Asymptotic stabilization of a 2x2 hyperbolic system in BV space

J.-M. Coron, S. Ervedoza†, S.S. Ghoshal,
O. Glass and V. Perrollaz

† Institut de Mathématiques de Toulouse

Benasque 2017
23/08/2017

Outline

Introduction – General setting

- Stabilization issues for one-dimensional hyperbolic systems of conservation laws:

$$\partial_t u + \partial_x(f(u)) = 0, \quad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \qquad \text{(SCL)}$$

satisfying the (strict) hyperbolicity condition that at each point

Strict hyperbolicity

df has n distinct real eigenvalues $\lambda_1 < \cdots < \lambda_n$.

- Typical examples: compressible fluid flows, fluid through a canal, traffic flow, etc.

Characteristic fields

- Corresponding to the characteristic speeds $\lambda_1 < \cdots < \lambda_n$, the Jacobian $A(u) := df(u)$ has n right eigenvectors $r_i(u)$.

- We denote $(\ell_i)_{i=1,\dots,n}$ the left eigenvectors of $df(u)$ satisfying $\ell_i \cdot r_j = \delta_{ij}$.

- The characteristic families will be supposed to be genuinely non-linear (GNL), that is:

$$\nabla\lambda_i \cdot r_i \neq 0 \quad \text{for all } u \text{ in } \Omega.$$

$\rightsquigarrow$ Convention: $\nabla\lambda_i \cdot r_i > 0$.

Boundary conditions

System of conservation laws in a bounded interval $(0, L)$:

$$\partial_t u + \partial_x(f(u)) = 0, \quad t \geq 0, x \in (0, L), \tag{SCL}$$

$\rightarrow$ Has to be completed with suitable boundary conditions.

- We suppose moreover that the characteristic speeds are stricly separated from 0:

$$\lambda_1 < \cdots < \lambda_m < 0 < \lambda_{m+1} < \cdots < \lambda_n.$$

- We will be interested in boundary conditions put in the following form:

$$\begin{pmatrix} u_+(t, 0) \\ u_-(t, L) \end{pmatrix} = G \begin{pmatrix} u_+(t, L) \\ u_-(t, 0) \end{pmatrix}$$

with

$$u_+ := (u_{m+1}, \dots, u_n) \text{ and } u_- := (u_1, \dots, u_m).$$

Stabilization problem

- We consider an equilibrium point $\overline{u}$ of the system. To simplify, we fix $\overline{u} = 0$ and $G(0) = 0$.
- The question is to design boundary conditions, i.e. G so that $\overline{u}$ becomes an asymptotically stable point for the resulting closed-loop system.

- We recall that a point $\overline{u}$ is called stable when for any neighborhood $\mathcal{V}$ of $\overline{u}$, there exists a neighborhood $\mathcal{U}$ of $\overline{u}$ such that any trajectory of the system starting from $\overline{u}$ stays in $\mathcal{V}$ for all $t \geq 0$.
- It is called asymptotically stable when moreover any trajectory starting from $\mathcal{U}$ satisfies $u(t, \cdot) \to \overline{u}$ as $t \to +\infty$.

Stabilization problem

- A point $\overline{u} = 0$ is called exponentially stable when any trajectory starting from some neighborhood $\mathcal{U}$ of $\overline{u} = 0$ satisfies

$$\|u(t,\cdot)\| \leq C \exp(-\gamma t)\|u(0,\cdot)\| \quad \text{for all } t \geq 0,$$

for some fixed $\gamma > 0$ and $C > 0$.

Careful...

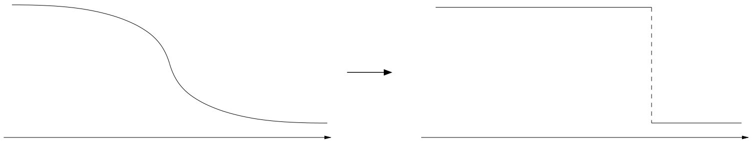
Stabilization properties may depend on the functional setting under consideration !

On the functional setting – Appearance of shocks

When considering the Burger's equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0, \quad t > 0, x \in \mathbb{R},$$

solutions with smooth initial data may develop singularities in finite time:



$\Rightarrow$ 2 possible functional settings:

- Smooth functions (e.g. C^1 or H^2) with small norms;
- Discontinuous functions, corresponding to weak solutions.

