A mean-field game model for pedestrian flow with minimal time

Guilherme Mazanti
joint work with Filippo Santambrogio

VII Partial differential equations,
optimal design and numerics
Benasque — August 25^{th}, 2017

LMO, Université Paris-Sud
Université Paris-Saclay

Outline

Introduction

Macroscopic models for pedestrian flow

- Goal: propose and study a nice mean-field game (MFG) model for pedestrian flow in a certain domain $\Omega \subset \mathbb{R}^d$ (or also a graph, a manifold, etc.)
- Macroscopic models for pedestrian flow:
$$\partial_t \rho + \mathrm{div}(\rho v) = 0.$$
  - $\rho(t, x)$: density of people at position $x \in \Omega$ in time t.
  - $v(t, x, \rho)$: velocity.
  - Conservation law for pedestrians (recall Tuesday's talk by R. Colombo).
- How do people choose v?
- The MFG approach: people solve an optimal control problem, which depends on the average behavior of other people.

Introduction

Mean field games

Mean field games are differential games with a continuum of players / agents, assumed to be rational, indistinguishable, and influenced only by the average behavior of other players.

Introduction
Mean field games

Mean field games are differential games with a continuum of players / agents, assumed to be rational, indistinguishable, and influenced only by the average behavior of other players.

- continuum of players: macroscopic model, density ρ.
- differential games: players' dynamics given by a controlled differential equation $\dot{\gamma}(t) = f(t, \gamma(t), u(t))$.
- rationallity: players minimize some cost.
- indistinguishability: f and the cost are the same for all players.
- average behavior: f and the cost depend on the current player's state $\gamma(t)$ and on ρ.

Introduction

Mean field games

Mean field games are differential games with a continuum of players / agents, assumed to be rational, indistinguishable, and influenced only by the average behavior of other players.

- continuum of players: macroscopic model, density ρ.
- differential games: players' dynamics given by a controlled differential equation $\dot{\gamma}(t) = f(t, \gamma(t), u(t))$.
- rationallity: players minimize some cost.
- indistinguishability: f and the cost are the same for all players.
- average behavior: f and the cost depend on the current player's state $\gamma(t)$ and on ρ.

Fixed point: given ρ, players evolve according to optimal trajectories, and this evolution gives ρ.

Introduction

Mean field games

Origins of MFGs: [Lasry, Lions; 2006], [Lasry, Lions; 2006], [Lasry, Lions; 2007], [Huang, Malhamé, Caines; 2006], [Huang, Caines, Malhamé; 2007]. Motivation from problems in economics and engineering.

Goal: provide an approximation for Nash equilibria of games with N symmetric players for large N.

Introduction

Minimal time mean field games

MFG model for pedestrian flow:

- People move in $\Omega \subset \mathbb{R}^d$, non-empty, open, and bounded.
- Goal: leave Ω through $\Gamma \subset \partial\Omega$, non-empty and closed.
- Initially: $\rho_0 \in \mathcal{P}(\overline{\Omega})$.
- Dynamics: people choose their speed up to a maximal value
$$\dot{\gamma}(t) = f(\rho_t, \gamma(t))u(t), \qquad u(t) \in \overline{B}(0, 1) = \text{closed unit ball.}$$

Introduction

Minimal time mean field games

MFG model for pedestrian flow:

- People move in $\Omega \subset \mathbb{R}^d$, non-empty, open, and bounded.
- Goal: leave Ω through $\Gamma \subset \partial\Omega$, non-empty and closed.
- Initially: $\rho_0 \in \mathcal{P}(\overline{\Omega})$.
- Dynamics: people choose their speed up to a maximal value
  $$\dot{\gamma}(t) = f(\rho_t, \gamma(t))u(t), \qquad u(t) \in \overline{B}(0, 1) = \text{closed unit ball.}$$
  Typically:
  $$f(\rho, x) = K\left[\int_{\overline{\Omega}} \chi(x - y)\, d\rho(y)\right],$$
  - χ: convolution kernel,
  - K: positive decreasing function.

Introduction

Minimal time mean field games

Most MFGs in the literature consider optimization criteria in fixed time T (same for all agents).

