Regularity for the minimum time function with Hörmander vector fields

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VII PARTIAL DIFFERENTIAL EQUATIONS, OPTIMAL DESIGN, AND NUMERICS

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Minimum time and eikonal equation

Let

- $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth (C^∞) boundary Γ
- $\{X_1, \dots, X_N\}$ be smooth (C^∞) vector fields $X_i : \overline{\Omega} \to \mathbb{R}^n$

Denote by $y^{x,u}(\cdot)$ be the unique solution of the controlled system

$$\begin{cases} y'(t) = \sum_{j=1}^N u_j(t) X_j(y(t)) & (t \geq 0) \\ y(0) = x \in \overline{\Omega} \end{cases}$$

where $u : [0, +\infty[\to \overline{B}_1(0)$ is a measurable function (control)

Define

- the *transfer time* to Γ as $\tau_\Gamma(x, u) = \inf \{ t \geq 0 \ : \ y^{x,u}(t) \in \Gamma \}$
- the *minimum time function* as $T(x) = \inf_{u(\cdot)} \tau_\Gamma(x, u) \quad (x \in \overline{\Omega})$

When continuous, T is the unique viscosity solution of Dirichlet problem

$$\begin{cases} \sum_{j=1}^N \langle X_j(x), DT(x) \rangle^2 = 1 & \text{in} \quad \Omega \\ T = 0 & \text{on} \quad \Gamma \end{cases}$$

Problems: regularity of T and structure of Sing T

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Problems: regularity of T and structure of $\operatorname{Sing} T$

Outline

1. Hörmander vector fields and continuity of T
2. Singular time-optimal trajectories and (lack of) Lipschitz continuity
3. Regularity of the minimum time function
4. Analysis of the singular sets of T
5. Work in Progress
   - Homotopy equivalence
   - Estimate of Hausdorff dimension
   - Nonsmooth targets

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Hörmander vector fields

Given smooth vector fields $X_1, \dots, X_N : \overline{\Omega} \to \mathbb{R}^n$ define

- $\mathrm{Lie}(\{X_i\}_{i=1}^N)$ = Lie algebra generated by $\{X_i\}_{i=1}^N$
- $\mathrm{Lie}(\{X_i\}_{i=1}^N)[x] = \{X(x) \;:\; X \in \mathrm{Lie}(\{X_i\}_{i=1}^N)\}$ for any $x \in \overline{\Omega}$

Definition

We say that $\{X_i\}_{i=1}^N$ is a system of Hörmander vector fields on $\overline{\Omega}$ if

$$Lie(\{X_i\}_{i=1}^N)[x] = \mathbb{R}^n \qquad \forall x \in \overline{\Omega}$$

$X_1, \dots, X_N$ need not be linearly independent nor we suppose $N < n$

Example (Grushin's system)

In $\mathbb{R}^2$ consider

$$X_1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad X_2(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \quad \textit{Then} \quad [X_1, X_2](x) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

So $\{X_1, X_2\}$ is a system of Hörmander vector fields

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So $\{X_1, X_2\}$ *is a system of Hörmander vector fields*

Characteristic points

The *Hamiltonian* associated with Hörmander vector fields $\{X_1, \ldots, X_N\}$ is given by

$$h(x,p) = \sum_{j=1}^{N} \langle X_j(x), p \rangle^2, \quad (x,p) \in \overline{\Omega} \times \mathbb{R}^n$$

Definition

The characteristic set *of* $\{X_1, \ldots, X_N\}$ *is given by*

$$\mathrm{Char}(X_1, \ldots X_N) = \{(x,p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus \{0\}) \ : \ h(x,p) = 0\}.$$

A point $x \in \Gamma$ *is called characteristic if*

$$span\{X_i(x)\}_{i=1}^{N} \subset T_\Gamma(x) \qquad \textit{(tangent space to } \Gamma \textit{ at } x\textit{)}$$

We denote by $E = E(\{X_i\}_{i=1}^{N}, \Gamma) \subset \Gamma$ *the set of all characteristic points*

Theorem (Derridj 1972)

Let $\{X_i\}_{i=1}^{N}$ *be Hörmander v.f. Then* E *is closed and* $\mathcal{H}^{(n-1)}(E) = 0$

