathfrak{p}_- = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1}$ be the negatively graded part of $\mathfrak{p}$. The cochains of the Spencer complex of $\mathfrak{p}$ are linear maps $\Lambda^\bullet \mathfrak{p}_- \to \mathfrak{p}$, where $\Lambda^\bullet$ is meant here in the super sense. We extend degree of $\mathfrak{p}$ to the space of q-cochains $C^q(\mathfrak{p}_-, \mathfrak{p}) = \mathfrak{p} \otimes \Lambda^q \mathfrak{p}_-^*$ for all q by declaring that $\mathfrak{p}_j^*$ has degree $-j$. It follows that $C^q(\mathfrak{p}_-, \mathfrak{p}) = \oplus_{p \in \mathbb{Z}} C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$ where $C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$ is space of q-cochains with $\deg = p$. The spaces in the complexes of even cochains of small degree $\deg = p$ are given in following Table (although for $p = 4$ there are cochains also for $q = 5, 6$ which are omitted).

Even q-cochains of small degree

	q				
p	0	1	2	3	4
0	$\mathfrak{so}(V)$	$S \to S$ $V \to V$	$\odot^2 S \to V$		
2		$V \to \mathfrak{so}(V)$	$\Lambda^2 V \to V$ $V \otimes S \to S$ $\odot^2 S \to \mathfrak{so}(V)$	$\odot^3 S \to S$ $\odot^2 S \otimes V \to V$	$\odot^4 S \to V$
4			$\Lambda^2 V \to \mathfrak{so}(V)$	$\odot^2 S \otimes V \to \mathfrak{so}(V)$ $\Lambda^2 V \otimes S \to S$ $\Lambda^3 V \to V$	$\odot^4 S \to \mathfrak{so}(V)$ $\odot^3 S \otimes V \to S$

Here $C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$ is the space of q-cochains with $\deg = p$. The Spencer differential

$$\partial : C^{p,q}(\mathfrak{p}_-, \mathfrak{p}) \to C^{p,q+1}(\mathfrak{p}_-, \mathfrak{p})$$

is the Chevalley–Eilenberg differential for the Lie superalgebra $\mathfrak{p}_-$ relative to its module $\mathfrak{p}$ with respect to adjoint action. We are interested in $q = 2$, $p = 2, 4$.

	q				
p	0	1	2	3	4
0	$\mathfrak{so}(V)$	$S \to S$ $V \to V$	$\odot^2 S \to V$		
2		$V \to \mathfrak{so}(V)$	$\Lambda^2 V \to V$ $V \otimes S \to S$ $\odot^2 S \to \mathfrak{so}(V)$	$\odot^3 S \to S$ $\odot^2 S \otimes V \to V$	$\odot^4 S \to V$
4			$\Lambda^2 V \to \mathfrak{so}(V)$	$\odot^2 S \otimes V \to \mathfrak{so}(V)$ $\Lambda^2 V \otimes S \to S$ $\Lambda^3 V \to V$	$\odot^4 S \to \mathfrak{so}(V)$ $\odot^3 S \otimes V \to S$

For example if $q = 1, 2$ the Spencer differential is explicitly given by:

$$\partial : C^{p,1}(\mathfrak{p}_-, \mathfrak{p}) \to C^{p,2}(\mathfrak{p}_-, \mathfrak{p})$$

$$\partial\zeta(X,Y) = [X, \zeta(Y)] - (-1)^{|X||Y|}[Y, \zeta(X)] - \zeta([X,Y]) \ , \tag{1}$$

$$\partial : C^{p,2}(\mathfrak{p}_-, \mathfrak{p}) \to C^{p,3}(\mathfrak{p}_-, \mathfrak{p})$$

$$\begin{aligned} \partial\zeta(X,Y,Z) = {} & [X, \zeta(Y,Z)] + (-1)^{|X|(|Y|+|Z|)}[Y, \zeta(Z,X)] \\ & + (-1)^{|Z|(|X|+|Y|)}[Z, \zeta(X,Y)] - \zeta([X,Y],Z) \\ & - (-1)^{|X|(|Y|+|Z|)}\zeta([Y,Z],X) - (-1)^{|Z|(|X|+|Y|)}\zeta([Z,X],Y) \end{aligned} \tag{2}$$

where $X, Y, Z \in \mathfrak{p}_-$ and $\zeta \in C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$.

