# Minimal controllability time for the heat equation under unilateral state constraint

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# The Problem

Consider the 1-D heat equation

$$\begin{aligned}
\dot{y}(t,x) &= \partial_x^2 y(t,x) && (t \in \mathbb{R}_+^*,\ x \in (0,1)),\\
\partial_x y(t,0) &= v_0(t) && (t \in \mathbb{R}_+^*),\\
\partial_x y(t,1) &= v_1(t) && (t \in \mathbb{R}_+^*),
\end{aligned}$$

with initial condition $\mathrm{y}^0 \geqslant 0$, given,

$$y(0,x) = \mathrm{y}^0(x) \qquad (x \in (0,1)).$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

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It is well known that

- for every time $T > 0$ there exists controls $v_0$ and $v_1 \in L^2(0,T)$ such that $y(T,\cdot) = \mathrm{y}^1$
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Is it possible to find $T > 0$ and controls $v_0$ and $v_1$ such that $y$ satisfies $y(T,\cdot) = \mathrm{y}^1$ together with,

$$y(t,x) \geqslant 0 \qquad ((t,x) \in (0,1) \times (0,T) \ \text{ a.e.})\,?$$

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If $\inf\limits_{x \in (0,1)} \mathrm{y}^0(x) > \mathrm{y}^1$, then $\mathrm{y}^1$ cannot be reached in arbitrarily small time $T$.

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- finally,

$$y(t,\tfrac{1}{2}) > \mathrm{y}^1 \qquad \text{for } t \in \left[0, \frac{1}{\pi^2} \ln \frac{\inf \mathrm{y}^0}{\mathrm{y}^1}\right) .$$

## First considerations II

Due to the comparison principle, the constraint

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Consequently, we will first consider the control problem

$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) && (t \in \mathbb{R}_+^*\,,\ x \in (0,1))\,,\\ y(t,0) &= u_0(t) && (t \in \mathbb{R}_+^*)\,,\\ y(t,1) &= u_1(t) && (t \in \mathbb{R}_+^*)\,,\end{aligned}$$

with the control constraints

$$u_0(t) \geqslant 0 \quad \text{and} \quad u_1(t) \geqslant 0 \qquad (t \geqslant 0 \ \text{ a.e.}).$$

1. Controllability of the heat equation with nonnegative Dirichlet controls
   - Existence of nonnegative controls
   - Minimal controllability time
   - Numerical examples
2. Consequences for the 1-D heat equation with nonnegative state constraint
3. Conclusion and open problems

# The constrained Dirichlet control problem

Consider the 1-D heat equation

$$\dot{y}(t,x) = \partial_x^2 y(t,x) \qquad (t>0\,,\ x\in(0,1))\,, \tag{1a}$$
$$y(t,0) = u_0(t) \qquad (t>0)\,, \tag{1b}$$
$$y(t,1) = u_1(t) \qquad (t>0)\,, \tag{1c}$$

with constant initial condition $\mathrm{y}^0 \in L^2(0,1)$, given,

$$y(0,x) = \mathrm{y}^0(x) \qquad (x\in(0,1))\,.$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

$$y(T,x) = \mathrm{y}^1 \qquad (x\in[0,1] \text{ a.e.})\,,$$

with the control constraints

$$u_0(t) \geqslant 0 \quad \text{and} \quad u_1(t) \geqslant 0 \qquad (t>0 \ \text{ a.e.}).$$

# Existence of controls

## Proposition

*For every* $\mathrm{y}^0 \in L^2(0,1)$ *and every* $\mathrm{y}^1 \in \mathbb{R}_+^*$, *there exists a time* $T > 0$ *large enough and controls* $u_0, u_1 \in H^1(0,T)$ *such that*

$$u_0(t) > 0 \quad \text{and} \quad u_1(t) > 0 \qquad (t \in [0,T])$$

*and the solution* $y$ *of* (1) *satisfies*

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This allows us to define

$$\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) = \inf\left\{T > 0\,,\ \exists u_0, u_1 \in L^1(0,T) \text{ s.t. } u_0 \geqslant 0\,,\ u_1 \geqslant 0 \text{ and } y(T,\cdot) = \mathrm{y}^1\right\} \geqslant 0\,,$$

