Reconstruction of coefficient in the wave equation : a Weighted Energy Inversion Procedure.

Lucie Baudouin
with M. de Buhan, S. Ervedoza & A. Osses.

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Coefficient inverse problem in the wave equation

In a smooth bounded domain $\Omega \subset \mathbb{R}^n$, it writes for instance,

$$\begin{cases} \partial_{tt} y(t,x) - \Delta_x y(t,x) + p(x) y(t,x) = f(t,x), & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in \Omega. \end{cases}$$

or with variable speed

$$\begin{cases} \partial_{tt} y - \nabla \cdot (a(x) \nabla y) = f, & \text{in } (0,T) \times \Omega, \\ y = g, & \text{on } (0,T) \times \partial\Omega, \\ (y(0), \partial_t y(0)) = (y_0, 0), & \text{in } \Omega, \end{cases}$$

- Given data : Source terms f, g ; initial data : (y^0, y^1) ;
- Unknown : the potential $p = p(x)$ or the speed $a = a(x)$;
- Additional measurement : the flux $\partial_\nu y(t,x)$ on $(0,T) \times \partial\Omega$.

Several comments

- The determination in Ω of p or a from an additional measurement are inverse problems for which uniqueness and stability are well-known and proved using Carleman estimates.
- Classical reconstruction method : minimizing

$$J(q) = \|\partial_\nu y[q] - \partial_\nu y[p]\|,$$

  generally not convex $\rightsquigarrow$ may have several local minima.
  Algorithms not guaranteed to converge to the global minimum.
- Klibanov, Beilina and co-authors have worked a lot on related questions...

The Weighted Energy Inversion Procedure

• Goal : propose an algorithm to compute the unknown coefficient, satisfying :

- some convergence estimates with no a priori guess ;
- easy implementation & numerical efficiency.

• Core idea of WEIP : build a convergent algorithm of reconstruction taking advantage of

- the appropriate Carleman estimates to build the cost functional ;
- the structure of the proof of stability to prove the convergence.

Outline

Inverse problem for the wave equation

- Generalities
- Reconstruction and goals
- Lipschitz stability result for the continuous wave equation

Globally convergent reconstruction algorithm

- A first algorithm
- First numerics
- New Algorithm

Reconstruction of the speed

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Determination of the potential in the wave equation

$$\begin{cases} \partial_{tt} y - \Delta y + p y = f, & (0,T) \times \Omega, \\ y = g, & (0,T) \times \partial\Omega \\ (y(0), \partial_t y(0)) = (y^0, y^1), & \Omega. \end{cases}$$

Is it possible to retrieve the potential $p = p(x),\ x \in \Omega$ *from measurement of the flux* $\partial_\nu y(t,x)$ *on* $(0,T) \times \partial\Omega$ *?*

- Uniqueness : Given $p_1 \neq p_2$, can we guarantee $\partial_\nu y[p_1] \neq \partial_\nu y[p_2]$?
- Stability : If $\partial_\nu y[p_1] \simeq \partial_\nu y[p_2]$, can we guarantee that $p_1 \simeq p_2$?
- Reconstruction : Given $\partial_\nu y[p]$, can we compute p ?

• Known results : Uniqueness (Klibanov '92) and stability (Yamamoto '99, Imanuvilov Yamamoto '01), using Carleman estimates.

• Main question : Reconstruction ; how to compute the potential from the boundary measurement ?

Natural idea for reconstruction

Given a continuous measurement $\mathscr{M}[p] = \partial_\nu y[p]|_{(0,T)\times\partial\Omega}$

- Discretize the wave equation

$$\begin{cases} \partial_{tt} y_h - \Delta_h y_h + p_h y_h = f_h \simeq f, \\ y_h|_{(0,T)\times\partial\Omega} = g_h \simeq g, \\ (y_h, \partial_t y_h)(t=0) = (y_h^0, y_h^1) \simeq (y^0, y^1). \end{cases}$$

- Solve the following discrete inverse problem : Find a potential p_h so that the corresponding discrete solution $y_h[p_h]$ approximates at best the measurement :

$$\partial_h y_h[p_h]|_{(0,T)\times\partial\Omega}(t,x) \simeq \mathscr{M}[p](t,x)$$

$$\text{i.e. } p_h = \text{Argmin}_{q_h} \|\partial_h y_h[q_h] - \mathscr{M}[p]\|_*$$

Question : Do we get $p_h \simeq p$?