Our model:

- Optimization criterion: agents want to leave Ω through Γ in minimal time.

$$\inf\{T \geq 0 \mid \dot\gamma(t) = f(\rho_t, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0,1),$$
$$\gamma(0) = x,\ \gamma(T) \in \Gamma,\ \gamma(t) \in \overline{\Omega} \text{ for } t \in [0, T],$$
$$\dot\gamma(t) = 0 \text{ for } t > T\}.$$

Introduction

Minimal time mean field games

Most MFGs in the literature consider optimization criteria in fixed time T (same for all agents).

Our model:

- Optimization criterion: agents want to leave Ω through Γ in minimal time.

$$\inf\{T \geq 0 \mid \dot{\gamma}(t) = f(\rho_t, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0,1), \\ \gamma(0) = x,\ \gamma(T) \in \Gamma,\ \gamma(t) \in \overline{\Omega} \text{ for } t \in [0, T], \\ \dot{\gamma}(t) = 0 \text{ for } t > T\}.$$

- For simplicity, $\Gamma = \partial\Omega$ in this talk (room with no walls).

Main question: characterize the evolution of the density ρ.

Existence of an equilibrium

The Lagrangian approach

- Eulerian approach: $\rho : \mathbb{R}_+ \to \mathcal{P}(\overline{\Omega})$ is a curve on the set of measures. Motion is described by the density and the velocity field of the population.

Existence of an equilibrium

The Lagrangian approach

- Eulerian approach: $\rho : \mathbb{R}_+ \to \mathcal{P}(\overline{\Omega})$ is a curve on the set of measures. Motion is described by the density and the velocity field of the population.
- Lagrangian approach: $Q \in \mathcal{P}(\mathcal{C})$, where $\mathcal{C} = \mathcal{C}(\mathbb{R}_+, \overline{\Omega})$, is a measure on the set of curves. Motion is described by the trajectory of each agent.

Lagrangian framework for mean field games already used in the literature, cf. e.g. the survey in [Benamou, Carlier, Santambrogio; 2017].

Link between Eulerian and Lagrangian: $\rho_t = e_{t\#} Q$, where $e_t : \mathcal{C} \to \overline{\Omega}$ is the evaluation at time t of a curve, $e_t(\gamma) = \gamma(t)$.

Existence of an equilibrium

The Lagrangian approach

Definition

A measure $Q \in \mathcal{P}(\mathcal{C})$ is a Lagrangian equilibrium of the mean field game if $e_{0\#}Q = \rho_0$ and Q-almost every $\gamma \in \mathcal{C}$ is optimal for

$$\begin{aligned}\inf\{T \geq 0 \mid \dot\gamma(t) = f(e_{t\#}Q, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0, 1),\\ \gamma(0) = x,\ \gamma(T) \in \partial\Omega,\ \gamma(t) \in \overline{\Omega} \text{ for } t \in [0, T],\\ \dot\gamma(t) = 0 \text{ for } t > T\}.\end{aligned}$$

In the sequel, we consider

- the existence of a Lagrangian equilibrium;
- the characterization of equilibria by the MFG system;
- open problems and simulations.

Existence of an equilibrium

Main result

Theorem

Assume that $f : \mathcal{P}(\overline{\Omega}) \times \overline{\Omega} \to \mathbb{R}_+$ *is Lipschitz continuous and*

$$f_{\max} = \sup_{\substack{\mu \in \mathcal{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} f(\mu, x) < +\infty, \qquad f_{\min} = \inf_{\substack{\mu \in \mathcal{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} f(\mu, x) > 0.$$

Then there exists a Lagrangian equilibrium $Q \in \mathcal{P}(\mathcal{C})$ *for this game.*

With no loss of generality (change in time scale): $f_{\max} = 1$.

Existence of an equilibrium

Main result

Theorem

Assume that $f : \mathcal{P}(\overline{\Omega}) \times \overline{\Omega} \to \mathbb{R}_+$ is Lipschitz continuous and

$$f_{\max} = \sup_{\substack{\mu \in \mathcal{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} f(\mu, x) < +\infty, \qquad f_{\min} = \inf_{\substack{\mu \in \mathcal{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} f(\mu, x) > 0.$$

Then there exists a Lagrangian equilibrium $Q \in \mathcal{P}(\mathcal{C})$ for this game.

