Inverse obstacle problem with partial Cauchy data: a shape optimization approach.

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VII Partial differential equations, optimal design and numerics
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Inverse obstacle problem

- **The data:**
  - \( \mathcal{D} \) open set of \( \mathbb{R}^d \) (\( d \geq 2 \)), with Lipschitz boundary
  - \( \Gamma \subset \partial \mathcal{D}, \ |\Gamma| > 0, \ \Gamma_c := \partial \mathcal{D} \setminus \Gamma \)
  - \((g_D, g_N)\): (possibly noisy) Cauchy data, \((g_D, g_N) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\).

- **Problem:**
  Find an inclusion \( \mathcal{O}, \ \overline{\mathcal{O}} \subset \mathcal{D}, \ \Omega := \mathcal{D} \setminus \overline{\mathcal{O}} \) connected, and \( u \in H^1(\Omega) \), s.t.

\[
(\mathcal{P}) \begin{cases} 
\Delta u = 0 & \text{in } \Omega \\
u = g_D & \text{on } \Gamma \\
\partial_n u = g_N & \text{on } \Gamma \\
u = 0 & \text{on } \partial \mathcal{O}
\end{cases}
\]
Inverse problems: typical questions

• Classical questions in context of inverse problems:

1) Identifiability - there exists at most one couple \((\mathcal{O}, u)\) solution of \((\mathcal{P})\).


3) Reconstruction.
Reconstruction methods (non-exhaustive list)


• Exterior approach, based on the Quasi-reversibility method: \textit{A quasi-reversibility approach to solve the inverse obstacle problem}, L. Bourgeois and J.D., Inverse problems and Imaging (2010).


\[ \Rightarrow \] detection of obstacles in fluids:


\[ \Rightarrow \] ...
Reconstruction methods (non-exhaustive list)


  \[ \rightsquigarrow \text{detection of obstacles in fluids:} \]


- ...
Shape optimization for inverse obstacle problems: general strategy

1. Initial situation
2. choose an arbitrary open set $\omega \Subset D$
3. compute $u$ solving the direct problem
   \[
   \begin{align*}
   \Delta u_\omega &= 0 \text{ in } D \setminus \overline{\omega} \\
   \partial_\nu u_\omega &= g_N \text{ on } \partial D \\
   u_\omega &= 0 \text{ on } \partial \omega
   \end{align*}
   \]
4. compute $J(\omega) = \int_{\partial D} (g_D - u_\omega)^2 \, ds \rightarrow$ if zero, ok, if not, compute the shape derivative of $J$ w.r.t. $\omega \rightarrow$ gradient algorithm.
Initial situation

choose an arbitrary open set $\omega \subseteq D$

compute $u$ solving the direct problem

$$\begin{cases} 
\Delta u_\omega = 0 \text{ in } D \setminus \overline{\omega} \\
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4. compute $J(\omega) = \int_{\partial D} (g_D - u_\omega)^2 \, ds \to$ if zero, ok, if not, compute the shape derivative of $J$ w.r.t. $\omega \to$ gradient algorithm.
In our computation, we will minimize a Kohn-Vogelius functional:

\[
\min_\omega \mathcal{K}(\omega) := \int_{\Omega^c} |\nabla (u^D_\omega - u^N_\omega)|^2 \, dx
\]

where \( u^D_\omega, u^N_\omega \) solve

\[
\begin{align*}
\Delta u^D_\omega &= 0 \text{ in } \Omega^c \\
u^D_\omega &= g_D \text{ on } \partial \Omega \\
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\end{align*}
\]

\[
\begin{align*}
\Delta u^N_\omega &= 0 \text{ in } \Omega^c \\
\partial_n u^N_\omega &= g_N \text{ on } \partial \Omega \\
\partial_n u^N_\omega &= 0 \text{ on } \partial \omega
\end{align*}
\]

Advantages:

- \((g_D, g_N)\) are treated symmetrically
- only volumic quantities
- numerically: better reconstructions.

Still a severely ill-posed problem! Regularization?
Example of reconstruction

An example of reconstruction with noisy data.
Incomplete data

- The whole strategy is possible only if the data \((g_D, g_N)\) are available on the whole boundary of the domain \(D\) (at least one of them).

- But in lots of practical applications, some parts of the boundary are unaccessible → no measurements on them (particularly true for fluid problems).

- ⇒ the whole strategy fails.

- Main objective: propose a *shape optimization* strategy to reconstruct the unknown inclusion when only Cauchy data are available *only on a subpart of the boundary of the domain of study*.

- Clearly, we have to reconstruct both \(\omega\) and the missing data → *data completion problem*. 

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Data completion problem

• **The data:**
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  - \( (g_D, g_N) \): (possibly noisy) Cauchy data, \( (g_D, g_N) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \).