First goal :

Analyze the convergence of the discrete inverse problems.

presented in Benasque 2011.

Remarks :

- Natural question for all inverse problems in infinite dimensions : Finding a source term, a conductivity...
- Depends *a priori* on the numerical scheme employed.

Main difficulty :

- Different dynamics for the wave equation and its discrete approximations, cf Ervedoza - Zuazua '11 :
  $\rightsquigarrow$ Numerical artefacts : High-frequency spurious waves, generated by the schemes.

Second goal :

Propose a globally convergent algorithm for reconstruction.

presented in Benasque 2013.

Remarks :

- Reconstruction of the potential, with a single boundary measurement (during a time T large enough) ;
- Using the observation $\mathscr{M}[p] = \partial_\nu y[p]$, a classical method for solving this inverse problem consists in minimizing

$$J(q) = \|\partial_t \left(\partial_\nu y[q] - \mathscr{M}[p]\right)\|^2_{L^2(\Gamma_0\times(0,T))}$$

$\rightsquigarrow$ not convex $\quad\rightsquigarrow$ local minima ;

- Our algorithm will be based on Carleman estimate and the proof scheme of the stability result.

Third goal :

Propose a numerically efficient algorithm.

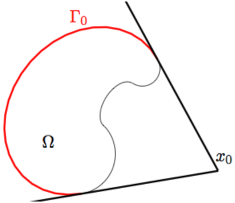
Remarks :

Minimizing a strictly convex and coercive quadratic functional based on a Carleman estimate means dealing with $e^{2se^{\lambda\psi}}$ for large parameters s and λ...

Idea : Our more recent algorithm will be based on

- a single parameter Carleman estimate,
- a preconditioning of the cost functional (conjugate variable),
- and the splitting of the observations by cut-off.

Stability Result (Yamamoto '99, LB-Puel '01)



Let $x_0 \in \mathbb{R}^N \setminus \Omega$ and let Γ_0 and T satisfy

$$\{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0 \quad ; \quad T > \sup_{x\in\Omega}\{|x - x_0|\}.$$

Let the potential p, the initial data y^0 and the solution $y[p]$ s.t.

$$\|p\|_{L^\infty(\Omega)} \le m, \quad \inf_{x\in\Omega}\{|y^0(x)|\} \ge \gamma > 0, \quad y[p] \in H^1(0,T;L^\infty(\Omega))$$

Then, one can prove uniqueness and local Lipschitz stability of the inverse problem for the wave equation : $\forall q \in L^\infty_{\le m}(\Omega)$,

$$\frac{1}{C}\,\|p - q\|_{L^2(\Omega)} \le \|\partial_\nu y[p] - \partial_\nu y[q]\|_{H^1((0,T);L^2(\Gamma_0))}\,.$$

Carleman Estimate (Imanuvilov '02, LB-Puel '01)

Assuming $\{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0$, there exists $s_0 > 0$, $\lambda > 0$ and $M = M(s_0, \lambda, T, \beta, x_0) > 0$ such that :

$$s \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}(|\partial_t w|^2 + |\nabla w|^2)\,dxdt + s^3 \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|w|^2\,dxdt$$
$$\leq M \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\partial_{tt}w - \Delta_x w|^2\,dxdt + M s \int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\,|\partial_\nu w|^2\,d\sigma dt$$

for all $s > s_0$, $w \in L^2(-T, T; H_0^1(\Omega))$ and φ satisfying

$$\begin{cases} \partial_{tt}w - \Delta_x w \in L^2(\Omega \times (-T,T)), \\ \partial_\nu w \in L^2(-T,T; L^2(\Gamma_0)), \\ w(\pm T) = \partial_t w(\pm T) = 0 \text{ in } \Omega\ ; \end{cases} \qquad \begin{cases} \beta \in (0,1), C_0 \text{ large enough}, \\ \psi(x,t) = |x - x_0|^2 - \beta t^2 + C_0, \\ \varphi(x,t) = e^{\lambda\psi(x,t)}. \end{cases}$$

$\rightsquigarrow$ but also Zhang, Klibanov,...