# Proof I

## Existence of controls

For the proof, we also refer to Schmidt 1980

Set

$$\tilde{y}(t,x) = y(t,x) - \mathrm{y}^1\,, \quad \tilde{u}_0(t) = u_0(t) - \mathrm{y}^1 \quad \text{and} \quad \tilde{u}_1 = u_1 - \mathrm{y}^1\,,$$

Then, $\tilde{y}$ is solution of (1) with controls $\tilde{u}_0$ and $\tilde{u}_1$ and initial condition

$$\tilde{y}(0,x) = \mathrm{y}^0(x) - \mathrm{y}^1 \qquad (x \in (0,1))\,.$$

Consequently, we aim to prove the existence of a time $T > 0$ and controls $\tilde{u}_0$ and $\tilde{u}_1$ satisfying,

$$\tilde{u}_0(t) > -\mathrm{y}^1 \quad \text{and} \quad \tilde{u}_1(t) > -\mathrm{y}^1 \qquad (t \in (0,T) \ \text{ a.e.})$$

such that

$$\tilde{y}(T,\cdot) = 0\,.$$

For any $T > 0$ the existence of controls $\tilde{u}_0, \tilde{u}_1 \in H^1(0,T)$ such that $\tilde{y}(T,\cdot) = 0$ is ensured by Fattorini-Russel 1971.

# Proof II
## Existence of controls

In terms of the adjoint system,

$$\begin{aligned} -\dot{z}(t,x) &= \partial_x^2 z(t,x) && (t>0\,,\ x\in(0,1))\,,\\ z(t,0) &= z(t,1)=0 && (t>0)\,,\\ z(T,x) &= \mathrm{z}^0(x) && (x\in(0,1))\,, \end{aligned}$$

there exists a constant $\tilde{c}(T)>0$ such that,

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \leqslant \tilde{c}(T)\left(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)}\right) \qquad (\mathrm{z}^0\in L^2(0,1))\,.$$

This inequality being true in any time interval, we also have

$$\|z(\tfrac{T}{2},\cdot)\|^2_{L^2(0,1)} \leqslant \tilde{c}(\tfrac{T}{2})\left(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)}\right)$$

Using the dissipativity properties,

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \leqslant e^{-C_0\frac{T}{2}}\|z(\tfrac{T}{2},\cdot)\|^2_{L^2(0,1)}\,,$$

# Proof III

## Existence of controls

we obtain

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \leqslant e^{-C_0\frac{T}{2}}\tilde{c}(\tfrac{T}{2})\left(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)}\right).$$

By duality this means that the controls $\tilde{u}_0$ and $\tilde{u}_1$ can be chosen such that

$$\|\tilde{u}_i\|^2_{H^1(0,T)} \leqslant e^{-C_0\frac{T}{2}}\tilde{c}(\tfrac{T}{2})\,\|\mathrm{y}^0 - \mathrm{y}^1\|^2_{L^2(0,1)} \qquad (i \in \{0,1\})$$

Using the embedding $H^1(0,T) \subset L^\infty(0,T)$,

$$\|\tilde{u}_i\|^2_{L^\infty(0,T)} \leqslant Ce^{-C_0\frac{T}{2}}\tilde{c}(\tfrac{T}{2})\,\|\mathrm{y}^0 - \mathrm{y}^1\|^2_{L^2(0,1)} \qquad (i \in \{0,1\})$$