Weak solutions

- Weak solutions can account for shock waves.
- In the context of weak solutions, uniqueness holds provided we consider entropy conditions.
- We thus consider bounded variation functions, with small total variation in x ("à la Glimm").

Entropy solutions

Definition

An entropy/entropy flux couple for a hyperbolic system of conservation laws (SCL) is defined as a couple of regular functions $(\eta, q) : \Omega \to \mathbb{R}$ satisfying:

$$\forall u \in \Omega, \quad D\eta(u) \cdot Df(u) = Dq(u).$$

Definition

A function $u \in L^\infty(0, T; BV(0, L)) \cap \mathrm{Lip}(0, T; L^1(0, L))$ is called an entropy solution of (SCL) when, for any entropy/entropy flux couple (η, q), with η convex, one has in the sense of measures

$$\partial_t(\eta(u)) + \partial_x(q(u)) \leq 0.$$

Entropy conditions, 2

- Of course $(\eta, q) = (\pm \mathrm{Id}, \pm f)$ are entropy/entropy flux couples. So entropy solutions are particular cases of weak solutions.

- The entropy inequalities are automatically satisfied by vanishing viscosity limits:

$$u^\varepsilon \to u \text{ with } \ \partial_t u^\varepsilon + \partial_x(f(u^\varepsilon)) - \varepsilon \partial_{xx} u^\varepsilon = 0.$$

- Glimm (1965) showed the existence of global entropy solutions with the assumption of small total variation, that is when $\partial_x u_0$ is small in the space of bounded measures.

References on stabilization in the context of classical solutions

- Slemrod, Greenberg-Li, ...
- Bastin-Coron, Bastin-Coron-d'Andrea-Novel, Bastin-Coron-d'Andrea-Novel-de Halleux-Prieur, Bastin-Coron-Krstic-Vazquez, ...
- Leugering-Schmidt, Dick-Gugat-Leugering, Gugat-Herty,...
- Ta-Tsien Li, Tie Hu Qin, ...
- Many others! $\rightsquigarrow$ See the recent book of Bastin-Coron.

The stabilization of (SCL) indeed depends on the functional setting at hand !
$\rightsquigarrow$ Coron-Nguyen 2015.

In the context of entropy solutions

- Scalar cases:
  - Ancona and Marson (1998), (reachable set)
  - Horsin (1998), (reachable set)
  - Perrollaz (2011), (Stabilization)
  - Adimurthi-Gowda-Ghoshal (2013), (reachable set)
  - Andreianov-Donadello-Marson (2015), (reachable set)
- Several works on the system case:
  - Bressan-Coclite (asymptotic result and a counterexample, 2002),
  - Ancona-Coclite (Temple systems, 2005, reachable set),
  - Ancona-Marson (one-side open loop stabilization, 2007),
  - Glass (Euler equations, 2007, 2014),
  - Andreianov-Donadello-Ghoshal-Razafison (2015, triangular system),
  - Coron-E.-Glass.-Ghoshal-Perrollaz (2017).

A simple framework

- Here we consider 2×2 systems of conservation laws:

$$\partial_t u + \partial_x(f(u)) = 0 \ \text{ in } \ [0, +\infty) \times [0, L],$$

with characteristic speeds $\lambda_1 < \lambda_2$ and satisfying the conditions:

  - each characteristic field is genuinely non-linear,
  - velocities are positive: $0 < \lambda_1 < \lambda_2$.

- The boundary conditions are as follows:

$$u(t, 0) = Ku(t, L),$$

where K is a 2×2 (real) matrix.

- The goal is to find conditions on K ensuring the (exponential) stability of the system.

Main result

Theorem [Coron-E.-Glass-Ghoshal-Perrollaz 2017]

Suppose the above assumptions satisfied. If K satisfies

$$\inf_{\alpha\in(0,+\infty)} \Big(\max\big\{|\ell_1(0)\cdot Kr_1(0)| + \alpha|\ell_2(0)\cdot Kr_1(0)|,$$
$$\alpha^{-1}|\ell_1(0)\cdot Kr_2(0)| + |\ell_2(0)\cdot Kr_2(0)|\big\}\Big) < 1,$$

$\exists$ positive constants C, ν, $\varepsilon_0 > 0$, such that $\forall u_0 \in BV(0,L)$ satisfying

$$|u_0|_{BV} \leq \varepsilon_0,$$

$\exists$ an entropy solution u in $L^\infty(0,\infty; BV(0,L))$ satisfying $u(0,\cdot) = u_0(\cdot)$, and the boundary conditions for almost all times, s.t.