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Towards a (re)constructive approach

It is easy to check that $Z = \partial_t \left(y[p] - y[q]\right)$ satisfies

$$\begin{cases} \partial_{tt} Z - \Delta_x Z + q(x) Z = (q-p)\partial_t y[p], & (t,x) \in (0,T) \times \Omega, \\ Z(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega \\ (Z(0,x), \partial_t Z(0,x)) = (0, (q-p)y^0), & x \in \Omega. \end{cases}$$

Main idea : source term $(q-p)\partial_t y[p]$ less relevant than initial data $(q-p)y^0$, thanks to the Carleman estimate, whereas

$$\partial_\nu Z = \partial_t \partial_\nu y[p] - \partial_t \partial_\nu y[q] \quad \text{on } (0,T) \times \Gamma_0 \quad \text{is known.}$$

$\rightsquigarrow$ Hence, we try to fit Z using this information, and apply the following new Carleman estimate.

A new Carleman estimate (LB, de Buhan, Ervedoza '13)

Assuming $q \in L^\infty_{\leq m}(\Omega)$, $L_q = \partial_{tt} - \Delta_x + q(x)$,

$$\{x \in \partial\Omega,\, (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0\ , \qquad \sup_{x\in\Omega} |x - x_0| < \beta T$$

$\exists s_0 > 0$, $\lambda > 0$ and $M = M(s_0, \lambda, T, \beta, x_0, m) > 0$ such that

$$s \int_0^T\!\!\int_\Omega e^{2s\varphi} \left(|\partial_t w|^2 + |\nabla w|^2 + s^2 |w|^2\right) dxdt \ + \ s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t w(0)|^2 \, dx$$

$$\leq \ M \int_0^T\!\!\int_\Omega e^{2s\varphi} |L_q w|^2 \, dxdt + M s \int_0^T\!\!\int_{\Gamma_0} e^{2s\varphi} \left|\partial_\nu w\right|^2 \, d\sigma dt,$$

for all $s > s_0$ and $w \in L^2(-T, T; H^1_0(\Omega))$ satisfying

$$\begin{cases} L_q w \in L^2(\Omega \times (-T,T)) \\ \partial_\nu w \in L^2((0,T)\times\Gamma_0), \\ w(0,x) = 0, \ \forall x \in \Omega. \end{cases}$$

Algorithm

Initialization : $q^0 = 0$ or any initial guess.

Iteration : Given q^k,

1 - Compute $w[q^k]$ the solution of

$$\begin{cases} \partial_t^2 w - \Delta w + q^k w = f, & \text{in } \Omega \times (0,T), \\ w = g, & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{cases}$$

and set $\mu^k = \partial_t \left(\partial_\nu w[q^k] - \partial_\nu w[p]\right)$ on $\Gamma_0 \times (0,T)$.

2 - Introduce the functional

$$J_0^k(z) = \int_0^T \int_\Omega e^{2s\varphi} |L_{q^k} z|^2 + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2,$$

on the space $\mathcal{T}^k = \{z \in L^2(0,T; H_0^1(\Omega)), z(t=0) = 0,$
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$L_{q^k} z \in L^2(\Omega \times (0,T)), \partial_\nu z \in L^2(\Gamma_0 \times (0,T)) \}$.

Theorem

Assume the **geometric and time conditions**. *Then, for all $s > 0$ and $k \in \mathbb{N}$, the functional J_0^k is continuous, strictly convex and coercive on $\mathcal{T}^k$ endowed with a suitable weighted norm.*

3 - Let Z^k be the unique minimizer of the functional J_0^k, and then set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{w_0} \iff (\tilde{q}^{k+1} - q^k)w_0 = \partial_t Z^k(0),$$

where w_0 is the initial condition of (1).

4 - Finally, set

$$q^{k+1} = T_m(\tilde{q}^{k+1}), \quad \text{where } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q)m, & \text{if } |q| \geq m. \end{cases}$$

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Algorithm's convergence (LB, de Buhan & Ervedoza 13')

Theorem

Assuming the geometric and time conditions (among others), there exists a constant $M > 0$ such that $\forall s \geq s_0(m)$ and $k \in \mathbb{N}$,

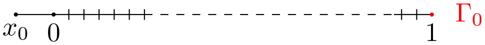
$$\int_\Omega e^{2s\varphi(0)}(q^{k+1} - Q)^2\, dx \leq \frac{M}{\sqrt{s}} \int_\Omega e^{2s\varphi(0)}(q^k - Q)^2\, dx.$$

In particular, when s is large enough, the algorithm converges.

Remark : This algorithm converges to the global minimum from any initial guess.