Thus, for $T$ large enough,

$$\|\tilde{u}_0\|_{L^\infty(0,T)},\ \|\tilde{u}_1\|_{L^\infty(0,T)} < \mathrm{y}^1$$

and hence,

$$\tilde{u}_0(t) > -\mathrm{y}^1 \quad \text{and} \quad \tilde{u}_1(t) > -\mathrm{y}^1 \qquad (t \in [0,T] \text{ a.e.}).$$

# Minimal control time

## Theorem

*Let* $y_0 \in L^2(0,1)$ *and* $y_1 \in \mathbb{R}_+^*$ *with* $y_0 \neq y_1$. *Then,*

1. $\underline{T} := \underline{T}\left(y^0, y^1\right) > 0$,
2. *there exist nonnegative controls* $\underline{u}_0, \underline{u}_1 \in \mathcal{M}(0, \underline{T})$ *such that the solution* $y$ *with controls* $\underline{u}_0$ *and* $\underline{u}_1$ *satisfies* $y(T, \cdot) = y^1$.

The solution $y$, of the Dirichlet control problem with controls in the set of Radon measures, is defined by transposition.

## Remark

$\underline{T}\left(y^0, y^1\right) > 0$ *even if* $y^0 < y^1$.

# Proof of $\underline{T} > 0$ I

Define $y_n(t) = \displaystyle\int_0^1 y(t,x)\sin(n\pi x)\,\mathrm{d}x$, where $y$ is solution of (1). We have

$$\dot{y}_n(t) = \int_0^1 \partial_x^2 y(t,x)\sin(n\pi x)\,\mathrm{d}x = -n\pi\int_0^1 \partial_x y(t,x)\cos(n\pi x)\,\mathrm{d}x$$
$$= n\pi\left(u_0(t) - (-1)^n u_1(t)\right) - (n\pi)^2 y_n(t)$$

with $y_n(0) = \displaystyle\int_0^1 \mathrm{y}^0(x)\sin(n\pi x)\,\mathrm{d}x := \mathrm{y}_n^0$. Thus,

$$y_n(T) = e^{-(n\pi)^2 T}\mathrm{y}_n^0 + n\pi\int_0^T e^{-(n\pi)^2(T-t)}\left(u_0(t) - (-1)^n u_1(t)\right)\,\mathrm{d}t\,.$$

If $y(T,x) \equiv \mathrm{y}_1$, we have $y_n(T) = \displaystyle\int_0^1 \mathrm{y}_1\sin(n\pi x)\,\mathrm{d}x = \frac{1-(-1)^n}{n\pi}\mathrm{y}_1$.
Consequently,

$$\frac{1-(-1)^n}{n\pi}\mathrm{y}^1 - e^{-(n\pi)^2 T}\mathrm{y}_n^0 = n\pi\int_0^T e^{-(n\pi)^2(T-t)}\left(u_0(t) - (-1)^n u_1(t)\right)\,\mathrm{d}t\,.$$

# Proof of $\underline{T} > 0$ II

For $n = 2p$,

$$\int_0^T e^{(2p\pi)^2 t} \left(u_0(t) - u_1(t)\right) \mathrm{d}t = \frac{\mathrm{y}^0_{2p}}{2p\pi},$$

For $n = 2p+1$,

$$\frac{2\,\mathrm{y}^1}{(2p+1)\pi} - e^{-(2p+1)^2\pi^2 T}\,\mathrm{y}^0_{2p+1} = (2p+1)\pi \int_0^T e^{-(2p+1)^2\pi^2(T-t)} \left(u_0(t) + u_1(t)\right) \mathrm{d}t\,.$$

But,

$$e^{-(2p+1)^2\pi^2 T} \leqslant e^{-(2p+1)^2\pi^2(T-t)} \leqslant 1 \qquad (t \in [0, T])\,.$$

$u_0$ and $u_1$ being nonnegative,

$$\begin{aligned} e^{-(2p+1)^2\pi^2 T} \int_0^T \left(u_0(t) + u_1(t)\right) \mathrm{d}t \leqslant \int_0^T e^{-(2p+1)^2\pi^2(T-t)} \left(u_0(t) + u_1(t)\right) \mathrm{d}t \\ \leqslant \int_0^T \left(u_0(t) + u_1(t)\right) \mathrm{d}t\,, \end{aligned}$$