Spencer cohomology and Killing spinors

We have obtained the following results.

Thm[Figueroa-O'Farrill,—] If $d = 11$ then:

(i) $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$;

(ii) $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^4 V$.

Thm[de Medeiros, Figueroa-O'Farrill, —] If $d = 4$ then:

(i) $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$;

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Remarks.

- We recover the 4-form of $d = 11$ supergravity through cohomology;
- In $d = 4$ group $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ corresponds exactly to the scalar, vector and pseudoscalar auxiliary fields in the old minimal off-shell formalism of $N = 1$ d= 4 supergravity.

Spencer cohomology and Killing spinors

But there is more. If we look at the β-component (remember $\beta : V \otimes S \longrightarrow S$) of the Spencer cocycle, we find:

d=11

$$\beta(v,s) = v \cdot \varphi \cdot s - 3\varphi \cdot v \cdot s \ ,$$

where $\varphi \in \Lambda^4 V$;

d=4

$$\beta(v,s) = v \cdot (a + b\,\mathrm{vol}) \cdot s - \frac{1}{2}(v \cdot \varphi + 3\varphi \cdot v) \cdot \mathrm{vol} \cdot s \ ,$$

where $(a, b, \varphi) \in \Lambda^0 V \oplus \Lambda^0 V \oplus \Lambda^1 V$.

We recognize them as the zero degree pieces of the gravitino variation in the respective theories of supergravity. In other words, they indicate what the relevant superconnection and Killing spinor equations are.

Observations

- In $d = 11$ supergravity, the relevant Killing spinor equation encodes *all* the information about bosonic bkgds. Indeed the Clifford trace

$$\sum_i e^i \cdot \mathcal{R}(e_i, -) : TM \longrightarrow \mathrm{End}\,(S(M))$$

of curvature $\mathcal{R} : \Lambda^2 TM \longrightarrow \mathrm{End}\,(S(M))$ of the superconnection D vanishes iff $dF = 0$ and the field equations are satisfied [Gauntlett, Pakis '03]. So, in a very real sense, Spencer cohomology knows about $d = 11$ supergravity and also knows about the off-shell formulation of $N = 1$ $d = 4$ supergravity;
- In principle one can play the same game in any dimension and signature and look for new Killing spinor equations. These equations are in general related to the geometries admitting rigid supersymmetric field theories;
- The vanishing of the Clifford trace is a condition which is clearly satisfied by the rigid maximally supersymmetric geometries. It turns out that solving first for this condition substantially simplifies computations.

Classifications

We classified maximally supersymmetric filtered deformations in $d = 11$ and $d = 4$, that is filtered deformations $\mathfrak{g}$ of subalgebras $\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0$ of $\mathfrak{p}$ with $\mathfrak{a}_{-2} = V$, $\mathfrak{a}_{-1} = S$ and $\mathfrak{a}_0 = \mathfrak{h}$ a subalgebra of $\mathfrak{so}(V)$. The fact that $\mathfrak{a}_{-1} = S$ means we have maximal supersymmetry, whereas the fact that $V' = V$ (which is forced) means we are describing (locally) homogeneous geometries. We bootstrapped the computation of $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ from that of $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ and obtained the following results.

Thm[Figueroa-O'Farrill,—] If $d = 11$ we find the Killing superalgebras $\mathfrak{g}$ of maximally supersymmetric bkgds and nothing else:

(i) $\mathfrak{p}$ itself for Minkowski spacetime;

(ii) $\mathrm{osp}(8|4)$ for $AdS_4 \times S^7$ [Freund, Rubin '80];

(iii) $\mathrm{osp}(6,2|4)$ for $S^4 \times AdS_7$ [Pilch, van Nieuwenhuizen, Townsend '84];

(iv) the Killing superalgebra of max. susy pp-wave [Kowalski-Glikman '84].

