Variant of Optimality Criteria Method for Multiple State Optimal Design Problems

Ivana Crnjac

J. J. STROSSMAYER UNIVERSITY OF OSIJEK
DEPARTMENT OF MATHEMATICS
Trg Ljudevita Gaja 6
31000 Osijek, Hrvatska
http://www.mathos.unios.hr

icrnjac@mathos.hr

Joint work with **K. Burazin and M. Vrdoljak**

WeConMApp

Stationary diffusion equation

Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{Sym_d})$ satisfying

$$\mathbf{A}\xi \cdot \xi \geq \alpha|\xi|^2, \quad \mathbf{A}^{-1}\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2, \qquad \xi \in \mathbf{R}^d$$

and $f \in \mathrm{H}^{-1}(\Omega)$. Stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Ω - mixture of two isotropic materials with conductivities $0 < \alpha < \beta$:

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I},$$

where $\chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$.

Stationary diffusion equation

Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{Sym_d})$ satisfying

$$\mathbf{A}\xi \cdot \xi \geq \alpha|\xi|^2, \quad \mathbf{A}^{-1}\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2, \qquad \xi \in \mathbf{R}^d$$

and $f \in \mathrm{H}^{-1}(\Omega)$. Stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Ω - mixture of two isotropic materials with conductivities $0 < \alpha < \beta$:

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I},$$

where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$.

Multiple state optimal design problem

$$\begin{cases} J(\chi) = \int_\Omega [\chi(\mathbf{x}) g_\alpha(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\chi(\mathbf{x})) g_\beta(\mathbf{x}, \mathsf{u}(\mathbf{x}))] \, d\mathbf{x} \longrightarrow \min, \\ \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}), \ \int_\Omega \chi \, d\mathbf{x} = q_\alpha, \end{cases}$$

$0 < q_\alpha < |\Omega|$, and g_α, g_β Caratheodory functions which satisfy growth condition

$$g_j(x, \mathsf{u}) \le a|\mathsf{u}|^s + b(x), \qquad j = \alpha, \beta,$$

for some $a > 0$, $b \in \mathrm{L}^1(\Omega)$ and $1 \le s < \frac{2d}{d-2}$,
and $\mathsf{u} = (u_1, \dots, u_m)$, $m \ge 2$, where u_i is the solution of

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases}, \qquad i = 1, \dots, m$$

where $f_i \in \mathrm{H}^{-1}(\Omega)$, $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, $0 < \alpha < \beta$.

Definition (Composite material)

If a sequence of characteristic functions $\chi_n \in \mathrm{L}^\infty(\Omega;\{0,1\})$ and conductivities $\mathbf{A}^n(x) = \chi_n(x)\alpha\mathbf{I} + (1-\chi_n(x))\beta\mathbf{I}$ satisfy

$$\begin{array}{ccc} \chi_n & \overset{*}{\rightharpoonup} & \theta \\ \mathbf{A}^n & \overset{H}{\rightharpoonup} & \mathbf{A}, \end{array}$$

then it is said that $\mathbf{A}$ is homogenised tensor of two-phase composite material with proportions θ of first material and microstructure defined by the sequence (χ_n).

Definition (H-convergence)

A sequence of matrix functions $\mathbf{A}^n$ is said to H-converge to $\mathbf{A}$ if for every f the sequence of solutions of

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}^n\nabla u_n) = f \\ u_n \in \mathrm{H}_0^1(\Omega) \end{array} \right.$$

satisfies $u_n \rightharpoonup u$ in $\mathrm{H}_0^1(\Omega)$, $\mathbf{A}^n\nabla u_n \rightharpoonup \mathbf{A}\nabla u$ in $\mathrm{L}^2(\Omega;\mathbf{R}^d)$, where u is the solution of the homogenised equation

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega)\,. \end{array} \right.$$

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A} = \mathrm{diag}(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+)\,,$$

where
$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym_d}) : \ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}$

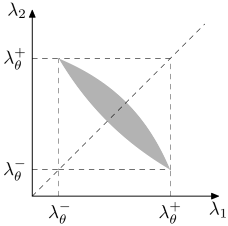
$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$\lambda_\theta^- \le \lambda_j \le \lambda_\theta^+ \quad j = 1, \dots, d$$