# Proof of $\underline{T} > 0$ III

We have obtained,

$$\frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - e^{-(2p+1)^2\pi^2 T}\,\frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} \leqslant \int_0^T (u_0(t)+u_1(t))\,\mathrm{d}t$$
$$\leqslant e^{(2p+1)^2\pi^2 T}\,\frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi}\,.$$

If for every $T > 0$ there exists nonnegative controls $u_0^T$ and $u_1^T$ steering $\mathrm{y}_0$ to $\mathrm{y}_1$ in time $T$, then

$$\lim_{T\to 0}\int_0^T \left(u_0^T(t)+u_1^T(t)\right)\,\mathrm{d}t = \frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} := \gamma \in \mathbb{R} \qquad (p\in\mathbb{N})\,.$$

Hence,

$$\mathrm{y}^0_{2p+1} = \frac{2\mathrm{y}^1}{(2p+1)\pi} - (2p+1)\pi\gamma \qquad (p\in\mathbb{N})\,.$$

$\mathrm{y}^0 \in L^2(0,1)$, ensures that $\displaystyle\sum_{n=0}^{\infty}\left|\mathrm{y}^0_n\right|^2 < \infty$ and hence $\gamma = 0$, $\mathrm{y}^0_{2p+1} = \dfrac{2\mathrm{y}^1}{(2p+1)\pi}$ and

$$\lim_{T\to 0}\int_0^T \left(u_0^T(t)+u_1^T(t)\right)\,\mathrm{d}t = 0\,.$$

# Proof of $\underline{T} > 0$ IV

Since $u_0^T \geqslant 0$ and $u_1^T \geqslant 0$, we can also conclude

$$\lim_{T\to 0} \int_0^T u_0^T(t)\,\mathrm{d}t = \lim_{T\to 0} \int_0^T u_1(t)\,\mathrm{d}t = 0\,.$$

consequently passing to the limit $T \to 0$ in

$$\int_0^T e^{(2p\pi)^2 t} \left( u_0^T(t) - u_1^T(t) \right) \mathrm{d}t = \frac{\mathrm{y}^0_{2p}}{2p\pi}\,,$$

we obtain

$$\mathrm{y}^0_{2p} = 0 \qquad (p \in \mathbb{N}^*)\,.$$

All in all, since the family $\left\{ \sqrt{2}\sin(n\pi\,\cdot) \right\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(0,1)$, we conclude that $\mathrm{y}^0$ can be steered to $\mathrm{y}^1$ in arbitrarily small time with nonnegative controls if and only if

$$\mathrm{y}^0(x) = \mathrm{y}^1 \qquad (x \in (0,1))\,.$$

# Proof of Controllability in the minimal time $\underline{T}$ I

Define $(\varepsilon_k)_{k\in\mathbb{N}}$ a sequence of positive numbers converging to 0.
For every $k \in \mathbb{N}$, there exist nonnegative controls $u_0^k, u_1^k \in L^1(0, \underline{T} + \varepsilon_k)$, so that the solution $y$ satisfies $y(\underline{T} + \varepsilon_k, \cdot) = \mathrm{y}^1$.
Define $\bar{\varepsilon} = \sup_{k\in\mathbb{N}} \varepsilon_k$.

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Define $\bar{\varepsilon} = \sup\limits_{k\in\mathbb{N}} \varepsilon_k$.