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Let $\{X_i\}_{i=1}^N$ be Hörmander v.f. and define for any $x \in \overline{\Omega}$

$$k(x) = \min\left\{k \geq 1 \ : \ \mathrm{Lie}^k(\{X_i\}_{i=1}^N)[x] = \mathbb{R}^n\right\}$$

where $\mathrm{Lie}^k(\{X_i\}_{i=1}^N)$ is defined recursively by $\mathrm{Lie}^1(\{X_i\}_{i=1}^N) = \mathrm{span}\,\{X_i\}_{i=1}^N$ and

$$\begin{aligned} &\mathrm{Lie}^{k+1}(\{X_i\}_{i=1}^N) \\ &\quad = \mathrm{span}\left(\mathrm{Lie}^k(\{X_i\}_{i=1}^N) \cup \left\{[X, X_j] \ : \ X \in \mathrm{Lie}^k(\{X_i\}_{i=1}^N),\ j = 1,\dots,N\right\}\right) \end{aligned}$$

Hölder regularity [Nagel, Stein, and Wainger 1985]

$$T \in C^{0,1/r}(\overline{\Omega}) \quad \text{where} \quad r = \max\{k(x) \ : \ x \in \overline{\Omega}\}$$

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Singular time-optimal trajectories

For any $z \in \Gamma$ we denote by $\nu(z)$ the outward unit normal to Γ at z and set

$$N_\Gamma(z) := \{\lambda \nu(z) \ : \ \lambda \geq 0\}$$

Let $x \in \Omega$ and let $y = y^{x,u}$ be a time-optimal trajectory with $u : [0, T(x)] \to \overline{B}_1(0)$

Definition

We say that y is singular if $\exists\, p : [0, T(x)] \to \mathbb{R}^n \setminus \{0\}$ absolutely continuous such that

$$\begin{cases} -p'(t) = \sum_{j=1}^N u_j(t) DX_j(y(t))^* p(t) & t \in [0, T(x)] \ \ a.e. \\ \langle X_j(y(t)), p(t)\rangle = 0 & t \in [0, T(x)]\,,\ j = 1, \dots, N \\ p(T(x)) \in N_\Gamma(y(T(x))) \end{cases}$$

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Strongly bracket generating systems

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We say that $\{X_i\}_{i=1}^N$ *is* strongly bracket generating *on* $\overline{\Omega}$ *if* $\forall\, v = (v_1, \ldots, v_N) \in \mathbb{R}^n \setminus \{0\}$

$$span\{X_i(x)\}_{i=1}^N + span\Big\{\sum_{j=1}^N v_j\,[X_j, X_i](x)\Big\}_{i=1}^N = \mathbb{R}^n \quad \forall x \in \overline{\Omega}$$

Example (Nonholonomic integrator)

A strongly bracket generating system of v.f. in $\mathbb{R}^3$ *is given by*

$$X_1(x) = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} \quad X_2(x) = \begin{pmatrix} 0 \\ 1 \\ -x_1 \end{pmatrix} \quad Then \quad [X_1, X_2](x) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Proposition

A strongly bracket generating system of v.f. admits no singular trajectory.

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A strongly bracket generating system of v.f. admits no singular trajectory.

Strongly bracket generating systems

Definition

We say that $\{X_i\}_{i=1}^N$ *is* strongly bracket generating *on* $\overline{\Omega}$ *if* $\forall\, v = (v_1, \ldots, v_N) \in \mathbb{R}^n \setminus \{0\}$

$$span\{X_i(x)\}_{i=1}^N + span\Big\{\sum_{j=1}^N v_j\,[X_j, X_i](x)\Big\}_{i=1}^N = \mathbb{R}^n \quad \forall x \in \overline{\Omega}$$

Example (Nonholonomic integrator)

A strongly bracket generating system of v.f. in $\mathbb{R}^3$ *is given by*

$$X_1(x) = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} \quad X_2(x) = \begin{pmatrix} 0 \\ 1 \\ -x_1 \end{pmatrix} \quad \text{Then} \quad [X_1, X_2](x) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Proposition

A strongly bracket generating system of v.f. admits no singular trajectory.