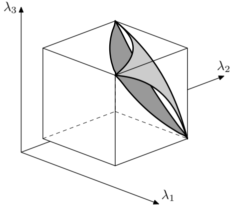
$$\sum_{j=1}^d \frac{1}{\lambda_j - \alpha} \le \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{j=1}^d \frac{1}{\beta - \lambda_j} \le \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

2D:



3D:



Original problem:

$$\left\{\begin{array}{c} J(\chi) = \int_\Omega [\chi(\mathbf{x}) g_\alpha(\mathbf{x}, \mathsf{u}) + (1-\chi(\mathbf{x})) g_\beta(\mathbf{x}, \mathsf{u})] \, d\mathbf{x} \longrightarrow \min \,, \\ \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \,, \int_\Omega \chi \, d\mathbf{x} = q_\alpha \,. \end{array}\right.$$

Generalized objective function is

$$J(\theta, \mathbf{A}) = \int_\Omega [\theta(\mathbf{x}) g_\alpha(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\theta)(\mathbf{x})) g_\beta(\mathbf{x}, \mathsf{u}(\mathbf{x}))] \, d\mathbf{x}$$

where $\mathsf{u} = (u_1, \ldots, u_m)$ and u_i is the solution of the state equation

$$\left\{\begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{array}\right. , \qquad i = 1, \ldots, m.$$

Relaxed problem:

$$\left\{\begin{array}{c} J(\theta, \mathbf{A}) \longrightarrow \min \,, \\ (\theta, \mathbf{A}) \in \mathcal{A} \,, \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,. \end{array}\right.$$

Optimality condition

- Let $(\theta^*, \mathbf{A}^*)$ be optimal point of the relaxed problem and $\varepsilon \mapsto (\theta^\varepsilon, \mathbf{A}^\varepsilon)$ be a smooth path with $(\theta^0, \mathbf{A}^0) = (\theta^*, \mathbf{A}^*)$.
- If $\varepsilon \mapsto J(\theta^\varepsilon, \mathbf{A}^\varepsilon)$ is smooth, then necessary condition of optimality is

$$\delta J(\theta^*, \mathbf{A}^*) := \frac{d}{d\epsilon} J(\theta^\varepsilon, \mathbf{A}^\varepsilon)_{|_{\varepsilon=0}} \geq 0$$

Let us introduce adjoint states $p_1, \ldots, p_m$ as solutions of

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla p_i) = \theta \dfrac{\partial g_\alpha}{\partial u_i}(\cdot, \mathsf{u}) + (1-\theta)\dfrac{\partial g_\beta}{\partial u_i}(\cdot, \mathsf{u}) \\ p_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m\,.$$

Then

$$\delta J(\theta^*, \mathbf{A}^*) = \int_\Omega \delta\theta(\mathbf{x})[g_\alpha(\mathbf{x}, \mathsf{u}^*(\mathbf{x})) - g_\beta(\mathbf{x}, \mathsf{u}^*(\mathbf{x})) + l]\, d\mathbf{x} - \int_\Omega \sum_{i=1}^m \delta\mathbf{A}\nabla u_i^* \cdot \nabla p_i^*\, d\mathbf{x}\,,$$

for any admissible variation $(\delta\theta, \delta\mathbf{A}) := \left(\dfrac{d\theta^\varepsilon}{d\varepsilon}, \dfrac{d\mathbf{A}^\varepsilon}{d\varepsilon}\right)\Bigg|_{\varepsilon=0}$.

Optimality conditions

How to choose a smooth path?

1. Let us fix θ^*.
   a) Then set $\mathcal{K}(\theta^*)$ is convex and we can choose segment

$$\mathbf{A}^\varepsilon = \mathbf{A}^* + \varepsilon(\mathbf{A} - \mathbf{A}^*), \qquad \mathbf{A} \in \mathcal{K}(\theta^*)$$

$$\vdots$$

[L.Tartar, G. Allaire, M. Vrdoljak]

   b) We choose segment in terms of inverse matrices

$$(\mathbf{A}^\varepsilon)^{-1} = (\mathbf{A}^*)^{-1} + \varepsilon(\mathbf{A}^{-1} - (\mathbf{A}^*)^{-1}), \qquad \mathbf{A}^{-1} \in \tilde{\mathcal{K}}(\theta^*)$$

Optimality conditions

How to choose a smooth path?