According to

$$\frac{2\,\mathrm{y}^1}{(2p+1)\pi} - e^{-(2p+1)^2\pi^2 T}\,\mathrm{y}^0_{2p+1} = (2p+1)\pi \int_0^T e^{-(2p+1)^2\pi^2(T-t)} \left(u_0^k(t) + u_1^k(t)\right) \mathrm{d}t\,,$$

we obtain,

$$\|u_0^k\|_{L^1(0,\underline{T}+\bar{\varepsilon})} + \|u_1^k\|_{L^1(0,\underline{T}+\bar{\varepsilon})} = \int_0^{\underline{T}+\varepsilon_k} \left(u_0^k(t) + u_1^k(t)\right) \mathrm{d}t$$

$$\leqslant \inf_{p\in\mathbb{N}} \left( e^{(2p+1)^2\pi^2(\underline{T}+\varepsilon_k)} \frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} \right)$$

$$\leqslant \frac{2e^{\pi^2(\underline{T}+\bar{\varepsilon})}\,|\mathrm{y}^1|}{\pi^2} + \frac{|\mathrm{y}^0_1|}{\pi} \leqslant \infty\,.$$

# Proof of Controllability in the minimal time $\underline{T}$ II

In conclusion,

- The sequences $(u_0^k)_k$ and $(u_1^k)_k$ are bounded in $L^1(0, \underline{T} + \bar{\varepsilon})$;
- $(u_0^k)_k$ and $(u_1^k)_k$ have their support contained in $[0, \underline{T} + \varepsilon_k]$, with $\varepsilon_k \to 0$;
- Thus, they are (up to a subsequence) weakly convergent in the sense of measures to some nonnegative controls $\underline{u}_i$ in $\mathcal{M}([0, \underline{T}])$;
- These limits ensure the control requirements in the minimal control time $\underline{T}$.

$\square$

# Lower bounds on $\underline{T}$

When $\mathrm{y}^0$ is a constant initial condition, $\underline{T} := \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ satisfies

1. if $\mathrm{y}^1 < \mathrm{y}^0$,

$$\underline{T} > \frac{1}{\pi^2} \log \frac{\mathrm{y}^0}{\mathrm{y}^1} \qquad \text{and} \qquad \sup_{p \in \mathbb{N}^*} \frac{1}{(2p+1)^2} \left( \frac{\mathrm{y}^1}{\mathrm{y}^0} - e^{-(2p+1)^2 \pi^2 \underline{T}} \right) \leqslant \frac{\mathrm{y}^1}{\mathrm{y}^0} e^{\pi^2 \underline{T}} - 1 \,.$$

For $\mathrm{y}^1 \equiv 1$ and $\mathrm{y}^0 \equiv 5$, we obtain (numerically): $\underline{T} \geqslant 0.165297$;

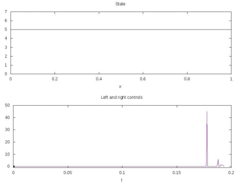
2. if $\mathrm{y}^1 > \mathrm{y}^0$,

$$\frac{\mathrm{y}^1}{\mathrm{y}^0} - e^{-\pi^2 \underline{T}} \leqslant \inf_{p \in \mathbb{N}^*} \frac{1}{(2p+1)^2} \left( \frac{\mathrm{y}^1}{\mathrm{y}^0} e^{(2p+1)^2 \pi^2 \underline{T}} - 1 \right) \,.$$

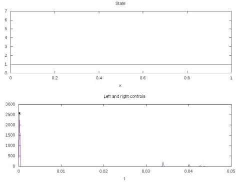
For $\mathrm{y}^1 \equiv 5$ and $\mathrm{y}^0 \equiv 1$, we obtain (numerically): $\underline{T} \geqslant 0.023076$;

# Numerical examples

- From $\mathrm{y}^0 \equiv 5$ to $\mathrm{y}^1 \equiv 1$, $\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) \simeq 0.1931$.




- From $\mathrm{y}^0 \equiv 1$ to $\mathrm{y}^1 \equiv 5$, $\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) \simeq 0.0438$.