With no loss of generality (change in time scale): $f_{\max} = 1$.

Distance in $\mathcal{P}(\overline{\Omega})$: Wasserstein distance

$$W_1(\mu, \nu) = \min_{\substack{\gamma \in \mathcal{P}(\overline{\Omega} \times \overline{\Omega}) \\ \pi_{1\#}\gamma = \mu,\ \pi_{2\#}\gamma = \nu}} \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \, d\gamma(x, y).$$

Existence of an equilibrium

Strategy of the proof

Strategy of the proof:

- For fixed $Q \in \mathcal{P}(\mathcal{C})$, let $\mathbf{\Gamma}_Q \subset \mathcal{C}$ be the set of all optimal trajectories for the measure Q. Define
$$F(Q) = \{\widetilde{Q} \mid e_{0\#}\widetilde{Q} = \rho_0 \text{ and } \widetilde{Q}(\mathbf{\Gamma}_Q) = 1\}.$$
Equilibrium $\iff$ fixed point of the set-valued map F, i.e., $Q \in F(Q)$.
- Prove required properties of F to apply Kakutani fixed point theorem. Needs some properties of the value function

$$\begin{aligned}\tau_Q(t_0, x_0) = \inf\{T \geq 0 \mid \dot\gamma(t) = f(e_{t\#}Q, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0,1),\\ \gamma(t_0) = x_0,\ \gamma(t_0 + T) \in \partial\Omega,\ \gamma(t) \in \overline{\Omega} \text{ for } t \in [t_0, t_0 + T],\\ \dot\gamma(t) = 0 \text{ for } t > t_0 + T\}.\end{aligned}$$

Existence of an equilibrium

Strategy of the proof

Dynamics: $\dot{\gamma}(t) = f(e_{t\#}Q, \gamma(t))u(t)$ with f and u bounded by 1 $\implies$ optimal trajectories are 1-Lipschitz continuous.

We consider only $Q \in \mathcal{P}(\mathcal{C})$ supported on 1-Lipschitz continuous trajectories. Let $\mathcal{Q}$ be the set of such Q.

Existence of an equilibrium

Strategy of the proof

Dynamics: $\dot{\gamma}(t) = f(e_{t\#}Q, \gamma(t))u(t)$ with f and u bounded by 1 $\Longrightarrow$ optimal trajectories are 1-Lipschitz continuous.

We consider only $Q \in \mathcal{P}(\mathcal{C})$ supported on 1-Lipschitz continuous trajectories. Let $\mathcal{Q}$ be the set of such Q.

Proposition

- *The* inf *in* τ_Q *is a* min, τ_Q *is bounded and globally Lipschitz continuous, and* $\mathcal{Q} \ni Q \mapsto \tau_Q(t, x)$ *is Lipschitz continuous, uniformly in* x *and locally uniformly in* t*;*
- *The set of optimal trajectories* Γ_Q *is compact* $\forall Q \in \mathcal{Q}$ *and* $\mathcal{Q} \ni Q \mapsto \Gamma_Q$ *is upper semi-continuous;*
- $F(Q)$ *is non-empty, compact, and convex* $\forall Q \in \mathcal{Q}$*, and* F *is upper semi-continuous.*

These properties yield the existence of a fixed point for F.

The MFG system

The Lagrangian approach

We have proved the existence of a Lagrangian equilibrium to the minimal time mean field game.

- Advantage: easier than to prove than in the Eulerian approach. Application of Kakutani fixed point theorem requires fewer properties of the optimal trajectories.
- Drawback: we have no information on $\rho_t = e_{t\#}Q$.

Goal: characterize τ_Q and ρ as solutions of a system of PDEs.

The MFG system
Hypotheses

We use extra assumptions:

Hypotheses

- $f : \mathcal{P}(\overline{\Omega}) \times \overline{\Omega} \to \mathbb{R}_+^*$ is given by $f(\mu, x) = K[E(\mu, x)]$, with
$$E(\mu, x) = \int_{\overline{\Omega}} \chi(x - y)\eta(y)\,d\mu(y),$$
$K \in \mathcal{C}^{1,1}(\mathbb{R}_+, \mathbb{R}_+^*)$ is bounded, $\chi \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathbb{R}_+)$, and $\eta \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathbb{R}_+)$ with $\eta(x) = 0$ and $\nabla\eta(x) = 0$ for $x \in \partial\Omega$.
- Ω satisfies the uniform exterior sphere condition: $\mathbb{R}^d \setminus \Omega$ is a union of closed balls with the same radius.