In all cases $\mathfrak{h} = \mathfrak{so}(V) \cap \mathrm{stab}(\varphi)$ where $\varphi \in \Lambda^4 V$.

Classifications

Thm[de Medeiros, Figueroa-O'Farrill,—] If $d = 4$ there are three families of filtered deformations $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ of subalgebras $\mathfrak{a} = V \oplus S \oplus \mathfrak{h}$ of Poincaré superalgebra $\mathfrak{p}$:

(i) $\mathfrak{p}$ itself for Minkowski spacetime;

(ii) In this case $\mathfrak{h} = \mathfrak{so}(V)$, there exist $a, b \in \mathbb{R}$ with $a^2 + b^2 > 0$ and associated homogeneous Lorentzian mnfd $(M = G/H, g)$, $Lie(G) = \mathfrak{g}_{\bar{0}}$, $Lie(H) = \mathfrak{h}$ is locally isometric to AdS_4;

(iii) There is nonzero $\varphi \in V$, $\mathfrak{h}$ is any subalgebra of $\mathfrak{so}(V) \cap \mathrm{stab}(\varphi)$ and the geometry is that of a Lie group with bi-invariant metric:

- If φ is spacelike then (M, g) is locally isometric to $\mathrm{AdS}_3 \times \mathbb{R}$;
- If φ is timelike then (M, g) is locally isometric to $\mathbb{R} \times S^3$;
- If φ is lightlike then we have the so-called Nappi-Witten group NW_4.

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Rem. Only (i) is max. susy bkgd of supergravity but (i)-(iii) are the Lorentzian $d = 4$ geometries admitting $N = 1$ rigid susy field theories [Festuccia-Seiberg '11], see also [Butter-Inverso-Lodato '15] for NW_4.

Conclusions and Outlook

- None of the geometries in our classification are new. The novelty lies in the approach which systematises the search for geometries on which one can define rigid supersymmetric field theories by mapping it to an algebraic problem on which we can bring to bear representation-theoric techniques;
- In Festuccia-Seiberg approach, the rigid supersymmetric non-linear sigma-models in curved space are obtained by applying Noether method to obtain some locally supersymmetric field theory coupled to supergravity and then taking a rigid limit where gravity supermultiplet is frozen out. One then looks for supersymmetric geometries and compute associated Killing superalg.;
- Our alternative strategy is to start with Poincaré superalgebra $\mathfrak{p}$ and get all relevant Killing superalgebras directly as filt. def. $\mathfrak{g}$ of subalgebras $\mathfrak{a}$ of $\mathfrak{p}$. This approach should work also for different dimensions d and amount N of supersymmetry for which there possibly are no (off-shell) formulations of sugra. We do not expect it however to be tamed for cases with a single supercharge [Klare, Tomasiello, Zaffaroni '13].
- We are trying to extend the methods in $d = 11$ to the case $\dim S' > 16$.

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Thanks!

Tanaka structures

An important source of inspiration for this work is theory of Tanaka structures, a generalization of the usual notion of G-structure $\pi : P \to M$, $G \subset \mathrm{GL}(V)$, which deals with geometries supported over non-integrable distributions of tangent bundle. It is classical result [Guillemin '65] that obstructions to integrability for a G-structure $\pi : P \to M$ can be expressed in terms of G-equiv. "curvatures" $\mathcal{R}^p : P \longrightarrow H^{p,2}(\mathfrak{g})$ with values in Spencer cohomology groups associated to Lie algebra $\mathfrak{g} = Lie(G)$ of G. The cohomology is in this case computed with respect to appropriate $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}_\infty = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots$ of depth one, where $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_0 = \mathfrak{g}$. There is an analogous result for Tanaka structures and $\mathbb{Z}$-graded Lie algebras of depth > 1.

I will now give a flavour of the geometric ideas behind previous results and make clear why in our case we used gradings of depth two. For simplicity of exposition I avoid technicalities on supermnfds and deal with them as if they were usual smooth mnfds. I will also present a (not so recent anymore) classification result on Tanaka structures on supermanifolds.