Proof of proposition

Let $x \in \Omega$, let $y = y^{x,u}$ be a singular trajectory, and let $p \neq 0$ satisfy

$$\begin{cases} -p'(t) = \sum_{j=1}^{N} u_j(t) DX_j\big(y(t)\big)^* p(t)\,, & p(T(x)) \in N_\Gamma(y(T(x))) \\ \langle X_k(y(t)), p(t) \rangle = 0 & t \in [0, T(x)]\,,\ k = 1, \dots, N \end{cases}$$

Then for all $k = 1, \dots, N$

$$\begin{aligned} 0 = \frac{d}{dt} \langle X_k(y), p \rangle &= \langle DX_k(y(t)) y'(t), p(t) \rangle + \langle X_k(y(t)), p'(t) \rangle \\ &= \sum_{j=1}^{N} u_j(t) \langle DX_k(y(t)) X_j(y(t)), p(t) \rangle - \sum_{j=1}^{N} u_j(t) \langle X_k(y(t)), DX_j(y(t))^* p(t) \rangle \\ &= \Big\langle \sum_{j=1}^{N} u_j(t) [X_k, X_j](y(t)), p(t) \Big\rangle \end{aligned}$$

Now, taking $t_0 \in [0, T(x)]$ such that $u(t_0) \neq 0$, we obtain the contradiction

$$p(t_0) \perp \Big(\mathrm{span}\big\{ X_i(y(t_0)) \big\}_{i=1}^{N} + \mathrm{span}\Big\{ \sum_{j=1}^{N} u_j(t_0)\, [X_k, X_j](y(t_0)) \Big\}_{k=1}^{N} \Big) = \mathbb{R}^n$$

Singular trajectories and characteristic points

Let $\{X_i\}_{i=1}^N$ be Hörmander v.f.

Proposition

Let $x \in \Omega$ and let $y = y^{x,u}$ be a time-optimal trajectory. Then

$$y \ \ singular \quad \Longleftrightarrow \quad y(T(x)) \in E\big(\{X_i\}_{i=1}^N, \Gamma\big)$$

Proof: By Pontryagin's Maximum Principle (PMP) $\exists p : [0, T(x)] \to \mathbb{R}^n \setminus \{0\}$ (adjoint state)

$$\begin{cases} y'(t) = D_p H(y(t), p(t), u(t)), \quad p'(t) = -D_x H(y(t), p(t), u(t)) \\ p(T(x)) \in N_\Gamma(y(T(x))) \\ H(y(t), p(t), u(t)) = \max_{u \in \overline{B}_1(0)} \sum_{j=1}^N u_j \langle X_j(y^{x,u}(t)), p(t)\rangle = \sqrt{h(y^{x,u}(t), p(t))} \end{cases}$$

So, the function $[0, T(x)] \ni t \mapsto h(y^{x,u}(t), p(t))$ is constant. This implies that

$$\begin{aligned} y(T(x)) \in E \quad &\Longleftrightarrow \quad h(y(T(x)), p(T(x))) = 0 \\ &\Longleftrightarrow \quad (y(t), p(t)) \in \mathrm{Char}(X_1, \dots X_N) \ \ \forall t \in [0, T(x)] \end{aligned}$$

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Singular trajectories and failure of Lipschitz continuity

$f : \overline{\Omega} \to \mathbb{R}$ is Lipschitz at $x_0 \in \overline{\Omega}$ if $\exists$ a neighbourhood U of x_0 and $L \geq 0$ such that

$$|f(x) - f(x_0)| \leq L|x - x_0| \qquad \forall x \in U \cap \overline{\Omega}$$

f can be Lipschitz continuous at x_0 without being Lipschitz on any neighbourhood of x_0

Theorem

Let $\{X_i\}_{i=1}^N$ be Hörmander v.f. Then

(a) *T fails to be Lipschitz at a point $x_0 \in \overline{\Omega}$*

if and only if

(b) *x_0 is the starting point of a singular time-optimal trajectory*

Remark

The fact that the presence of singular optimal trajectories may destroy the regularity (i.e., subanalyticity of the point-to-point distance associated with real analytic distributions) of a solution of a first order Hamilton-Jacobi equation was observed by Sussmann (1992), Agrachev (1998), and Trélat (2006) for solutions of the Dirichlet problem

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Proximal normals

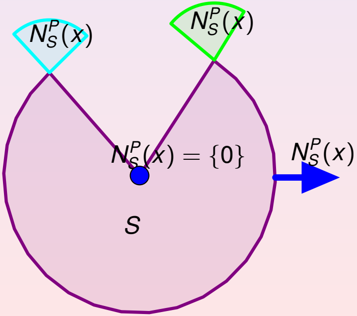
Let $S \subset \mathbb{R}^n$ be a closed set

Definition (Proximal normals)