1. Let us fix θ^*.
   a) Then set $\mathcal{K}(\theta^*)$ is convex and we can choose segment

$$\mathbf{A}^\varepsilon = \mathbf{A}^* + \varepsilon(\mathbf{A} - \mathbf{A}^*), \qquad \mathbf{A} \in \mathcal{K}(\theta^*)$$

$$\vdots$$

[L.Tartar, G. Allaire, M. Vrdoljak]

   b) We choose segment in terms of inverse matrices

$$(\mathbf{A}^\varepsilon)^{-1} = (\mathbf{A}^*)^{-1} + \varepsilon(\mathbf{A}^{-1} - (\mathbf{A}^*)^{-1}), \qquad \mathbf{A}^{-1} \in \tilde{\mathcal{K}}(\theta^*)$$

Optimality conditions

How to choose a smooth path?

1. Let us fix θ^*.
   a) Then set $\mathcal{K}(\theta^*)$ is convex and we can choose segment

$$\mathbf{A}^\varepsilon = \mathbf{A}^* + \varepsilon(\mathbf{A} - \mathbf{A}^*), \qquad \mathbf{A} \in \mathcal{K}(\theta^*)$$

$$\vdots$$

[L.Tartar, G. Allaire, M. Vrdoljak]

   b) We choose segment in terms of inverse matrices

$$(\mathbf{A}^\varepsilon)^{-1} = (\mathbf{A}^*)^{-1} + \varepsilon(\mathbf{A}^{-1} - (\mathbf{A}^*)^{-1}), \qquad \mathbf{A}^{-1} \in \tilde{\mathcal{K}}(\theta^*)$$

Sets $\mathcal{K}(\theta)$ and $\tilde{\mathcal{K}}(\theta)$

One can show equvalence of the following sets

$\mathcal{K}(\theta)$ - Set of all symmetric matrices $\mathbf{A}$ with eigenvalues $\lambda_1, \ldots, \lambda_d$ which satisfy inequalities

$$\begin{aligned} \lambda_\theta^- \leq \quad & \lambda_j \quad \leq \lambda_\theta^+, \quad j = 1, \ldots, d \\ \sum_{j=1}^d \frac{1}{\lambda_j - \alpha} \quad & \leq \quad \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha} \\ \sum_{j=1}^d \frac{1}{\beta - \lambda_j} \quad & \leq \quad \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+} \end{aligned}$$

$\tilde{\mathcal{K}}(\theta)$ - Set of all symmetric matrices $\mathbf{A}^{-1}$ with eigenvalues $\nu_1, \ldots, \nu_d$ which satisfy inequalities

$$\begin{aligned} \nu_\theta^+ \leq \quad & \nu_j \quad \leq \nu_\theta^-, \quad j = 1, \ldots, d \\ \sum_{j=1}^d \frac{1}{\alpha^{-1} - \nu_j} \quad & \leq \quad \frac{1}{\alpha^{-1} - \nu_\theta^-} + \frac{d-1}{\alpha^{-1} - \nu_\theta^+} \\ \sum_{j=1}^d \frac{1}{\nu_j - \beta^{-1}} \quad & \leq \quad \frac{1}{\nu_\theta^- - \beta^{-1}} + \frac{d-1}{\nu_\theta^+ - \beta^{-1}} \end{aligned}$$

$$\mathbf{A} \in \mathcal{K}(\theta) \Leftrightarrow \mathbf{A}^{-1} \in \tilde{\mathcal{K}}(\theta)$$

Optimality conditions

From necessary condition of optimality

$$-\int_\Omega \sum_{i=1}^m \delta \mathbf{A} \nabla u_i^* \cdot \nabla p_i^* \, d\mathbf{x} \geq 0$$

then follows (for a.e. in Ω)

$$(\mathbf{A}^*)^{-1} : \mathbf{N} = \min_{\mathbf{A} \in \mathcal{K}(\theta^*)} \mathbf{A}^{-1} : \mathbf{N},$$

where $\mathbf{N} = \text{Sym} \sum_{i=1}^m \sigma_i^* \otimes \tau_i^*$, $\sigma_i^* = \mathbf{A}^* \nabla u_i^*$ and $\tau_i^* = \mathbf{A}^* \nabla p_i^*$, $i = 1, \ldots, m$.