• **Problem:** find \( u \in H^1(\Omega) \), s.t. \( (\mathcal{P}_c) \)
  \[
  \begin{cases}
  \Delta u = 0 & \text{in } \mathcal{D} \\ 
  u = g_D & \text{on } \Gamma \\ 
  \partial_n u = g_N & \text{on } \Gamma 
  \end{cases}
  \]

• **Problem** is severely ill-posed (exponentially ill-posed), it has at most one solution that does not depend continuously on the data.
In particular, the set of data for which the problem has no solution is dense in \( H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \) \( \Rightarrow \) high instability \( \Rightarrow \) it is mandatory to propose a regularization method to solve the problem numerically.

• In the sequel, we denote by \( u_{ex} \) the exact solution corresponding to exact data \( (g_D, g_N) \).
Data completion problem

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- **Problem:** find $u \in H^1(\Omega)$, s.t. $\left\{ \begin{array}{l}
\Delta u = 0 \quad \text{in } \mathcal{D} \\
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\end{array} \right.$

- This problem is severely ill-posed (exponentially ill-posed), it has at most one solution that does not depend continuously on the data.
  In particular, the set of data for which the problem has no solution is dense in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ $\Rightarrow$ high instability $\Rightarrow$ it is mandatory to propose a regularization method to solve the problem numerically.

- In the sequel, we denote by $u_{ex}$ the exact solution corresponding to exact data $(g_D, g_N)$. 
Kohn-Vogelius minimization strategy

- Main idea: minimize the energy functional

\[
    \mathcal{K}(\varphi, \psi) := \frac{1}{2} \int_{\mathcal{D}} |\nabla (u_\varphi - u_\psi)|^2 \, dx
\]

over all \((\varphi, \psi) \in H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)\), where \(u_\varphi\) and \(u_\psi\) verify

\[
\begin{align*}
\Delta u_\varphi &= 0 \text{ in } \mathcal{D} \\
\partial_\nu u_\varphi &= \varphi \text{ on } \Gamma_c \\
\partial_\nu u_\psi &= \psi \text{ on } \Gamma_c
\end{align*}
\]

and

\[
\begin{align*}
\Delta u_\psi &= 0 \text{ in } \mathcal{D} \\
\partial_\nu u_\psi &= g_N \text{ on } \Gamma \\
u_\psi &= \psi \text{ on } \Gamma_c.
\end{align*}
\]

- Easy remark: \(\mathcal{K}(\varphi, \psi) = 0 \iff u_\varphi = u_{ex} = u_\psi + \text{cte.}\)

**Property**

\[
\inf_{H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)} \mathcal{K}(\varphi, \psi) = 0
\]
Regularization of the K-V functional

- Regularized Kohn-Vogelius functional: for $\varepsilon > 0$, for $(\varphi, \psi) \in H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)$,

$$
K_\varepsilon (\varphi, \psi) = K(\varphi, \psi) + \frac{\varepsilon}{2} \left( \|v_\varphi\|_{H^1(\Omega)}^2 + \|v_\psi\|_{H^1(\Omega)}^2 \right).
$$

with

$$
\begin{align*}
\Delta v_\varphi &= 0 \quad \text{in } D \\
v_\varphi &= 0 \quad \text{on } \Gamma \\
\partial_\nu v_\varphi &= \varphi \quad \text{on } \Gamma_c
\end{align*}
$$

and

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\Delta v_\psi &= 0 \quad \text{in } D \\
\partial_\nu v_\psi &= 0 \quad \text{on } \Gamma \\
v_\psi &= \psi \quad \text{on } \Gamma_c
\end{align*}
$$

**Property**

There exists a unique $(\varphi_\varepsilon, \psi_\varepsilon) \in H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)$ s.t.

$$
K_\varepsilon (\varphi_\varepsilon, \psi_\varepsilon) = \arg\min_{(\varphi, \psi) \in H^{-1/2}(\Gamma)} K_\varepsilon (\varphi, \psi).
$$
Convergence results

Property

The sequence \((\varphi_\varepsilon, \psi_\varepsilon)\) is a minimizing sequence for \(\mathcal{K}\).

Theorem

Suppose \((\mathcal{P}_c)\) admits a (necessarily unique) solution \(u_{ex}\). Then \((\varphi_\varepsilon, \psi_\varepsilon)\) converges to \((\partial_{\nu} u_{ex}, u_{ex} + \text{cte})\) \(\Leftrightarrow u_{\varphi_\varepsilon} \xrightarrow{\varepsilon \to 0} u_{ex}\).

Furthermore, the convergence is monotonic: the map \(\varepsilon \mapsto \|u_{\varphi_\varepsilon} - u_{ex}, u_{\psi_\varepsilon} - u_{ex}\|_{H^1(\Omega) \times H^1(\Omega)}\) is strictly increasing.