A vector $v \in \mathbb{R}^n$ is called a proximal normal to S at x if $\exists\, \delta > 0$ and $C > 0$ such that

$$\langle v, y - x\rangle \leq C|y - x|^2 \qquad \forall\, y \in B_\delta(x) \cap S. \tag{2}$$

The set of all proximal normals to S at x will be denoted by $N_S^P(x)$



Horizontal supergradients

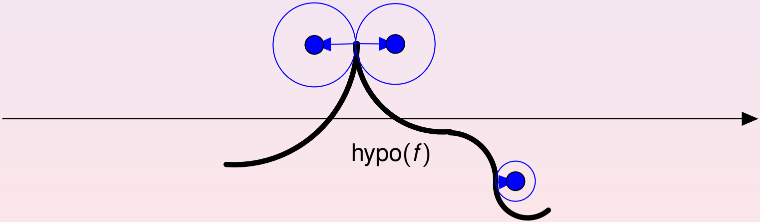
The hypograph of a function $f : \overline{\Omega} \to \mathbb{R}$ is the set

$$\mathrm{hypo}(f) = \{(x, \alpha) \in \overline{\Omega} \times \mathbb{R} \ : \ \alpha \leq f(x)\}$$

A vector $p \in \mathbb{R}^n$ is a horizontal proximal supergradient of a function f at $x \in \Omega$ if

$$(-p, 0) \in N_{\mathrm{hypo}(f)}(x, f(x))$$

The set of all such supergradient is denoted by $\partial^{P,\infty} f(x)$



Important: f Lipschitz at $x_0 \in \Omega \implies \partial^{P,\infty} f(x_0) = \{0\}$

Proof of Theorem: sufficiency

Theorem

T fails to be Lipschitz at a point $x_0 \in \overline{\Omega}$ if and only if x_0 is the starting point of a singular time-optimal trajectory

Want to show

$$\exists\, y_0 = y^{x_0,u_0} \ \text{ singular} \quad \Longrightarrow \quad -p(0) \in \partial^{P,\infty} T(x_0) \quad (\Rightarrow T \text{ not Lipschitz at } x_0)$$

where $p \neq 0$ is the dual arc given by PMP

Recall that p satisfies the transversality condition at $T(x_0)$

$$(p(T(x_0)), 0) \in N_{\mathrm{hypo}(T)}(y_0(T(x_0)), 0)$$

or

$$\langle p(T(x_0)), x - y_0(T(x_0)) \rangle \leq \frac{1}{\delta} \left| x - y_0(T(x_0)) \right|^2 \quad \forall x \in \overline{\Omega}$$

So, we want to propagate such property up to $t = 0$, that is,

$$(p(0), 0) \in N_{\mathrm{hypo}(T)}(x_0, T(x_0))$$

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Proof of sufficiency: $T(x) \leq T(x_0)$

We have to prove that $\exists\, C > 0$ such that, $\forall\, x \in \overline{\Omega}$ with $|T(x) - T(x_0)| < 1$,

$$\langle p(0), x - x_0 \rangle \leq C\big(|x - x_0|^2 + (\alpha - T(x_0))^2\big) \quad \forall\, \alpha \leq T(x)$$

Suppose $\boxed{T(x) \leq T(x_0)}$ and let $\boxed{y(t) = y^{x,u_0}(t)}$ Then

$$\langle p(0), x - x_0 \rangle = \langle p(T(x)), y(T(x)) - y_0(T(x)) \rangle - \int_0^{T(x)} \frac{d}{dt} \langle p(t), y(t) - y_0(t) \rangle \, dt$$

By using the adjoint system

$$\begin{aligned} \left| \frac{d}{dt} \langle p, y - y_0 \rangle \right| &= \left| \sum_{j=1}^{N} u_j \langle p, X_j(y) - X_j(y_0) - DX_j(y_0)(y - y_0) \rangle \right| \\ &\leq C|p|\,|y - y_0|^2 \leq |x - x_0|^2 \end{aligned}$$

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By using the adjoint system

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Proof of sufficiency: $T(x) \leq T(x_0)$ completed

We now compute

$$
\begin{aligned}
&\langle p(T(x)), y(T(x)) - y_0(T(x))\rangle \\
&= \langle p(T(x_0)), y(T(x)) - y_0(T(x))\rangle + \langle p(T(x)) - p(T(x_0)), y(T(x)) - y_0(T(x))\rangle \\
&\qquad \leq \langle p(T(x_0)), y(T(x)) - y_0(T(x))\rangle + C\big(|T(x) - T(x_0)|^2 + |x - x_0|^2\big)
\end{aligned}
$$