Theorem (von Neumann)

For symmetric matrices $\mathbf{A}$ and $\mathbf{M}$ following inequality is valid

$$A : M \leq \lambda(\mathbf{A}) \cdot \lambda(\mathbf{M}),$$

where $\lambda(\mathbf{A})$ and $\lambda(\mathbf{M})$ are vectors of eigenvalues of $\mathbf{A}$ and $\mathbf{M}$ in nondecreasing order. Equality holds if and only if $\mathbf{A}$ and $\mathbf{M}$ are diagonalizable in same basis.

$$\begin{cases} \displaystyle\sum_{i=1}^{d} \nu_i \eta_i \longrightarrow \min, \\ \nu_\theta^+ \leq \nu_j \leq \nu_\theta^- \ , \quad j = 1, \dots, d \\ \displaystyle\sum_{j=1}^{d} \frac{1}{\alpha^{-1} - \nu_j} \leq \frac{1}{\alpha^{-1} - \nu_\theta^-} + \frac{d-1}{\alpha^{-1} - \nu_\theta^+} \\ \displaystyle\sum_{j=1}^{d} \frac{1}{\nu_j - \beta^{-1}} \leq \frac{1}{\nu_\theta^- - \beta^{-1}} + \frac{d-1}{\nu_\theta^+ - \beta^{-1}} \end{cases}$$

where ν_i are increasing, while η_i decreasing for $i = 1..d$

Optimality conditions

2. Let us now consider variation of θ. We take smooth path $(\theta^\varepsilon, \mathbf{A}^\varepsilon)$ such that a.e. $\mathbf{x} \in \Omega$

$$(\mathbf{A}^\varepsilon)^{-1}(\mathbf{x}) : \mathbf{N}(\mathbf{x}) = \min_{\mathbf{A} \in \mathcal{K}(\theta^\varepsilon)} (\mathbf{A}^{-1}(\mathbf{x}) : \mathbf{N}(\mathbf{x}))$$

Again, from neccesary condition of optimality it follows (a.e. $\mathbf{x} \in \Omega$)

$$\delta\theta(\mathbf{x}) \left(g_\alpha(\mathbf{x}, \mathsf{u}^*(\mathbf{x})) - g_\beta(\mathbf{x}, \mathsf{u}^*(\mathbf{x})) + l + \frac{\partial g}{\partial \theta}(\theta^*(\mathbf{x}), \mathbf{N}(\mathbf{x})) \right) \geq 0.$$

Theorem

Let $(\theta^*, \mathbf{A}^*)$ be minimizer of objective functional $J(\theta, \mathbf{A})$. We introduce symmetric matrix $\mathbf{N} = Sym \sum_{i=1}^{m} \sigma_i^* \otimes \tau_i^*$, for $\sigma_i^* = \mathbf{A}\nabla u_i^*$, $\tau_i^* = \mathbf{A}\nabla p_i^*$ and define function $g(\theta, \mathbf{N}) = \min_{\mathbf{A}\in\mathcal{K}(\theta)} (\mathbf{A}^{-1} : \mathbf{N})$. Then

$$(\mathbf{A}^*)^{-1}(\mathbf{x}) : \mathbf{N}(\mathbf{x}) = g(\theta^*(\mathbf{x}), \mathbf{N}(\mathbf{x})), \qquad \textit{a.e } x \in \Omega.$$

Moreover, if we define function

$$R^*(\mathbf{x}) = g_\alpha(\mathbf{x}, \mathsf{u}) - g_\beta(\mathbf{x}, \mathsf{u}) + l + \frac{\partial g}{\partial \theta}(\theta^*(\mathbf{x}), \mathbf{N}^*(\mathbf{x}))$$

the optimal θ^* satisfies (a. e. on Ω)

$$\begin{array}{rl} \theta^*(\mathbf{x}) = 0 & \textit{if } R^*(\mathbf{x}) > 0 \\ \theta^*(\mathbf{x}) = 1 & \textit{if } R^*(\mathbf{x}) < 0 \\ 0 \le \theta^*(\mathbf{x}) \le 1 & \textit{if } R^*(\mathbf{x}) = 0\,. \end{array}$$