Manifolds with a Poincaré superstructure

Let $\mathcal{D} \subset TM$ be a distribution on a mnfd M. The *Levi form* at $x \in M$ is

$$\mathcal{L}_x : \mathcal{D}_x \times \mathcal{D}_x \longrightarrow T_x M/\mathcal{D}_x \ , \quad \mathcal{L}_x(s, s') = [X^{(s)}, X^{(s')}]_x/\mathcal{D}_x$$

where $X^{(s)}, X^{(s')} \in \Gamma(\mathcal{D})$ such that $X^{(s)}_x = s$, $X^{(s')}_x = s'$.

The Flat Model. For a fixed Poincaré superalgebra $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ let $M^{flat} = G/H$ where G is a connected Lie group with $Lie(G) = \mathfrak{p}$ and $H \subset G$ is the subgroup with $Lie(H) = \mathfrak{so}(V)$. Let $\mathcal{D}^{flat}$ be the unique G-invariant distribution on M^{flat} s.t. at point $o = eH$

$$\mathcal{D}^{flat}_o = S \subset T_o M^{flat} \qquad \text{(here } T_o M^{flat} \simeq \mathfrak{p}/\mathfrak{so}(V) = V \oplus S) \ .$$

The Levi form of this distribution is G-invariant. In $o = eH$ it is equal to

$$\mathcal{L}_o : S \times S \longrightarrow T_o M^{flat}/S \simeq V \ , \qquad \mathcal{L}_o(s, s') = k(s, s') \ ,$$

where k is the Dirac current.

Manifolds with a Poincaré superstructure

Def. A *Poincaré superstructure* on a mnfd M of $\dim(M) = \dim(V) + \dim(S)$ is the datum of a distribution $\mathcal{D}$ with Levi form

$$\mathcal{L}_x : \mathcal{D}_x \times \mathcal{D}_x \longrightarrow T_x M / \mathcal{D}_x$$

identifiable at all $x \in M$ with the Levi-form of $\mathcal{D}^{flat} \subset TM^{flat}$ that is with k.

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Motivation: Poincaré superstructures $(M, \mathcal{D})$ appear naturally in superspace presentations of supergravity (Roslyi-Schwarz et al.; Ogievetsky; A. S.-Spiro).

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Questions.

- Is the automorphism group $\mathrm{Aut}(M, \mathcal{D})$ a Lie group?
- How large can $\mathrm{Aut}(M, \mathcal{D})$ be?
- To classify the maximally homogeneous mnfds endowed with a Poincaré superstructure (i.e. the cases when $\mathrm{Aut}(M, \mathcal{D})$ is as big as possible).

Tanaka's method

We consider the point of view of Tanaka (series of several papers in '60–'70). Let M be mnfd with distribution $\mathcal{D}$. Consider filtration of Lie algebra $\mathfrak{X}(M)$ of all vector fields defined by $\Gamma(\mathcal{D})_{-1} = \Gamma(\mathcal{D})$ and for any $i > 1$ by

$$\Gamma(\mathcal{D})_{-i} = \Gamma(\mathcal{D})_{-i+1} + [\Gamma(\mathcal{D}), \Gamma(\mathcal{D})_{-i+1}] \ .$$

Evaluating at $x \in M$, we get flag

$$T_x M = \mathcal{D}_{-d}(x) \supset \mathcal{D}_{-d+1}(x) \supset \cdots \supset \mathcal{D}_{-2}(x) \supset \mathcal{D}_{-1}(x) = \mathcal{D}_x$$

and the commutator in $\mathfrak{X}(M)$ induces structure of graded Lie algebra on

$$\mathfrak{m}(x) = \mathrm{gr}(T_x M) = \mathfrak{m}(x)_{-d} \oplus \cdots \oplus \mathfrak{m}(x)_{-1} \ .$$

Def. $\mathcal{D}$ is *of type* $\mathfrak{m}$ if all $\mathfrak{m}(x)$ are isomorphic to a given $\mathfrak{m} = \mathfrak{m}_{-d} \oplus \cdots \oplus \mathfrak{m}_{-1}$.