Moreover, since $\langle p(T(x_0)), x - y_0(T(x_0))\rangle \leq \frac{1}{\delta}|x - y_0(T(x_0))|^2$,

$$
\begin{aligned}
&\langle p(T(x_0)), y(T(x)) - y_0(T(x))\rangle \\
&\quad = \langle p(T(x_0)), y_0(T(x_0)) - y_0(T(x))\rangle + \langle p(T(x_0)), y(T(x)) - y_0(T(x_0))\rangle \\
&\quad \leq \sum_{j=1}^{N}\int_{T(x)}^{T(x_0)} u_j(t)\langle p(T(x_0)), X_j(y_0(t))\rangle dt + \frac{1}{\delta}\big|y(T(x)) - y_0(T(x_0))\big|^2 \\
&\quad \leq \sum_{j=1}^{N}\int_{T(x)}^{T(x_0)} u_j(t)\langle p(T(x_0)), \underbrace{X_j(y_0(t)) - X_j(y_0(T(x_0)))}_{\leq |t - T(x_0)|}\rangle dt \\
&\qquad\qquad + C\big(|x - x_0|^2 + |T(x) - T(x_0)|^2\big)
\end{aligned}
$$

This proves $\langle p(0), x - x_0\rangle \leq C\big(|x - x_0|^2 + (\alpha - T(x_0))^2\big)$ with $\alpha = T(x)$

Proof of sufficiency: $T(x) \leq T(x_0)$ completed

We now compute

$$\begin{aligned}
&\langle p(T(x)), y(T(x)) - y_0(T(x))\rangle \\
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&\qquad \leq \langle p(T(x_0)), y(T(x)) - y_0(T(x))\rangle + C\big(|T(x) - T(x_0)|^2 + |x - x_0|^2\big)
\end{aligned}$$

Moreover, since $\langle p(T(x_0)), x - y_0(T(x_0))\rangle \leq \frac{1}{\delta}\,|x - y_0(T(x_0))|^2$,

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&\quad \leq \quad \sum_{j=1}^{N} \int_{T(x)}^{T(x_0)} u_j(t)\langle p(T(x_0)), X_j(y_0(t))\rangle dt + \frac{1}{\delta}\,\big|y(T(x)) - y_0(T(x_0))\big|^2 \\
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&\qquad\qquad + C\big(|x - x_0|^2 + |T(x) - T(x_0)|^2\big)
\end{aligned}$$

This proves $\langle p(0), x - x_0\rangle \leq C\big(|x - x_0|^2 + (\alpha - T(x_0))^2\big)$ with $\alpha = T(x)$

Proof of sufficiency: $T(x) \le T(x_0)$ completed

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&\qquad \le \langle p(T(x_0)), y(T(x)) - y_0(T(x))\rangle + C\left(|T(x) - T(x_0)|^2 + |x - x_0|^2\right)
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&\le \sum_{j=1}^{N} \int_{T(x)}^{T(x_0)} u_j(t)\langle p(T(x_0)), X_j(y_0(t))\rangle dt + \frac{1}{\delta}\left|y(T(x)) - y_0(T(x_0))\right|^2 \\
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Proof of sufficiency: $T(x) > T(x_0)$

if $\boxed{T(x) > T(x_0)}$ taking $\boxed{y(t) = y^{x,u_0}(t)}$ once again we obtain

$$\langle p(0), x - x_0 \rangle = \langle p(T(x_0)), y(T(x_0)) - y_0(T(x_0)) \rangle - \int_0^{T(x_0)} \frac{d}{dt} \langle p(t), y(t) - y_0(t) \rangle \, dt$$

The first term on the right-hand side is controlled by $|x - x_0|^2$ because of transversality

$$\langle p(T(x_0)), y(T(x_0)) - y_0(T(x_0)) \rangle \leq C|x - x_0|^2$$

The second term is controlled by $|x - x_0|^2$ as before

Proof of Theorem: necessity

Theorem

T fails to be Lipschitz at $x_0 \in \overline{\Omega}$ if and only if x_0 is the starting point of a singular time-optimal trajectory

We will derive a contradiction assuming that

T fails to be Lipschitz at x_0 but no singular optimal trajectory starts from x_0