Algorithm

Take some initial θ^0 and $\mathbf{A}^0$. For k from 0 to N:

1. Calculate u_i^k, $i = 1, \dots, m$, the solution of

$$\begin{cases} -\text{div}\,(\mathbf{A}^k \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega)\,. \end{cases}$$

2. Calculate p_i^k, $i = 1, \dots, m$, the solution of

$$\begin{cases} -\text{div}\,(\mathbf{A}^k \nabla p_i) = \theta^k \dfrac{\partial g_\alpha}{\partial u_i}(\cdot, \mathsf{u}^k) + (1-\theta^k)\dfrac{\partial g_\beta}{\partial u_i}(\cdot, \mathsf{u}^k) \\ p_i \in \mathrm{H}_0^1(\Omega),\ \mathsf{u}^k = (u_1^k, \dots, u_m^k) \end{cases}$$

   and define $\sigma_i^k := \mathbf{A}^k \nabla u_i^k$, $\tau_i^k := \mathbf{A}^k \nabla p_i^k$ and $\mathbf{N}^k := \mathrm{Sym}\sum_{i=1}^m (\sigma_i^k \otimes \tau_i^k)$.

3. For $\mathbf{x} \in \Omega$, let $\theta^{k+1}(\mathbf{x})$ be the zero of function

$$\theta \mapsto g_\alpha(\mathbf{x}, \mathsf{u}^k(\mathbf{x})) - g_\alpha(\mathbf{x}, \mathsf{u}^k(\mathbf{x})) + l + \frac{\partial g}{\partial \theta}(\theta, \mathbf{N}^k(\mathbf{x}))\,,$$

   and if a zero doesn't exist, take 0 (or 1) in case when this function is positive (or negative) on $\langle 0, 1\rangle$.

4. Let $(\mathbf{A}^{k+1})^{-1}(\mathbf{x})$ be the minimizer in the definition of $g(\theta^{k+1}(\mathbf{x}), \mathbf{N}^k(\mathbf{x}))$.

Theorem (d=2)

If dimension $d = 2$, then for given $\theta \in [0,1]$ and matrix $\mathbf{N}$ with eigenvalues $\eta_1 \geq \eta_2$ we have

A. *If $\eta_2 > 0$ and $\theta^A := \left(\alpha \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} - \beta\right) \frac{1}{\alpha - \beta}$, then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} \frac{1}{\beta}\,(\beta^2 - \alpha^2)\left(\frac{\sqrt{\eta_1} + \sqrt{\eta_2}}{(\theta(\alpha - \beta) + \beta + \alpha}\right)^2, & \theta < \theta^A \\ \frac{(\beta - \alpha)\,\eta_1}{(\theta(\alpha - \beta) + \beta)^2} + \eta_2 \left(\frac{1}{\alpha} - \frac{1}{\beta}\right), & \theta \geq \theta^A \end{cases}.$$

B. *If $\eta_1 < 0$ and $\theta^B := \left(\frac{\sqrt{-\eta_1}}{\sqrt{-\eta_2}} - 1\right) \frac{\beta}{\alpha - \beta}$, then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} -\frac{1}{\alpha}\,(\beta^2 - \alpha^2)\left(\frac{\sqrt{-\eta_1} + \sqrt{-\eta_2}}{\theta(\alpha - \beta) + 2\beta}\right)^2, & \theta > \theta^B \\ \frac{(\beta - \alpha)\,\eta_1}{(\theta(\alpha - \beta) + \beta)^2} + \eta_2 \left(\frac{1}{\alpha} - \frac{1}{\beta}\right), & \theta \leq \theta^B \end{cases}.$$

C. *If $\eta_1 \geq 0$ and $\eta_2 \leq 0$, then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \frac{(\beta - \alpha)\,\eta_1}{(\theta(\alpha - \beta) + \beta)^2} + \eta_2 \left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

Theorem (d=3)