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Example. Poincaré superstructures $(M, \mathcal{D})$ are precisely the distributions of type $\underbrace{\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1} = V \oplus S}_{\text{non-trivial bracket is } [s,s'] := k(s,s')}$ in Tanaka's sense.

Tanaka's method

Def. The *maximal prolongation* of Lie (super)algebra $\mathfrak{m} = \mathfrak{m}_{-d} \oplus \cdots \oplus \mathfrak{m}_{-1}$ is the maximal graded Lie (super)alg. $\mathfrak{g}_\infty = \mathfrak{g}_{-d} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots$ s.t.

i) $\mathfrak{g}_{-d} \oplus \cdots \oplus \mathfrak{g}_{-1} = \mathfrak{m}_{-d} \oplus \cdots \oplus \mathfrak{m}_{-1}$,

ii) for any $k \geq 0$ and $X \in \mathfrak{g}_k$ one has $[X, \mathfrak{g}_{-1}] = \{0\} \Rightarrow X = 0$.

It can be defined by $\mathfrak{g}_i = \{A \in \mathrm{Der}\,(\mathfrak{m}, \sum_{h<i} \mathfrak{g}_h) \mid A(\mathfrak{m}_j) \subseteq \mathfrak{m}_{j+i}\}$ for $i \geq 0$, where $\mathrm{Der}\,(\mathfrak{m}, W)$ is space of derivations of $\mathfrak{m}$ with values in $\mathfrak{m}$-module W.

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Theorem[Tanaka] M endowed with a distribution $\mathcal{D}$ of type $\mathfrak{m}$. Then

1. If $\dim(\mathfrak{g}_\infty) < \infty$ then $\mathrm{Aut}\,(M, \mathcal{D})$ is a Lie group of dim. $\leq \dim(\mathfrak{g}_\infty)$,
2. If $\dim(\mathrm{Aut}\,(M, \mathcal{D})) = \dim(\mathfrak{g}_\infty) < \infty$, then $(M, \mathcal{D}) \underset{\text{locally}}{\simeq} (G/P, \mathcal{D}^{\text{std}})$,

– G is Lie group with $Lie(G) = \mathfrak{g}_\infty$, P subgroup with $Lie(P) = \sum_{i \geq 0} \mathfrak{g}_i$,
– $\mathcal{D}^{\text{std}}$ is the unique G-invariant distribution on G/P s.t. $\mathcal{D}^{\text{std}}_o \simeq \mathfrak{m}_{-1}$ at $o = eP$, where $T_oG/P \simeq \mathfrak{m}_{-d} \oplus \cdots \oplus \mathfrak{m}_{-1}$.

The classification of maximally homogeneous $(M, \mathcal{D})$

Let $\mathfrak{g}_\infty = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots$ be max. prolong. of $\mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1} = V \oplus S$ and $(M, \mathcal{D})$ mnfd with Poincaré superstructure of type $\mathfrak{m}$. By Tanaka:

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Let $\mathfrak{g}_\infty = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots$ be max. prolong. of $\mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1} = V \oplus S$ and $(M, \mathcal{D})$ mnfd with Poincaré superstructure of type $\mathfrak{m}$. By Tanaka:

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2. If $\dim(\mathrm{Aut}\,(M, \mathcal{D})) = \dim(\mathfrak{g}_\infty) < \infty$, then $(M, \mathcal{D}) \underset{\text{locally}}{\simeq} (G/P, \mathcal{D}^{\text{std}})$, where $Lie(G) = \mathfrak{g}_\infty$, $Lie(P) = \sum_{i \geq 0} \mathfrak{g}_i$.

Leading problem

1. When $\dim \mathfrak{g}_\infty$ is finite?
2. Describe explicitly $\mathfrak{g}_\infty$ for any dimension and signature of V (and actually also for any amount N of supersymmetry i.e. # of copies of spinor repres.).