Since T fails to be Lipschitz at x_0, there exists $\{x_j\} \subset \Omega$ such that

$$\frac{|T(x_j) - T(x_0)|}{|x_j - x_0|} \geq j \quad \text{and} \quad x_j \to x_0 \quad \text{as} \quad j \to \infty$$

Moreover, for any time-optimal trajectory $y_0 = y^{x_0,u_0}$ we have that:

- $y_0(T(x_0)) \in \Gamma \setminus E$ by Proposition
- $\exists\, \delta > 0 \ : \ B_\delta(y_0(T(x_0))) \cap E = \emptyset$
- $T(x) \leq C d_\Gamma(x) \quad \forall\, x \in B_\delta(y_0(T(x_0))) \cap \overline{\Omega}$

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Proof of Theorem: necessity

Theorem

T fails to be Lipschitz at $x_0 \in \overline{\Omega}$ if and only if x_0 is the starting point of a singular time-optimal trajectory

We will derive a contradiction assuming that

T fails to be Lipschitz at x_0 but no singular optimal trajectory starts from x_0

Since T fails to be Lipschitz at x_0, there exists $\{x_j\} \subset \Omega$ such that

$$\frac{|T(x_j) - T(x_0)|}{|x_j - x_0|} \geq j \quad \text{and} \quad x_j \to x_0 \quad \text{as} \quad j \to \infty$$

Moreover, for any time-optimal trajectory $y_0 = y^{x_0,u_0}$ we have that:

- $y_0(T(x_0)) \in \Gamma \setminus E$ by Proposition
- $\exists\, \delta > 0 \ : \ B_\delta(y_0(T(x_0))) \cap E = \emptyset$
- $T(x) \leq C d_\Gamma(x) \quad \forall\, x \in B_\delta(y_0(T(x_0))) \cap \overline{\Omega}$

Proof of necessity: $T(x_j) \geq T(x_0)$

Suppose $\boxed{T(x_j) \geq T(x_0)}$ and let $\boxed{y^{x_0,u_0}(t)}$ be time-optimal
By dynamic programming

$$j \leq \frac{T(x_j) - T(x_0)}{|x_j - x_0|} \leq \frac{T\big(y^{x_j,u_0}(T(x_0))\big)}{|x_j - x_0|}$$

For large j we have that $y^{x_j,u_0}(T_0) \in \overline{\Omega} \cap B_\delta(y^{x_0,u_0}(T_0))$. So

$$j \leq \frac{C d_\Gamma\big(y^{x_j,u_0}(T(x_0))\big)}{|x_j - x_0|} \leq C\,\frac{\big|y^{x_j,u_0}(T(x_0)) - y^{x_0,u_0}(T(x_0))\big|}{|x_j - x_0|} \leq C$$

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Proof of necessity: $T(x_j) < T(x_0)$

Suppose now $\boxed{T(x_j) < T(x_0)}$ and let $\boxed{y^{x_j,u_j}(t)}$ be time-optimal

Without loss of generality we can assume that

$$y^{x_j,u_j} \longrightarrow y^{x_0,\bar{u}} \quad \text{uniformly on} \quad [0, T_0]$$

Then $y^{x_0,\bar{u}}$ is optimal and so $y^{x_0,\bar{u}}(T(x_0)) \notin E$

By dynamic programming

$$\begin{aligned} j \quad &\leq \quad \frac{T(x_0) - T(x_j)}{|x_j - x_0|} \leq \frac{T(y^{x_0,u_j}(T(x_j)))}{|x_j - x_0|} \\ &\leq \quad \frac{C d_\Gamma\left(y^{x_0,u_j}(T(x_j))\right)}{|x_j - x_0|} \leq C\,\frac{\left|y^{x_0,u_j}(T(x_j)) - y^{x_j,u_j}(T(x_j))\right|}{|x_j - x_0|} \leq C \end{aligned}$$

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Interior regularity

Theorem

Let $\{X_i\}_{i=1}^N$ be Hörmander v.f. Then the following properties are equivalent

1. *The minimum time problem has no singular optimal trajectory*
2. *T is locally semiconcave in Ω*
3. *T is locally Lipschitz in Ω*

Remark

The local semiconcavity of the sub-riemannian distance to a point x_0, in the absence of singular trajectories, was proved by C – Rifford (2008) without giving any estimate of the dependance on x_0 of the semiconcavity constant of $d_{SR}(x_0, \cdot)$. This does not imply the semiconcavity of $T(x) = \min_{y \in \Gamma} d_{SR}(y, x)$