The MFG system

Main result

Theorem

Under the previous assumptions, τ_Q and ρ solve the MFG system

$$\begin{cases} \partial_t \rho(t,x) - \operatorname{div}_x \left[f(\rho_t, x) \dfrac{\nabla_x \tau_Q(t,x)}{\left|\nabla_x \tau_Q(t,x)\right|} \rho(t,x) \right] = 0, & \mathbb{R}_+ \times \Omega, \\ -\partial_t \tau_Q(t,x) + \left|\nabla_x \tau_Q(t,x)\right| f(\rho_t, x) - 1 = 0, & \mathbb{R}_+ \times \Omega, \\ \rho(0,x) = \rho_0(x), & \overline{\Omega}, \\ \tau_Q(t,x) = 0, & \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

Continuity equation satisfied in the sense of distributions,

Hamilton–Jacobi equation satisfied in the viscosity sense.

The MFG system

Main result

Theorem

Under the previous assumptions, τ_Q and ρ solve the MFG system

$$\begin{cases} \partial_t \rho(t,x) - \mathrm{div}_x \left[f(\rho_t, x) \frac{\nabla_x \tau_Q(t,x)}{|\nabla_x \tau_Q(t,x)|} \rho(t,x) \right] = 0, & \mathbb{R}_+ \times \Omega, \\ -\partial_t \tau_Q(t,x) + |\nabla_x \tau_Q(t,x)| \, f(\rho_t, x) - 1 = 0, & \mathbb{R}_+ \times \Omega, \\ \rho(0,x) = \rho_0(x), & \overline{\Omega}, \\ \tau_Q(t,x) = 0, & \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

Continuity equation satisfied in the sense of distributions,
Hamilton–Jacobi equation satisfied in the viscosity sense.
Velocity field: $v(t,x,\rho_t) = -f(\rho_t, x) \frac{\nabla_x \tau_Q(t,x)}{|\nabla_x \tau_Q(t,x)|}$.

The MFG system

Strategy of the proof

Hamilton–Jacobi equation can be obtained by standard techniques on optimal control using a dynamic programming principle. But the situation is more subtle for the continuity equation. We need some properties of τ_Q.

The MFG system
Strategy of the proof

Hamilton–Jacobi equation can be obtained by standard techniques on optimal control using a dynamic programming principle. But the situation is more subtle for the continuity equation. We need some properties of τ_Q.

$$-\partial_t \tau_Q(t, x) + \left|\nabla_x \tau_Q(t, x)\right| f(\rho_t, x) - 1 = 0.$$

Proposition

- *There exists $c > 0$ such that, if $\partial_t \tau_Q(t, x)$ exists, then $\partial_t \tau_Q(t, x) \geq c - 1$.*
- *If τ_Q is differentiable at (t, x), then $\left|\nabla_x \tau_Q(t, x)\right| \geq c > 0$.*
- $\tau_Q(t + h, \gamma(t + h)) + h = \tau_Q(t, \gamma(t))$ *for every $\gamma \in \Gamma_Q$ (= optimal).*

The MFG system
Strategy of the proof

Corollary

If $\gamma \in \Gamma_Q$ and τ_Q is differentiable at $(t, \gamma(t))$, then
$$\dot{\gamma}(t) = -f(\rho_t, \gamma(t)) \frac{\nabla_x \tau_Q(t, \gamma(t))}{\left|\nabla_x \tau_Q(t, \gamma(t))\right|}.$$

τ_Q is Lipschitz, hence differentiable a.e., but it may be nowhere differentiable along a particular trajectory... we still need more properties of τ_Q and the optimal trajectories, which we obtain by applying Pontryagin Maximum Principle.