Numerical Simulations

- $\Omega = [0,1]$, $x_0 = -0.1$, $\Gamma_0 = \{x = 1\}$, $g = 0$, $\beta = 0.99$, $T = 1.5$, $\lambda = 0.1$, $s = 1$;

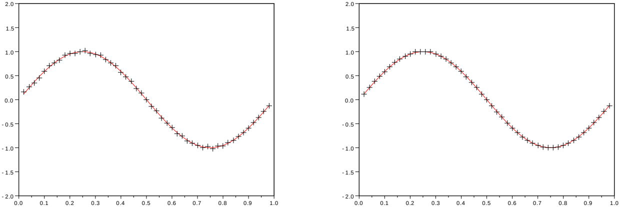


- Discretization with the finite-difference method : $N + 1 = \frac{1}{h}$, $(\Delta_h y_h)_j = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}$, $\quad \forall j \in \{1, \cdots, N\}$
- Penalization of high-frequencies with an extra regularization term in the cost funct. : $\displaystyle\int_0^T \int_0^1 e^{2s\varphi} |h\partial_h^+ \partial_t z_h|^2 \, dt$, coming from the discrete Carleman estimates, to have uniformity with respect to the discretization parameter h. Constraint : sh small enough.

$\rightsquigarrow$ 1D convergence result (LB & Ervedoza '13)

$\rightsquigarrow$ 2D case (LB & Ervedoza & Osses '15)

- Without noise, for $p(x) = \sin(2\pi x)$, one has



- Noise parameter $\alpha = 10\%$

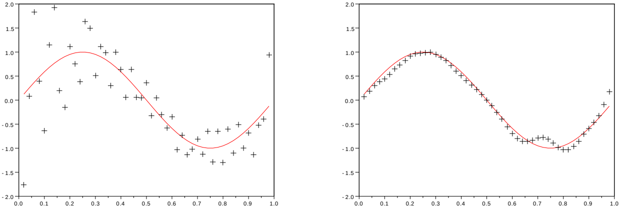


FIGURE – Without (left) and with (right) regularization term.

- s and λ should be large to ensure the convergence of the algorithm. But for $\lambda = 1$ and $s = 3$,

$$\max(\exp(2s\varphi))/\min(\exp(2s\varphi)) = 10^{110}!$$

- This first version of our theoretical algorithm is totally useless in practice. We made several improvements to be able to implement it numerically...

$\rightsquigarrow$ leading to a new numerically efficient algorithm.

New Algorithm

The algorithm is modified according to the following items...

- Single parameter Carleman estimate ;
- Preconditioning of the cost functional ;
- Splitting of the observations by cut-off ;

... and the convergence result remains the same.

A single parameter Carleman estimate

(Lavrentiev Romanov Shishatskii '86)

Assuming the geometric condition on Γ_0, $L_q = \partial_{tt} - \Delta_x + q(x)$, $q \in L^\infty_{\leq m}(\Omega)$, $\sup_{x\in\Omega} |x - x_0| < \beta T$ and $\varphi(t,x) = |x - x_0|^2 - \beta t^2$, then $\exists s_0 > 0$ and $M = M(s_0, T, \beta, x_0, m) > 0$ such that

$$\underbrace{s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t w(0)|^2 \, dx}_{\text{initial energy}} \leq M \underbrace{\int_0^T\!\!\int_\Omega e^{2s\varphi} |L_q w|^2 \, dx dt}_{\text{source}}$$

$$+ \; M s \underbrace{\int_0^T\!\!\int_{\Gamma_0} e^{2s\varphi} \left|\partial_\nu w\right|^2 \, d\sigma dt}_{\text{observation}} + M s^3 \iint_{\{\varphi<0\}} e^{2s\varphi} |z|^2 \, dx dt$$

Preconditioning the new cost functional

We remove some exponential factors by introducing the conjugate variable $y = e^{\varphi} z$ in the new functional

$$\widetilde{J}^k(y) = \int_0^T \int_\Omega |\mathscr{L}_{s,q^k} y|^2 + s \int_0^T \int_{\Gamma_0} |\partial_\nu y - e^{2s\varphi} \mu^k|^2 + s^3 \iint_{\{\varphi<0\}} |y|^2,$$

which is minimized on the same set $\mathcal{T}^k$ as before and where the conjugate operator is $\mathscr{L}_{s,q} = e^{s\varphi}(\partial_t^2 - \Delta + q)e^{-s\varphi}$.