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Boundary regularity

$f : \overline{\Omega} \to \mathbb{R}$ is Hölder of exponent $\alpha \in]0,1]$ at $x_0 \in \overline{\Omega}$ if $\exists$ a neighbourhood U of x_0 and $K \geq 0$ such that

$$|f(x) - f(x_0)| \leq K|x - x_0|^\alpha \qquad \forall x \in U \cap \overline{\Omega}$$

Theorem

Let $\{X_i\}_{i=1}^N$ be Hörmander v.f. Then

1. $x \in \Gamma \setminus E \quad \Longrightarrow \quad T \in C^\infty$ *on a neighborhood of x*
2. $x \in E \quad \Longrightarrow \quad T$ *Hölder of exponent $\frac{1}{k(x)}$ at x, with*

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Lipschitz singular sets

Define the Lipschitz singular set of T by

$$\mathrm{Sing}_{Lip}\, T = \left\{ x \in \Omega \ : \ \limsup_{\Omega \ni y \to x} \frac{|T(y) - T(x)|}{|y - x|} = \infty \right\}$$

and the Lipschitz singular support of T by

$$x \notin \mathrm{Sing\,supp}_{Lip}\, T \quad \Longleftrightarrow \quad \exists\, \Omega \overset{\text{open}}{\supset} U \ni x \ : \ T \in Lip(U)$$

Proposition (Properties of $\mathrm{Sing}_{Lip}\, T$)

(a) $\mathrm{Sing}_{Lip}\, T$ *is closed in* Ω

(b) T *is locally semiconcave in* $\Omega \setminus \mathrm{Sing}_{Lip}\, T$

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Proof

(a) $\mathrm{Sing}_{Lip}\, T$ is closed in Ω

Let $\mathrm{Sing}_{Lip}\, T \ni x_j \to x \in \Omega$ and let $y_j := y^{x_j, u_j}$ be a singular time-optimal trajectories. One can assume that $y_j \xrightarrow{\text{unif}} y$ optimal at x. Then

$$E \ni \lim_{j\to\infty} y_j(T(x_j)) = y(T(x)) \quad \Longrightarrow \quad y \text{ singular} \quad \Longrightarrow \quad x \in \mathrm{Sing}_{Lip}\, T$$

(b) T is locally semiconcave in $\Omega \setminus \mathrm{Sing}_{Lip}\, T$

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Singular support

Definition

For any $x \in \Omega$ and $k \in \mathbb{N} \cup \{\infty\}$ we say that

$$x \notin \operatorname{Sing\,supp}_{C^k} T \iff \exists\, \Omega \supset U \ni x \ : \ T \in C^k(U)$$

Theorem

Let $\{X_i\}_{i=1}^N$ be Hörmander v.f. Then

$$\operatorname{Sing\,supp}_{C^\infty} T = \operatorname{Sing\,supp}_{C^{1,1}} T =: \operatorname{Sing\,supp} T$$

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- there is concave paraboloid touching graph T at $(x, T(x))$
- T is of class C^2 in a neighborhood of any optimal trajectory starting at x [C–, Frankowska, and Scarinci (2015)] and [C– and Scarinci (2015)]

Characteristics can then be used to recover regularity of initial data

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Measure of Sing supp T

Theorem

Let $\{X_i\}_{i=1}^N$ be Hörmander v.f. Then $\mathcal{L}^n(\,\text{Sing supp } T) = 0$

Proof: We show that $\boxed{\mathcal{L}^n(\,\text{Sing supp}_{C^{1,1}} T) = 0}$ Indeed

$$\mathcal{L}^n(\,\text{Sing supp}_{C^{1,1}} T) = \underbrace{\mathcal{L}^n(\,\text{Sing}_{Lip} T)}_{=0 \text{ by K.T.Nguyen 2010}} + \mathcal{L}^n\Big(\text{Sing supp}_{C^{1,1}} T \setminus \text{Sing}_{Lip} T\Big)$$

T is locally semiconcave in $\Omega \setminus \text{Sing}_{Lip} T$. So

- T has second order Taylor expansion at a.e. $x \in \Omega \setminus \text{Sing}_{Lip} T$ by Alexandroff
- any such x belongs to $\Omega \setminus \text{Sing supp}_{C^{1,1}} T$

Therefore

$$\mathcal{L}^n\Big(\text{Sing supp}_{C^{1,1}} T \setminus \text{Sing}_{Lip} T\Big) = 0$$

