Some estimates for the higher eigenvalues of sets close to the ball

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PDEs, optimal design and numerics
Benasque, 21 August 2017

1. Spectral shape optimization
2. Quantitative estimates: a brief overview
3. A quantitative spectral problem

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- Numerics;
- Properties of sets "close" to optimality.

Spectral shape optimization

We are interested in spectral shape optimization problems:

$$\min\Big\{F(\lambda_1(\Omega),\dots,\lambda_k(\Omega))\ :\ \Omega\subset\mathbb{R}^N,\ \text{open},\ \ |\Omega|=1\Big\},$$

where $\lambda_i(\cdot)$ are the eigenvalues of Dirichlet-Laplacian.

Brief recalls about Dirichlet eigenvalues

Given an open set Ω with finite measure, there exist infinitely many eigenvalues:

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \to \infty$$

The couple $(u_i, \lambda_i(\Omega))$ solves the problem

$$\begin{cases} -\Delta u_i = \lambda_i(\Omega) u_i & \text{in } \Omega \\ u_i \in H_0^1(\Omega). \end{cases}$$

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Variational characterization for the eigenvalues:
min-max principle,

$$\lambda_i(\Omega) = \min_{E_i \subset H_0^1(\Omega), dim(E_i)=i} \max_{u \in E_i \setminus \{0\}} \left\{ \frac{\int |Du|^2}{\int u^2} \right\}$$

Some basic properties following from the min-max principle:

- Monotonicity:

$$\Omega_1 \subset \Omega_2 \qquad \Longrightarrow \qquad \lambda_i(\Omega_2) \leq \lambda_i(\Omega_1), \quad \forall i \in \mathbb{N}.$$

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- If Ω is a disconnected union of two sets Ω_1 and Ω_2, then:

$$\lambda_1(\Omega) = \min\Big\{\lambda_1(\Omega_1), \lambda_1(\Omega_2)\Big\},$$
$$\lambda_2(\Omega) = \min\Big\{\max\{\lambda_1(\Omega_1), \lambda_1(\Omega_2)\}, \lambda_2(\Omega_1), \lambda_2(\Omega_2)\Big\}.$$

A fundamental spectral shape optimization problem

Faber-Krahn, 1920s

$$\lambda_1(\Omega) \geq \lambda_1(B), \qquad \forall\, \Omega \subset \mathbb{R}^N,\ |\Omega| = 1,$$

where B is the ball of unit measure in $\mathbb{R}^N$, and with equality if and only if Ω is the unit ball.

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In a scale-invariant form it can be rewritten as:

$$\lambda_1(\Omega)|\Omega|^{2/N} \geq \lambda_1(B_r)|B_r|^{2/N}, \qquad \forall\, \Omega \subset \mathbb{R}^N.$$

Idea of the proof: Rearrangements, Polya-Szego inequality.

Main issue of this talk

If $\Omega \subset \mathbb{R}^N$ is an open set of unit measure with $\lambda_1(\Omega) \approx \lambda_1(B)$, what can we say about Ω and $\lambda_k(\Omega)$ for $k \geq 2$?

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A measure of closeness to the ball for a set Ω of unit measure, the **Fraenkel asymmetry**:

$$d(\Omega) := \inf \left\{ \left|\Omega \Delta (x + B)\right|, x \in \mathbb{R}^N \right\}.$$

The sharp quantitative Faber-Krahn inequality

Theorem (Brasco-De Philippis-Velichkov,...)

There exists a dimensional constant C such that for every open set $\Omega \subseteq \mathbb{R}^N$ with unit measure, one has

$$\lambda_1(\Omega) - \lambda_1(B) \geq \frac{1}{C}\, d(\Omega)^2 \,.$$

Remark: the exponent 2 is the sharp one.

Another kind of quantitative question

If $\Omega \subset \mathbb{R}^N$ is an open set of unit measure with $\lambda_1(\Omega) \approx \lambda_1(B)$, are the higher eigenvalues of Ω close to those of the ball?"

Remark

1. *This is not a trivial consequence of the previous quantitative estimate: you can have* $|\Omega \Delta B| \ll 1$ *but very different eigenvalues, think ball with small holes!*

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1. *This is not a trivial consequence of the previous quantitative estimate: you can have* $|\Omega \Delta B| \ll 1$ *but very different eigenvalues, think ball with small holes!*
2. *For generic sets the answer is clearly NO! One can have* $\lambda_1(\Omega_1) = \lambda_1(\Omega_2)$ *but* $\lambda_k(\Omega_1) \gg \lambda_k(\Omega_2)$*!*

A positive answer

Theorem (M.-Pratelli)

Let $k, N \in \mathbb{N}$, and let $\Omega \subseteq \mathbb{R}^N$ be an open set of unit measure with $\lambda_1(\Omega) \leq \lambda_1(B) + 1$. Then we have

$$-\frac{1}{C}(\lambda_1(\Omega) - \lambda_1(B))^{\beta'} \leq \lambda_k(\Omega) - \lambda_k(B) \leq C(\lambda_1(\Omega) - \lambda_1(B))^{\beta},$$

where β and β' are explicit exponents, not depending on k nor on N and $C = C(k, N)$ is an explicit constant.

A stronger upper bound for $k = 2$

Theorem (Ashbaugh-Benguria)

For all open sets $\Omega \subset \mathbb{R}^N$ of unit measure, we have

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B)}{\lambda_1(B)}.$$

Then it is easy to see that

$$\lambda_2(\Omega) - \lambda_2(B) \leq \frac{\lambda_2(B)}{\lambda_1(B)}(\lambda_1(\Omega) - \lambda_1(B)).$$

Bound from below

Step 1: If we have

$$\lambda_k(B) - \lambda_k(\Omega) \leq Cd(\Omega)^{\alpha}, \tag{1}$$

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Step 2: Prove (1) by reducing to the case when $\Omega \subset B_{d^\alpha}$.

Bound from above

Idea: We need upper bound for $\lambda_k(\Omega)$.
Look for functions in $H_0^1(\Omega)$ which are almost orthogonal and with Rayleigh quotient close to the eigenvalues of the ball.

Thank you for your attention!