For $d=3$, given $\theta \in [0,1]$ and matrix $\mathbf{N}$ with eigenvalues $\eta_1 \geq \eta_2 \geq \eta_3$ we have

A. *If $\eta_3 = 0$ then $\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \frac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2)$.*

B. *If $\eta_3 > 0$ and additionally $\sqrt{\eta_2}+\sqrt{\eta_3}-\sqrt{\eta_1} > 0$, it holds*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} \beta^{-1}(\beta-\alpha)(\alpha+2\beta)\left(\frac{\sqrt{\eta_1}+\sqrt{\eta_2}+\sqrt{\eta_3}}{2\theta(\alpha-\beta)+\alpha+2\beta}\right)^2, & \theta < \theta_1^B, \\ \beta^{-1}(\beta^2-\alpha^2)\left(\frac{\sqrt{\eta_2}+\sqrt{\eta_3}}{\theta(\alpha-\beta)+\alpha+\beta}\right)^2 + (\beta-\alpha)\frac{\eta_1}{(\theta\alpha+(1-\theta)\beta)^2}, & \theta_1^B \leq \theta < \theta_2^B, \\ \eta_3(\alpha^{-1}-\beta^{-1}) + \frac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2), & \theta \geq \theta_2^B, \end{cases}$$

where $\theta_1^B = 1 - \frac{\alpha(\sqrt{\eta_2}+\sqrt{\eta_3}-2\sqrt{\eta_1})}{(\alpha-\beta)(\sqrt{\eta_2}+\sqrt{\eta_3}-\sqrt{\eta_1})}$ and $\theta_2^B = 1 - \frac{\alpha(\sqrt{\eta_3}-\sqrt{\eta_2})}{(\alpha-\beta)\sqrt{\eta_3}}$.
If $\sqrt{\eta_2}+\sqrt{\eta_3}-\sqrt{\eta_1} \leq 0$ then we omit the first case in the above formula.

C. *If $\eta_3 < 0$ then, if η_2 and η_1 are negative as well then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} -\alpha^{-1}(\beta-\alpha)(2\alpha+\beta)\left(\frac{\sqrt{-\eta_1}+\sqrt{-\eta_2}+\sqrt{-\eta_3}}{2\theta(\alpha-\beta)+3\beta}\right)^2, & \theta > \theta_1^C, \\ -\alpha^{-1}(\beta^2-\alpha^2)\left(\frac{\sqrt{-\eta_2}+\sqrt{-\eta_3}}{\theta(\alpha-\beta)+2\beta}\right)^2 + \eta_1\frac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}, & \theta_2^C < \theta \leq \theta_1^C, \\ (\alpha^{-1}-\beta^{-1})\eta_3 + \frac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2), & \theta \leq \theta_2^C, \end{cases}$$

where $\theta_1^C = \frac{\beta(\sqrt{-\eta_2}+\sqrt{-\eta_3}-2\sqrt{-\eta_1})}{(\beta-\alpha)(\sqrt{-\eta_2}+\sqrt{-\eta_3}-\sqrt{-\eta_1})}$ and $\theta_2^C = \frac{\beta(\sqrt{-\eta_3}-\sqrt{-\eta_2})}{(\beta-\alpha)\sqrt{-\eta_3}}$.
If $\eta_2 < 0$ and $\eta_1 \geq 0$ then θ_1^C is not defined and we can express $\frac{\partial g}{\partial \theta}(\theta, \mathbf{N})$ by the second and third term in the formula above, omitting the assumption $\theta \leq \theta_1^C$ in the second case.
If $\eta_2 \geq 0$ then both θ_1^C and θ_2^C are not defined and $\frac{\partial g}{\partial \theta}$ is given by formula in the third case above, for any $\theta \in [0,1]$.

Example 1.

We consider two dimensional problem of weighted energy minimization

$$J(\theta, \mathbf{A}) = 2\int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \longrightarrow \min,$$

where $\Omega \subseteq \mathbf{R}^2$ is a ball $B(\mathbf{0}, 2)$, $\alpha = 1$, $\beta = 2$, while u_1 and u_2 are state functions for

$$\left\{ \begin{array}{l} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{array} \right. , \quad i = 1, 2,$$

where we take $f_1 = \chi_{B(\mathbf{0},1)}$ and $f_2 \equiv 1$ for right-hand sides.