$$|u(t)|_{BV} \leq C\exp(-\nu t)|u_0|_{BV}, \qquad t \geq 0.$$

Rewriting the condition

Denoting for $p \in [1, \infty)$

$$\|(x_1, \ldots, x_n)\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad \|(x_1, \ldots, x_n)\|_\infty := \max_{i=1\ldots n} |x_i|$$

$$\|M\|_p := \max_{\|x\|_p = 1} \|Mx\|_p \ \text{ for } \ M \in R^{n \times n},$$

one defines

$$\rho_p(K) := \inf\{\|\Delta K \Delta^{-1}\|_p, \ \Delta \text{ diagonal with positive entries}\}.$$

It is easy to check that

$$\inf_{\alpha \in (0,+\infty)} \big(\max\big\{|\ell_1(0)\cdot Kr_1(0)| + \alpha|\ell_2(0)\cdot Kr_1(0)|, \\ \alpha^{-1}|\ell_1(0)\cdot Kr_2(0)| + |\ell_2(0)\cdot Kr_2(0)|\big\}\big) = \rho_1(K),$$

so that the condition can be written as $\rho_1(K) < 1$.

Analogous conditions

- For the same question for classical solutions in C^m-norm ($m \geq 1$), a sufficient condition is:
$$\rho_\infty(K) < 1.$$
Cf. T. H. Qin, Y. C. Zhao, T. Li and Bastin-Coron.

- In the case of Sobolev spaces $W^{m,p}([0,L])$ with $m \geq 2$ and $p \in [1, +\infty]$, a sufficient condition is:
$$\rho_p(K) < 1.$$
Cf. Coron-d'Andréa-Novel-Bastin for $p = 2$, Coron-Nguyen for general p.

- One can actually show that
$$\rho_1(K) = \rho_\infty(K).$$

Remarks: Cauchy problem with boundary

- The known results on the existence of a standard Riemann semigroup for initial-boundary problem do not seem to cover our situation exactly and uniqueness of solutions in the spirit of Bressan-LeFloch or Bressan-Goatin seems open.

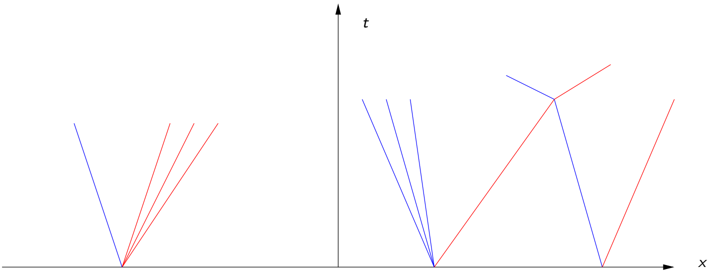
  Cf. Amadori, Amadori-Colombo, Colombo-Guerra, Donadello-Marson, Sablé-Tougeron,. . .

A general idea of the proof

- One constructs solutions using the wave-front tracking approach (here, DiPerna's approach since we consider 2×2 systems)
- Then the result relies on a Lyapunov function.
- This Lyapunov function is mainly inspired by two sources:
  - Lyapunov functions constructed in the classical case, cf. Coron-Bastin-d'Andrea-Novel, Coron-Bastin, ...
  - Glimm's functional used to construct entropy solutions in BV

1. Wave-front tracking algorithm

- Solutions are constructed directly using a wave-front tracking approach (cf. Dafermos, DiPerna, Bressan, . . .):
  - one constructs a sequence of approximations of a solutions,
  - these approximations are piecewise constant functions on $\mathbb{R}_+ \times \mathbb{R}$ where the discontinuities are straight lines separating states connected by shocks or rarefactions,



The Riemann problem... far from the boundary

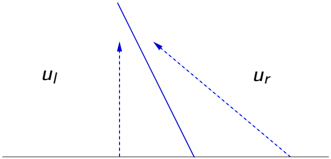
- Find autosimilar solutions $u = \overline{u}(x/t)$ to

$$\begin{cases} u_t + (f(u))_x = 0 \\ u_{|\mathbb{R}^-} = u_l \text{ and } u_{|\mathbb{R}^+} = u_r. \end{cases}$$

- Solved by introducing Lax's curves which consist of points that can be joined starting from u_l (in the case of GNL fields):
  - either by a shock,
  - or by a rarefaction wave.