The MFG system

Strategy of the proof

Proposition

If $\gamma \in \Gamma_Q$*, then* $\gamma \in \mathcal{C}^{1,1}([0, \tau_Q(0, \gamma(0))), \Omega)$*, the optimal control* $u \in \mathcal{C}^{1,1}([0, \tau_Q(0, \gamma(0))), \mathbb{S}^{d-1})$*, and*

$$\begin{cases} \dot\gamma(t) = f(\rho_t, \gamma(t)) u(t), \\ \dot u(t) = -\operatorname{Proj}_{T_{u(t)}\mathbb{S}^{d-1}} \nabla_x f(\rho_t, \gamma(t)). \end{cases}$$

The MFG system

Strategy of the proof

Proposition

If $\gamma \in \Gamma_Q$, then $\gamma \in \mathcal{C}^{1,1}([0, \tau_Q(0, \gamma(0))), \Omega)$, the optimal control $u \in \mathcal{C}^{1,1}([0, \tau_Q(0, \gamma(0))), \mathbb{S}^{d-1})$, and

$$\begin{cases} \dot\gamma(t) = f(\rho_t, \gamma(t)) u(t), \\ \dot u(t) = -\operatorname{Proj}_{T_{u(t)} \mathbb{S}^{d-1}} \nabla_x f(\rho_t, \gamma(t)). \end{cases}$$

Recall: $f(\mu, x) = K[E(\mu, x)]$, $E(\mu, x) = \int_{\overline{\Omega}} \chi(x - y) \eta(y) d\mu(y)$.

Proposition

Suppose $Q \in \mathcal{Q}$ is a Lagrangian equilibrium.

- $(t, x) \mapsto f(e_{t\#} Q, x)$ *is also* $\mathcal{C}^{1,1}$.
- τ_Q *is locally semiconcave.*

w semiconcave: $x \mapsto w(x) - C|x|^2$ is concave for some $C > 0$.

The MFG system
Strategy of the proof

Proposition

Let $Q \in \mathcal{Q}$ be a Lagrangian equilibrium and $\gamma \in \Gamma_Q$ (= optimal). Then, for every $t \in (0, \tau_Q(0, \gamma(0)))$, τ_Q admits a normalized gradient at $(t, \gamma(t))$ and the optimal trajectory satisfies

$$\dot\gamma(t) = -f(e_{t\#}Q, \gamma(t)) \frac{\nabla_x \tau_Q(t, \gamma(t))}{|\nabla_x \tau_Q(t, \gamma(t))|}.$$

$\Longrightarrow$ Continuity equation with velocity $-f(e_{t\#}Q, x) \frac{\nabla_x \tau_Q(t,x)}{|\nabla_x \tau_Q(t,x)|}$.

The MFG system

Strategy of the proof

Proposition

Let $Q \in \mathcal{Q}$ be a Lagrangian equilibrium and $\gamma \in \Gamma_Q$ (= optimal). Then, for every $t \in (0, \tau_Q(0, \gamma(0)))$, τ_Q admits a normalized gradient at $(t, \gamma(t))$ and the optimal trajectory satisfies

$$\dot{\gamma}(t) = -f(e_{t\#}Q, \gamma(t)) \frac{\nabla_x \tau_Q(t, \gamma(t))}{|\nabla_x \tau_Q(t, \gamma(t))|}.$$

$\Longrightarrow$ Continuity equation with velocity $-f(e_{t\#}Q, x)\frac{\nabla_x \tau_Q(t,x)}{|\nabla_x \tau_Q(t,x)|}$.

Definition

We say that τ_Q admits a normalized gradient at (t, x) if the set

$$\left\{ \frac{p_1}{|p_1|} \in \mathbb{S}^{d-1} \;\middle|\; p_1 \neq 0 \text{ and } \exists p_0 \in \mathbb{R} \text{ s.t. } (p_0, p_1) \in D^+\tau_Q(t, x) \right\}$$

contains exactly one element ($D^+\tau_Q$ is the super-differential of τ_Q). The unique element of this set is denoted by $\frac{\nabla_x \tau_Q(t,x)}{|\nabla_x \tau_Q(t,x)|}$.

Open problems and ongoing work

Ongoing work:

- $\Gamma \subsetneq \partial\Omega$. Existence of Lagrangian equilibrium and Hamilton–Jacobi equation with no extra difficulty. But the optimization problem now has state constraints.
- (with Samer Dweik) Regularity properties of ρ: $\rho_0 \in L^p \overset{?}{\Rightarrow} \rho_t \in L^p$.