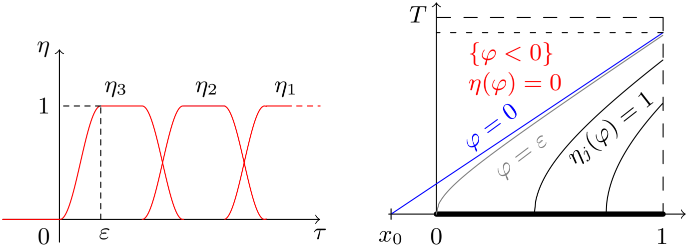
Nevertheless, there is still an exponential factor in the measurements.

Dealing finally with the observations

We split the observations in several parts and consider intervals in which the weight function does not significantly change. To do that :

$$\mu_j^k = \eta_j(\varphi)\mu^k, \qquad \forall \tau \in \mathbb{R}, \quad \sum_{j=1}^{N} \eta_j(\tau) = \eta(\tau),$$

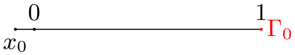
where the η_j are the following cut-off functions ($\varepsilon = \inf_\Omega |x - x_0|^2$) :



Y_j minimizer of $\widetilde{J}^k[\mu_j^k] \;\Rightarrow\; Y = \sum_{j=1}^{N} Y_j$ minimizer of $\widetilde{J}^k[\mu^k]$.

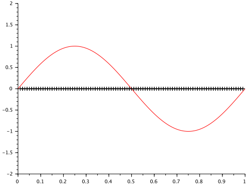
Discretization of the problem

- $\Omega = [0,1]$, $x_0 = -0.3$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.3$, $s = 100$, $f = 0$, $g = 2$, $u_0(x) = 2 + \sin(x\pi)$ and $u_1 = 0$.

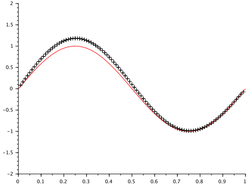


- To avoid the inverse crime, we use $\neq$ schemes and $\neq$ meshes in the direct and inverse problems :
  - direct problem : finite differences in space $h = 0.00025$, implicit theta scheme in time $\tau = 0.00033$;
  - inverse problem : finite differences in space $h = 0.05$, explicit Euler scheme in time $\tau = 0.05$, that is $CFL = 1$.

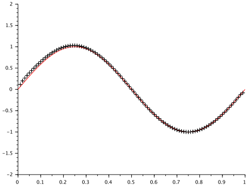
Illustration of the convergence of the algorithm