Shocks and rarefaction waves (GNL fields)

Shocks



Discontinuities satisfying:

- Rankine-Hugoniot (jump) relations

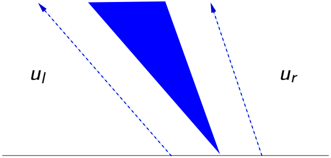
$$\big[\, f(u) \,\big] = s \big[\, u \,\big],$$

- Lax's inequalities:

$$\lambda_i(u_r) < s < \lambda_i(u_l)$$

Propagates at speed $s \sim \frac{1}{u_r - u_l} \int_{u_l}^{u_r} \lambda_i$

Rarefaction waves



Regular solutions,
obtained with integral curves of r_i:

$$\begin{cases} \dfrac{d}{d\sigma} R_i(\sigma) = r_i(R_i(\sigma)), \\ R_i(0) = u_l, \end{cases}$$

with $\sigma \geq 0$.

Propagates at speed $\lambda_i(R_i(\sigma))$

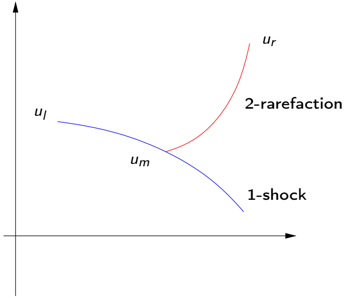
Lax's curves (GNL fields)

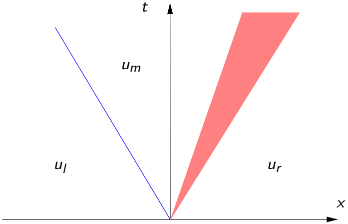
- We call $\Phi_i(\cdot, u_l)$ the *i-th Lax curve* consisting of points u_r that can be connected
  - by a i-shock ($\sigma < 0$)
  - or by a i-rarefaction wave ($\sigma \geq 0$).
- When $u_+ = \Phi_i(\sigma_i, u_-)$, we call σ_i the strength of the simple wave (u_-, u_+).
- By convention, $\sigma_i > 0$ for rarefactions and $\sigma_i < 0$ for shocks.
- Lax's theorem asserts that for u_l and u_r sufficiently close, one can find (σ_i) such that

$$u_r = \Phi_2(\sigma_2, \cdot) \circ \Phi_1(\sigma_1, \cdot) u_l.$$

- This allows to solve the Riemann problem.

Solving the Riemann problem

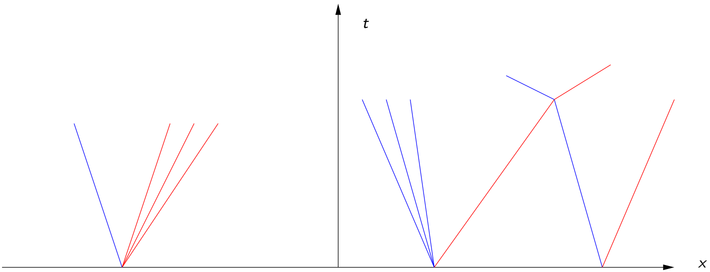




- Lax's Theorem proves that one can solve (at least locally) the Riemann problem by first following the 1-curve, then the 2-curve.

Front-tracking algorithm

- Approximate initial condition by piecewise constant functions.
- Solve the Riemann problems and replace rarefaction waves by rarefaction fans.
- For small times, one obtains a piecewise constant function where states are separated by straight lines called fronts.



- At each interaction point (points where fronts meet), iterate the process without splitting again rarefaction fronts

Estimates, convergence, etc.

- One shows than this defines a piecewise constant function, with a finite number of fronts and discrete interaction points.

- A central argument is due to Glimm: consider

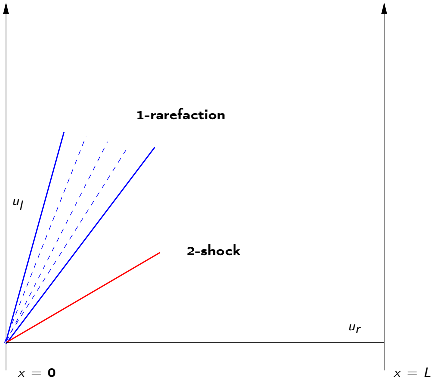
$$V(\tau) = \sum_{\alpha \text{ wave at time } t} |\sigma_\alpha| \ ; \quad Q(\tau) = \sum_{\substack{\alpha,\beta \\ \text{approaching waves}}} |\sigma_\alpha|.|\sigma_\beta|,$$

- Analyzing interactions $\alpha + \beta \to \alpha' + \beta'$ one shows that: for some $C > 0$, if $TV(u_0)$ is small enough, then $V(t) + CQ(t)$ is non-increasing. (Glimm's functional)

- One deduces bounds in $L^\infty_t BV_x$, then in $\mathrm{Lip}_t L^1_x$, so we have compactness. . .