Suppose \((\mathcal{P}_c)\) does not admit a solution. Then

\[\lim_{\varepsilon \to 0} \|\varphi_\varepsilon, \psi_\varepsilon\|_{H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)} = +\infty.\]

• It is mandatory to propose a strategy to deal with noisy data.
Derivatives of $\mathcal{K}_\varepsilon$

- We define $w_N, w_D \in H^1(\mathcal{D})$ solutions of

\[
\begin{align*}
\Delta w_N &= \varepsilon v_\psi & \text{in } \mathcal{D} \\
\partial_\nu w_N &= \partial_\nu u_\varphi - g_N & \text{on } \Gamma \\
w_N &= 0 & \text{on } \Gamma_c,
\end{align*}
\]

\[
\begin{align*}
\Delta w_D &= \varepsilon v_\varphi & \text{in } \mathcal{D} \\
w_D &= u_\psi - g_D & \text{on } \Gamma \\
\partial_\nu w_D &= 0 & \text{on } \Gamma_c.
\end{align*}
\]

Property

For all $(\varphi, \psi), (\tilde{\varphi}, \tilde{\psi})$ in $H^{-1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)$, we have

\[
\frac{\partial \mathcal{K}_\varepsilon}{\partial \varphi}(\varphi, \psi)[\tilde{\varphi}] = \langle \tilde{\varphi}, u_\varphi + \varepsilon v_\varphi + w_D - \psi \rangle_{\Gamma_c}
\]

and

\[
\frac{\partial \mathcal{K}_\varepsilon}{\partial \psi}(\varphi, \psi)[\tilde{\psi}] = \langle \partial_\nu u_\psi + \varepsilon \partial_\nu v_\psi + \partial_\nu w_N - \varphi, \tilde{\psi} \rangle_{\Gamma_c}.
\]
Inverse obstacle problem with partial Cauchy data

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- **Kohn-Vogelius strategy:** minimization of the regularized Kohn-Vogelius functional w.r.t to $\omega, \varphi$ and $\psi$.

\[
\mathcal{K}_\varepsilon(\omega, \varphi, \psi) := \int_{\mathcal{D} \setminus \overline{\omega}} |\nabla (u_\varphi - u_\psi)|^2 \, dx + \frac{\varepsilon}{2} \left( \|v_\varphi\|_{H^1(\mathcal{D} \setminus \overline{\omega})}^2 + \|v_\psi\|_{H^1(\mathcal{D} \setminus \overline{\omega})}^2 \right).
\]

→ Existence of a minimizer?
Computation of the shape derivative

- As usual, for $\mathbf{V} \in W^{2,\infty}(\mathbb{R}^d)$, compactly supported in $D$, we note

$$DK_\varepsilon(\omega) := \lim_{t \to 0} \frac{\mathcal{K}_{\varepsilon}((I + t\mathbf{V})\omega) - \mathcal{K}_{\varepsilon}(\omega)}{t}.$$ 

**Property**

We have

$$DK_\varepsilon(\omega) \cdot \mathbf{V} = -\int_{\partial \omega} (\partial_{\nu} \rho^u_D \partial_{\nu} u_\varphi + \partial_{\nu} \rho^\gamma_D \partial_{\nu} v_\varphi)(\mathbf{V} \cdot \nu)$$

$$- \int_{\partial \omega} (\partial_{\nu} \rho^u_D \partial_{\nu} u_\varphi + \partial_{\nu} \rho^\gamma_D \partial_{\nu} v_\psi)(\mathbf{V} \cdot \nu)$$

$$+ \frac{1}{2} \int_{\partial \omega} |\nabla(u_\varphi - u_\psi)|^2(\mathbf{V} \cdot \nu)$$

$$+ \frac{\varepsilon}{2} \int_{\partial \omega} \left(|\nabla v_\varphi|^2 + |\nabla v_\psi|^2 + |v_\varphi|^2 + |v_\psi|^2\right)(\mathbf{V} \cdot \nu)$$
Global algorithm

- choose an initial guess \((ω_0, ϕ_0, ψ_0)\)

- at step \(n\),

1. solve 10 (!) elliptic problems in \(D \setminus \overline{ω_n}\) to obtain \(u_ϕ_n, u_ψ_n, v_ϕ_n, v_ψ_n, w_N, w_D, ρ_D^u, ρ_N^u, ρ_D^v\) and \(ρ_N^v\)
2. compute the descent directions \(\tilde{ϕ}\) and \(\tilde{ψ}\)
3. compute the \(∇K_ε(ω_n)\)
4. update \(ϕ_n, ψ_n, ω_n\) (line search) \(→ ϕ_{n+1}, ψ_{n+1}, ω_{n+1}\).

- repeat until stopping criterion is reached.
Reconstructions - easy case

![Graph showing initial shape, real shape, and algorithm output with labels for each curve.]

Jéremi Dardé (IMT-Université Toulouse 3)  Inv. Obst. Prob. with Partial Cauchy Data  Benasque 2017 17 / 19
Reconstructions - hard case

![Graph showing initial shape, real shape, full data algorithm output, and partial data algorithm output.]

- initial shape
- real shape
- full data algorithm output
- partial data algorithm output
Future works

• In case of noisy data, propose a strategy to set the parameter of regularization w.r.t. noise amplitude
  → for the data completion problem, ok
  → for the inverse obstacle problem, ?.

• Reconstruction of objects in fluids (Stokes and Navier-Stokes equations)
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Thank you for your attention