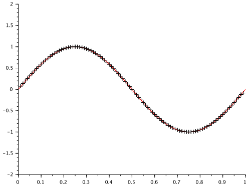
(a) q^0



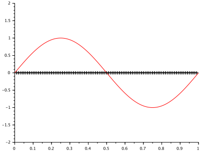
(b) q^1



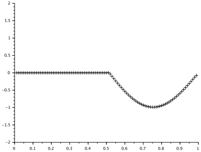
(c) q^2



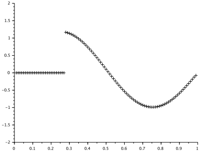
(d) q^3



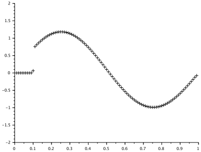
(a) $q_0^0 = q^0$



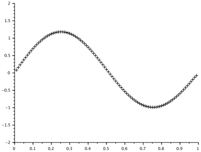
(b) q_1^0



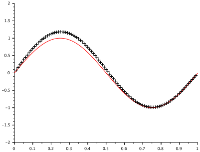
(c) q_2^0



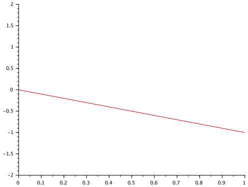
(d) q_3^0



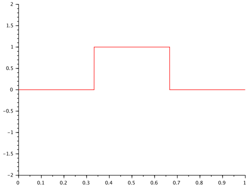
(e) q_4^0



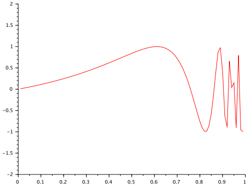
(f) $q_5^0 = q^1$



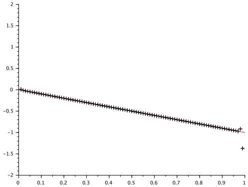
(a) $p = -x$



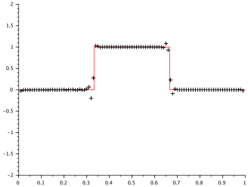
(b) p heaviside



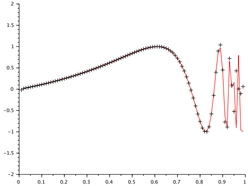
(c) $p(x) = \sin(\frac{x}{1-x})$



(a) $p = -x$

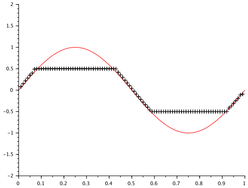


(b) $p = \text{gate}(x)$

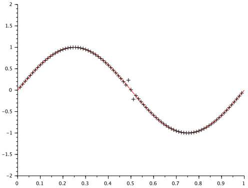


(c) $p(x) = \sin(\frac{x}{1-x})$

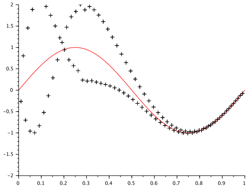
Wrong choices of the parameters



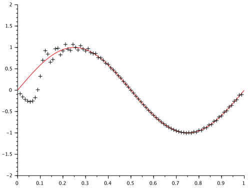
(a) Wrong choice of m



(b) u_0 vanishes at $x = 0.5$

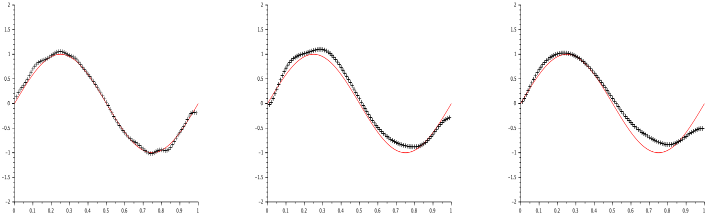


(c) No viscous term



(d) $T = 0.9 < 1$

With noise on the measurement of the flux

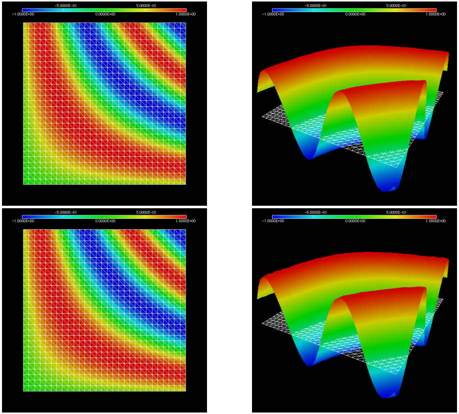


$s = 10$ and the noise is multiplicative : 1%, 5%, 10%.

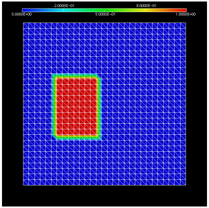
Taking s too large seems to amplify the effects of the noise...

Numerical results in 2D

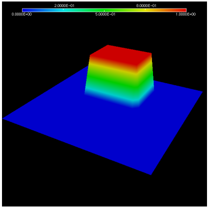
$\Omega = [0,1]^2$, $x_0 = (-0.3, -0.3)$ and $\Gamma_0 = \{x = 1\} \cup \{y = 1\}$



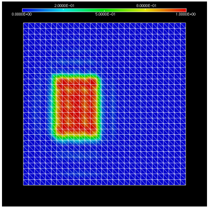
Exact potentials (top) vs Numerical potentials (bottom).



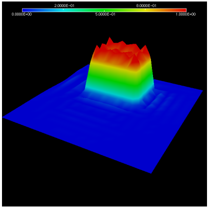
(a) Exact potential



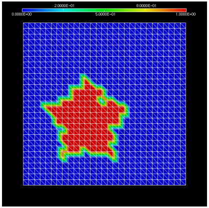
(b) 3D view



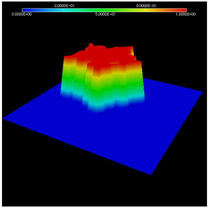
(c) Numerical



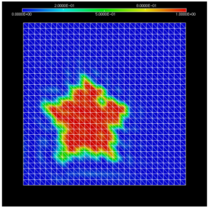
(d) 3D view



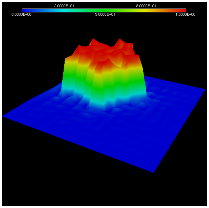
(a) Exact potential



(b) 3D view



(c) Numerical



(d) 3D view

Outline

Inverse problem for the wave equation

Generalities

Reconstruction and goals

Lipschitz stability result for the continuous wave equation

Globally convergent reconstruction algorithm

A first algorithm

First numerics

New Algorithm

Reconstruction of the speed

Recovery of the main coefficient

Wave equation with variable speed :