Open problems:

- Uniqueness.
- More general costs with free final time.
- Obtain this model as limit of microscopic models with large population.
- Numerical methods for this model.
- Stochastic dynamics (more classical in MFGs).

Simulations

One-dimensional case

$$\Omega = (0,1);$$
$$\Gamma = \partial\Omega;$$
$$\rho_0 = \delta_\ell;$$
$$\ell \in (0,1).$$

$$\chi(x) = \begin{cases} \frac{1+\cos\left(\frac{\pi x}{\varepsilon}\right)}{2\varepsilon}, & \text{if } |x| < \varepsilon, \\ 0, & \text{if } |x| \geq \varepsilon, \end{cases}$$

$$\eta(x) = \begin{cases} \frac{1-\cos\left(\frac{\pi d(x,\partial\Omega)}{\varepsilon}\right)}{2}, & \text{if } d(x,\partial\Omega) < \varepsilon, \\ 1, & \text{if } d(x,\partial\Omega) \geq \varepsilon, \end{cases}$$

$$K(x) = \frac{1}{1+\left(\frac{2x}{15}\right)^4},$$

$$\varepsilon = \frac{1}{10}.$$

Simulations

One-dimensional case

$\Omega = (0, 1)$;
$\Gamma = \partial\Omega$;
$\rho_0 = \delta_\ell$;
$\ell \in (0, 1)$.

$\rho_t = m\delta_{x(t)} + (1 - m)\delta_{y(t)}$;
$x(t)$: going left;
$y(t)$: going right;
$m \in [0, 1]$.

$$\chi(x) = \begin{cases} \frac{1+\cos\left(\frac{\pi x}{\varepsilon}\right)}{2\varepsilon}, & \text{if } |x| < \varepsilon, \\ 0, & \text{if } |x| \geq \varepsilon, \end{cases}$$

$$\eta(x) = \begin{cases} \frac{1-\cos\left(\frac{\pi d(x, \partial\Omega)}{\varepsilon}\right)}{2}, & \text{if } d(x, \partial\Omega) < \varepsilon, \\ 1, & \text{if } d(x, \partial\Omega) \geq \varepsilon, \end{cases}$$

$$K(x) = \frac{1}{1 + \left(\frac{2x}{15}\right)^4},$$

$$\varepsilon = \frac{1}{10}.$$

Simulations

One-dimensional case

$$\Omega = (0,1);\quad \Gamma = \partial\Omega;\quad \rho_0 = \delta_\ell;\quad \ell \in (0,1).$$

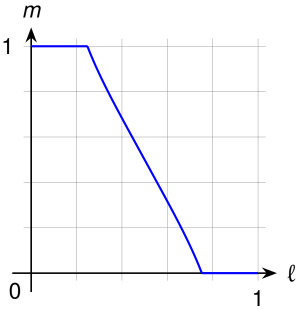
$$\rho_t = m\delta_{x(t)} + (1-m)\delta_{y(t)};$$
$x(t)$: going left;
$y(t)$: going right;
$m \in [0,1]$.

$$\chi(x) = \begin{cases} \frac{1+\cos\left(\frac{\pi x}{\varepsilon}\right)}{2\varepsilon}, & \text{if } |x| < \varepsilon, \\ 0, & \text{if } |x| \geq \varepsilon, \end{cases}$$

$$\eta(x) = \begin{cases} \frac{1-\cos\left(\frac{\pi d(x,\partial\Omega)}{\varepsilon}\right)}{2}, & \text{if } d(x,\partial\Omega) < \varepsilon, \\ 1, & \text{if } d(x,\partial\Omega) \geq \varepsilon, \end{cases}$$

$$K(x) = \frac{1}{1 + \left(\frac{2x}{15}\right)^4},$$

$$\varepsilon = \frac{1}{10}.$$



Simulations

One-dimensional case

$$\Omega = (0,1);\quad \Gamma = \partial\Omega;\quad \rho_0 = \delta_\ell;\quad \ell \in (0,1).$$

$$\rho_t = m\delta_{x(t)} + (1-m)\delta_{y(t)};$$

$x(t)$: going left;

$y(t)$: going right;

$m \in [0,1]$.