The classification of maximally homogeneous $(M, \mathcal{D})$

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Theorem I[Altomani,—] If $\dim V \geq 3$ then the maximal prolongation $\mathfrak{g}_\infty$ has always finite dimension. If $\dim(V) = 1, 2$ then $\mathfrak{g}_\infty$ is identifiable with (infinite dim.) algebra of infinitesimal automorphisms of contact manifolds.

Theorem II[Altomani,—] The Lie algebra $\mathfrak{g}_0$ decomposes in sum of ideals

$$\mathfrak{g}_0 = \mathfrak{so}(V) \oplus \mathbb{R}E \oplus \mathfrak{h}_0\,,$$

where E is the so-called grading element, determined by $\mathrm{ad}(E)|_{\mathfrak{g}_{-p}} = -p\,\mathrm{Id}$ and $\mathfrak{h}_0 = \{X \in \mathfrak{g}_0 \,|\, [X, \mathfrak{g}_{-2}] = 0\}$. Moreover, one can explicitly describe $\mathfrak{h}_0$.

The classification of maximally homogeneous $(M, \mathcal{D})$

Theorem III[Altomani,–]
Let $\dim V \geq 3$ and $\mathfrak{g}_\infty = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots$ the maximal prolongation of $\mathfrak{m} = V \oplus S$. Then exactly one of the following two cases occurs:

1. $\mathfrak{g}_p = 0$ for all $p \geq 1$;
2. $\mathfrak{g}_\infty$ is semisimple with a simple socle $\mathfrak{s}$. In this case $\mathfrak{s}$ is basic.

The classification of maximally homogeneous $(M, \mathcal{D})$

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1. $\mathfrak{g}_p = 0$ for all $p \geq 1$;
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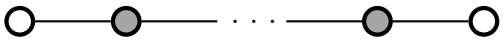
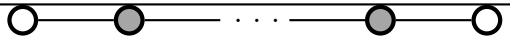
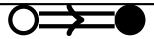
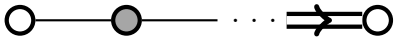
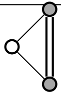
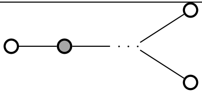
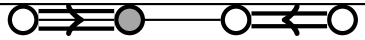
A basic Lie superalgebra is a simple Lie superalgebra which, in addition, can be described in terms of a (in general not unique) "Dynkin diagram". There are even ○, odd isotropic ⊗ and odd nonisotropic ● roots.

The classification of maximally homogeneous $(M, \mathcal{D})$

Theorem[Altomani,–] Let $\mathfrak{g}_\infty$ and $\mathfrak{s}$ be as in (2) of Thm III. Then (the complexification of) $\mathfrak{s}$ is isomorphic to one of the Lie superalgebras

$\mathfrak{s}$	**Dynkin diagram**	$\dim \mathfrak{s}_{-2}$	$\dim \mathfrak{s}_{-1}$	N	$\mathfrak{h}_0$
$\mathfrak{sl}(m+1\|4)$, $m \neq 3$		4	$4N$	$m+1$	$\mathfrak{gl}(N, \mathbb{C})$
$\mathfrak{psl}(4\|4)$		4	16	4	$\mathfrak{sl}(4, \mathbb{C})$
$\mathfrak{osp}(1\|4)$		3	2	1	0
$\mathfrak{osp}(2m+1\|4)$, $m \geq 1$		3	$2N$	$2m+1$	$\mathfrak{so}(N, \mathbb{C})$
$\mathfrak{osp}(2\|4)$		3	4	2	$\mathfrak{so}(2, \mathbb{C})$
$\mathfrak{osp}(2m\|4)$, $m \geq 2$		3	$2N$	$2m$	$\mathfrak{so}(N, \mathbb{C})$
$F(4)$		5	8	2	$\mathfrak{sl}(2, \mathbb{C})$

and the corresponding maximal prolongation $\mathfrak{g}_\infty \simeq \mathrm{der}(\mathfrak{s})$.

Outlook

– The mathematical development of Tanaka structures on supermanifolds is open and quite desirable, also in view of possible applications to supergravity and rigid supersymmetric field theories.

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References

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