Outline

Invariance for gradient flow

Denote by $\partial^D T(x)$ the Dini superdifferential of T at $x \in \Omega$, that is,

$$\partial^D T(x) = \left\{ p \in \mathbb{R}^n \ : \ \limsup_{\Omega \ni y \to x} \frac{T(y) - T(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}$$

Used in definition of viscosity solution together with Dini subdifferential $\partial_D T(x)$

Theorem

Let $\{X_i\}_{i=1}^N$ be Hörmander v.f. and suppose T is locally Lipschitz in Ω
Then Sing supp T is invariant for the generalized gradient flow

$$\dot{\gamma}(t) \in \sum_{j=1}^{N} \left\langle X_j(\gamma(t)), \partial^D T(\gamma(t)) \right\rangle X_j(\gamma(t)) \quad (t \geq 0)$$

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Homotopy equivalence

Corollary

Let $\{X_i\}_{i=1}^N$ be Hörmander v.f. and suppose T is locally Lipschitz in Ω
Then $\mathrm{Sing\,supp}\, T$ and Ω are homotopy equivalent

References

- Albano, C–, Nguyen, and Sinestrari (2013) invariance of $\mathrm{Sing}\, T$ for Riemannian systems and homotopy equivalence
- Albano (2016) invariance of $\mathrm{Sing\,supp}\, T$ for Riemannian systems and homotopy equivalence
- C–, Cheng, and Fathi (2017) singularities of weak KAM solutions, homotopy equivalence (with complement of Aubry set), local path-wise connectedness

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Outline

Hausdorff measure of $\mathrm{Sing}_{Lip}\, T$

Joint project with A. Marigonda and Khai T. Nguyen

Assume Ω satisfies a uniform Exterior Sphere Condition (ESC) of radius $\rho > 0$

$$\text{for all } \ x \in \Gamma \text{ there exists } \ B_\rho(y_x) \subset \mathbb{R}^n \setminus \Omega \ \text{ such that } \ x \in \overline{B}_\rho(y_x)$$

Equivalently, $\forall x \in \Gamma \; \exists p \in N^P_\Omega(x)$ with $|p| = \rho$

Define $T_0 = \max_{\overline{\Omega}} T$ and

$$\Gamma_t = \left\{ x \in \overline{\Omega} \ : \ T(x) = t \right\} \quad t \in [0, T_0]$$

Then we expect the following to be true:

- $\Gamma_t \cap \mathrm{Sing}_{Lip} T$ countably $(n-2)$-rectifiable
- $\mathrm{Sing}_{Lip} T$ has σ-finite $\mathcal{H}^{(n-1)}$ measure and

$$\mathcal{H} - \dim\left(\mathrm{Sing}_{Lip} T\right) \leqslant n - 1$$

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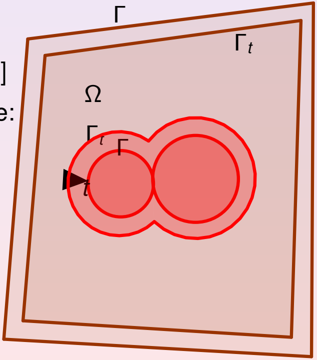

Outline

Extension to general targets

Joint project with H. Frankowska and V. Basco

Our goals

- To study minimum time function T for a system of (smooth) Hörmander verctor fields with a general (nosmooth) target (even a point)
- To recover a.e. differentiability of T
- To obtain necessary and sufficient conditions for Lipschitz continuity and semiconcavity of T
- To derive optimality conditions (sensitivity relations, optimal feedback)
- To study regularity of singular time optimal trajectories

What we have obtained so far

- Lipschitz continuity and semiconcavity in absence of singular trajectories
- Co-state inclusions when Ω satisfies uniform (ESC)
- Co-state inclusion in absence of singular trajectories

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- To obtain necessary and sufficient conditions for Lipschitz continuity and semiconcavity of T
- To derive optimality conditions (sensitivity relations, optimal feedback)
- To study regularity of singular time optimal trajectories

What we have obtained so far

- Lipschitz continuity and semiconcavity in absence of singular trajectories
- Co-state inclusions when Ω satisfies uniform (ESC)
- Co-state inclusion in absence of singular trajectories

Extension to general targets

Joint project with H. Frankowska and V. Basco

Our goals

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Thank you for your attention