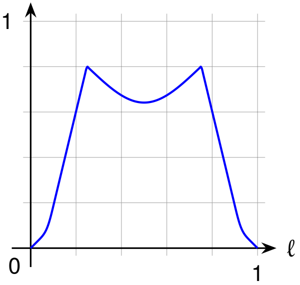
$$\chi(x) = \begin{cases} \frac{1+\cos\left(\frac{\pi x}{\varepsilon}\right)}{2\varepsilon}, & \text{if } |x| < \varepsilon, \\ 0, & \text{if } |x| \geq \varepsilon, \end{cases}$$

$$\eta(x) = \begin{cases} \frac{1-\cos\left(\frac{\pi d(x,\partial\Omega)}{\varepsilon}\right)}{2}, & \text{if } d(x,\partial\Omega) < \varepsilon, \\ 1, & \text{if } d(x,\partial\Omega) \geq \varepsilon, \end{cases}$$

$$K(x) = \frac{1}{1 + \left(\frac{2x}{15}\right)^4},$$

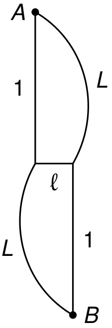
$$\varepsilon = \frac{1}{10}.$$

Exit time



Simulations

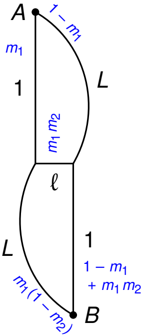
Braess-type paradox



Ω: the network;
$\Gamma = \{B\}$;
$\rho_0 = \delta_A$;
$L \in (1, +\infty)$, $\ell \in (0, 1)$;
χ, K, and ε as before;
$\eta(y) = 1$ for all $y \in \Omega$.

Simulations

Braess-type paradox



Ω: the network;
$\Gamma = \{B\}$;
$\rho_0 = \delta_A$;
$L \in (1, +\infty)$, $\ell \in (0, 1)$;
χ, K, and ε as before;
$\eta(y) = 1$ for all $y \in \Omega$.

If $0 < t < T_1$:

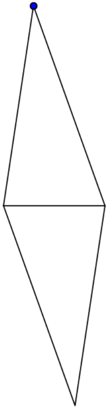
$$\rho_t = m_1 \delta_{x(t)} + (1 - m_1)\delta_{y(t)}.$$

If $T_1 < t < T_2$:

$$\rho_t = m_1(1 - m_2)\delta_{x(t)} + m_1 m_2 \delta_{z(t)} + (1 - m_1)\delta_{y(t)}.$$

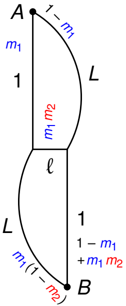
If $t > T_2$:

$$\rho_t = m_1(1 - m_2)\delta_{x(t)} + (1 - m_1 + m_1 m_2)\delta_{y(t)}.$$



Simulations

Braess-type paradox



By symmetry,

$$m_1 = 1 - m_1 + m_1 m_2 \quad \text{and} \quad 1 - m_1 = m_1(1 - m_2)$$

$$\implies m_1 = \frac{1}{2 - m_2}.$$

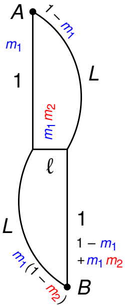
Simulations

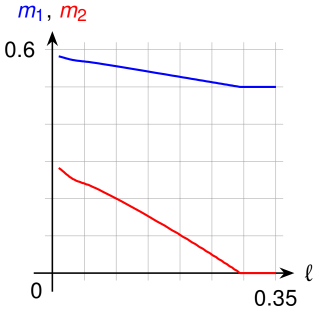
Braess-type paradox

By symmetry,

$$m_1 = 1 - m_1 + m_1 m_2 \quad \text{and} \quad 1 - m_1 = m_1(1 - m_2)$$

$$\implies m_1 = \frac{1}{2 - m_2}.$$





Simulations

Braess-type paradox

By symmetry,

$$m_1 = 1 - m_1 + m_1 m_2 \quad \text{and} \quad 1 - m_1 = m_1(1 - m_2)$$

$$\Longrightarrow \quad m_1 = \frac{1}{2 - m_2}.$$