# Heat equation with nonnegative state constraint I

Consider the 1-D heat equation

$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) + \mathbf{1}_\omega(x) w(t,x) && (t>0\,,\ x\in(0,1))\,,\\ \partial_x y(t,0) &= v_0(t) && (t>0)\,,\\ \partial_x y(t,1) &= v_1(t) && (t>0)\,, \end{aligned}$$

with given initial condition $\mathrm{y}^0 \geqslant 0$,

$$y(0,\cdot) = \mathrm{y}^0 \in L^2(0,1)\,.$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

$$y(T,x) = \mathrm{y}^1 \qquad (x \in (0,1) \text{ a.e.})\,,$$

with the state constraint,

$$y(t,x) \geqslant 0 \qquad (t \geqslant 0\,,\ x\in(0,1)\ \text{ a.e.}).$$

We assume $\omega \subset (0,1)$ is such that there exists an interval $(a,b) \subset (0,1) \setminus \omega$.

## Heat equation with nonnegative state constraint II

For $v_0, v_1 \in L^2(0,T)$ and $w \in L^2((0,T)\times\omega)$, define

$$u_a = y(\cdot, a) \in L^2(0,T) \quad \text{and} \quad u_b = y(\cdot, b) \in L^2(0,T)\,.$$

Furthermore, $y|_{(a,b)}$ is solution of

$$\begin{aligned}
&\dot{y}(t,x) = \partial_x^2 y(t,x) && (t>0\,,\ x\in(a,b))\,,\\
&y(t,a) = u_a(t) && (t>0)\,,\\
&y(t,b) = u_b(t) && (t>0)\,,
\end{aligned}$$

Consequently, if $v_0, v_1$ and $w$ are controls in time $T>0$ such that

$$y(t,x) \geqslant 0 \qquad \text{and} \qquad y(T,x) = \mathrm{y}^1\,,$$

then we have

$$u_a(t) \geqslant 0 \quad \text{and} \quad u_b(t) \geqslant 0 \qquad (t\in[0,T] \ \text{a.e.})$$

and hence $T$ cannot be arbitrarily small unless $\mathrm{y}^0|_{(0,1)\setminus\omega} = \mathrm{y}^1|_{(0,1)\setminus\omega}$.

# Numerical example I

Consider the 1-D heat equation with Neumann controls

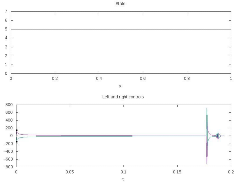
$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) && (t>0\,,\ x\in(0,1))\,, \\ \partial_x y(t,0) &= v_0(t) && (t>0)\,, \\ \partial_x y(t,1) &= v_1(t) && (t>0)\,, \end{aligned}$$

with the state constraint,

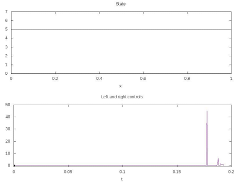
$$y(t,x) \geqslant 0 \qquad (t \geqslant 0\,,\ x\in(0,1)\ \text{ a.e.}).$$

# Numerical example II

- From $\mathrm{y}^0 \equiv 5$ to $\mathrm{y}^1 \equiv 1$, $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) \simeq 0.1938$.




Remind that with Dirichlet controls, we had,



Our proofs are based on spectral decomposition and this can be used to prove similar results for:

- Controllability to any kind of steady state;
- Parabolic equation of the form $\dot{y} = \partial_x \left(a(x)\partial_x y\right) - p(x)\partial_x y$ with internal and/or boundary control;
- $n$-D heat equations with constant coefficients;
- Finite dimensional versions of the heat equation.

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But cannot be used for

- Non-linear heat equations;
- Linear heat equation with time and space dependent parameters.

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Some over open questions

- Structure and uniqueness of the nonnegative Dirichlet controls in the minimal time $\underline{T}$?
- How the time optimal control is related to the adjoint state?
  Numerical examples: $1 \to 5$ $5 \to 1$

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THANK YOU FOR YOUR ATTENTION!