Boundary Riemann problem

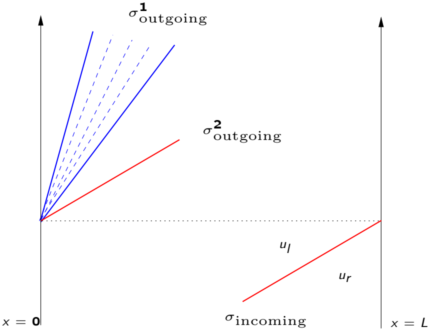
- In our case we have to take the boundary into account, and to be able to solve the boundary Riemann problem.
- Cf. Dubois-LeFloch, Amadori, Amadori-Colombo, Colombo-Guerra, Donadello-Marson, etc.



Boundary "interactions"

- One can then take "boundary interactions" into account.
- One can measure the size of the oungoing fronts in terms of the size of the incoming one. This highly depends on K!
- Roughly speaking, our condition ensures

$$|\sigma^1_{\text{outgoing}}| + |\sigma^2_{\text{outgoing}}| \leq \kappa |\sigma_{\text{incoming}}|, \quad 0 < \kappa < 1.$$



2. Using Lyapunov functions

Let $\lambda > 0$, and consider here, for sake of simplicity,

$$\begin{cases} \partial_t u + \lambda \partial_x u = 0, & (t,x) \in (0,\infty) \times (0,L), \\ u(t,0) = k u(t,L), & t \geq 0. \end{cases}$$

Exponential decay $\quad \Leftrightarrow \quad |k| < 1$
An easy way to prove $\Leftarrow$: Introduce

$$J(t) = \int_0^L |u(t,x)|^2 e^{-2\gamma x}\, dx,$$

which satisfies

$$\frac{d}{dt} J(t) = -2\gamma\lambda J(t) - \lambda \left(u(t,L)^2 e^{-2\gamma L} - u(t,0)^2 \right) \leq -2\gamma\lambda J(t)$$

if $\exp(-\gamma L) > |k|$, so that $\sqrt{J(t)} \leq e^{-\gamma\lambda t} \sqrt{J(0)}$.

Can be generalized to many (much more intricate) settings, see Bastin-Coron's book.

In our context

Our Lyapunov functional is as follows:

$$J := V + CQ$$

where

$$\begin{aligned} V(U) &= \sum_{i=0}^{n} \left(|\sigma_{i,1}| + |\sigma_{i,2}|\right) e^{-\gamma x_i}, \\ Q(U) &= \sum_{(x_i,\sigma_i)} |\sigma_i| e^{-\gamma x_i} \left(\sum_{(x_j,\sigma_j) \text{ approaching } (x_i,\sigma_i)} |\sigma_j| e^{-\gamma x_j} \right), \end{aligned}$$

for suitable constants, where

- $\sigma_{i,k}$ is the strength of the k-wave at x_i (σ_i when there is no ambiguity, i.e. for $i \geq 1$),
- $x_1, \ldots, x_n$ are the discontinuities in $(0, L)$,
- $u(t, 0+) = \Psi_2(\sigma_{0,2}, \Psi_1(\sigma_{0,1}, Ku(t, L-)))$.

Our Lyapunov functional, 2

Analyzing in particular interactions of fronts with the boundary, one shows that for suitable constants and provided that

$$TV(u_0) \text{ is small enough,}$$

one has for proper $\nu > 0$:

$$J(t) \leq J(0) \exp(-\nu t).$$

This allows to construct approximations and the solutions globally in time

and to get the result.

Open problems

- Considering a less particular case:
  - speeds with different signs,
  - $n \times n$ systems,
  - nonlinear boundary conditions,
  - non GNL characteristic fields, etc.
- What about source terms?

Thank you for your attention!

Ref: Dissipative boundary conditions for 2x2 hyperbolic systems of conservation laws for entropy solutions in BV.
J.D.E. 262 (2017), no. 1, 1–30.
J.-M. Coron, S. Ervedoza, S.S. Ghoshal, O. Glass, V. Perrollaz.