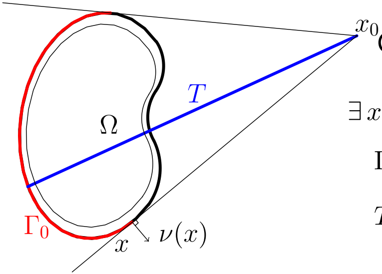
$$\begin{cases} \partial_{tt} y - \nabla \cdot (a(x) \nabla y) = f, & \text{in } (0,T) \times \Omega, \\ y = g, & \text{on } (0,T) \times \partial\Omega, \\ y(0) = y_0, \quad \partial_t y(0) = 0, & \text{in } \Omega, \end{cases}$$

- Given data : Source terms f, g, initial data : $(y^0, 0)$,
  boundary value $a = \mathbf{a}$ on $\partial\Omega$.
- Unknown : the speed $a = a(x)$, inside Ω.
- Additional measurement : $\partial_\nu y(t,x)$ on $(0,T) \times \partial\Omega$.

Goal : Find the variable speed $a = a(x)$.

$\rightsquigarrow$ Application in medical imaging.

Setting and assumptions



Geometric and time conditions :

$\exists\, x_0 \notin \overline{\Omega}$, such that

$$\Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) \geq 0\},$$

$$T > \frac{\sup_{x\in\Omega}|x - x_0|}{\sqrt{\beta}}.$$

- Regularity assumption on the solution $y[a]$
- Initial orientation condition : $|\nabla w_0(x)\cdot(x - x_0)| \geq r_0 > 0$ in Ω.
- $\mathcal{V}_{\alpha_0,\alpha_1,\beta_0,\mathbf{a}} = \{a \in C^1(\overline{\Omega}),\ 0 < \alpha_0 \leq a \leq \alpha_1,$
  $a + \frac{1}{2}\nabla a\cdot(x - x_0) \geq \beta_0 > 0$ ae in $\Omega,\ a = \mathbf{a}$ on $\partial\Omega\}$.

$\rightsquigarrow$ Ongoing work.

Idea

The speed reconstruction algorithm is based on the fact that if $y[a]$, $y[a^k]$, are the solution of the wave equation, then

$$z^k = \partial_t^2 \left(y[a^k] - y[a] \right)$$

solves

$$\begin{cases} \partial_t^2 z^k - \nabla \cdot (a^k \nabla z^k) = g^k, & \text{in } (0,T) \times \Omega, \\ z^k = 0, & \text{on } (0,T) \times \partial\Omega, \\ z^k(0,\cdot) = z_0^k, \quad \partial_t z^k(0,\cdot) = 0, & \text{in } \Omega, \end{cases}$$

where

$$g^k = \nabla \cdot ((a^k - a) \nabla \partial_t^2 y[a]), \qquad z_0^k = \nabla \cdot ((a^k - a) \nabla w_0),$$

The algorithm is constructed on the minimization of

$$J_{s,a^k}[\mu](z) = \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi}|\partial_t^2 z - \nabla\cdot(a^k \nabla z)|^2 \, dxdt + \frac{s}{2}\int_0^T \int_{\Gamma_0} e^{2s\varphi}|\partial_\nu z - \mu|^2 \, d\sigma dt + \frac{s^3}{2}\iint_{\mathcal{Q}} e^{2s\varphi}|z|^2 \, dxdt$$

in order to approximate $\tilde{z}^k = \eta(\varphi) z^k$, that satisfies :

- $\tilde{z}^k(0,\cdot) = \eta(\varphi(0,\cdot)) z_0^k = \nabla\cdot((a^k - a)\nabla w_0)$;
- $\tilde{z}^k = \eta(\varphi) z^k$ vanishes in $\mathcal{Q}$;
- $\partial_n \tilde{z}^k = \tilde{\mu}^k$ in $(0,T)\times\Gamma_0$.

Finally, we will need to study the first order differential equation that encapsulate $a^k - a$.

⋆ ⋆ ⋆

Thank you for your attention.

⋆

Related articles

- *Weighted Energy Inversion Procedure*,
  L. B., M. de Buhan, S. Ervedoza & A. Osses, *in preparation*.
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