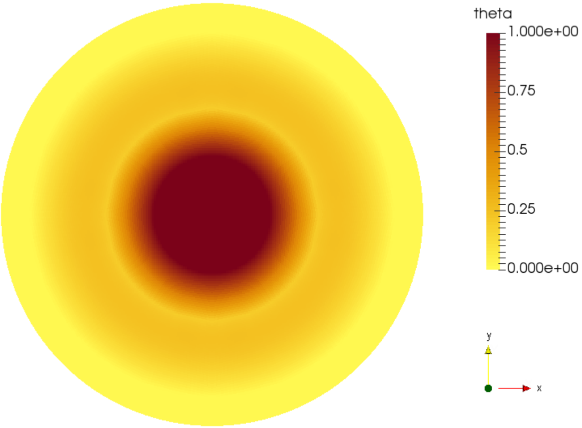
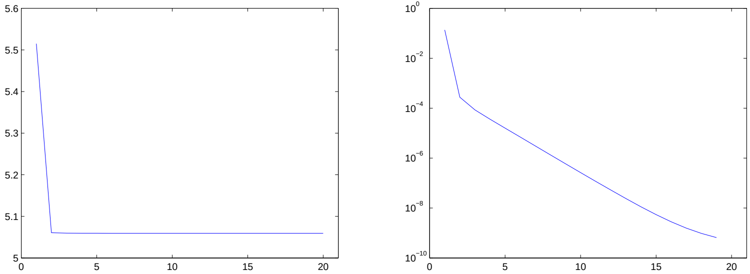


Figure: Optimal distribution of materials in circle with volume constraint 25% of the first material.

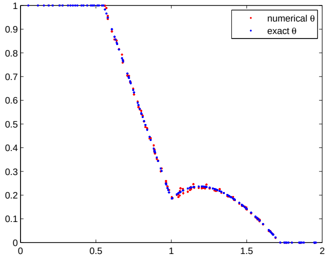


(a) Cost functional J.

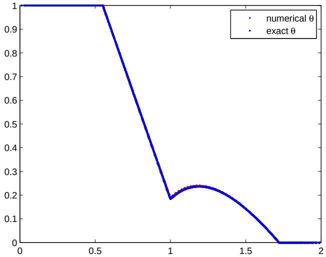
(b) $\|\theta^k - \theta^{k+1}\|_{L^2}$ in terms of the iteration number k.

Figure: Convergence history for energy minimization in circle.

Comparison between numerical and exact solution



(a) Mesh refinement 4



(b) Mesh refinement 6

Figure: Values of exact and numerical θ.

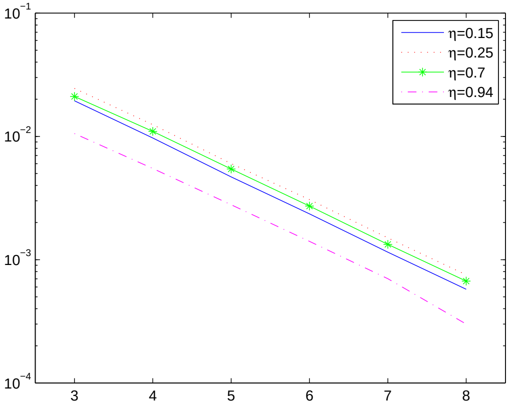


Figure: Dependence of L^1 error between numerical and exact solution with respect to mesh refinement.

Example 2.

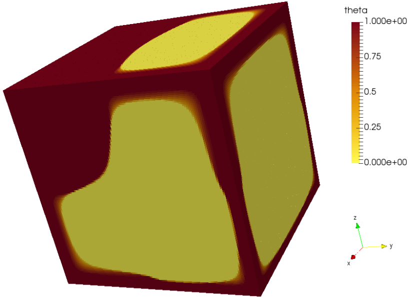
Second example is three dimensional energy minimization problem

$$J(\theta, \mathbf{A}) = \int_\Omega (f_1 u_1 + f_2 u_2)\, d\mathbf{x} \longrightarrow \min,$$

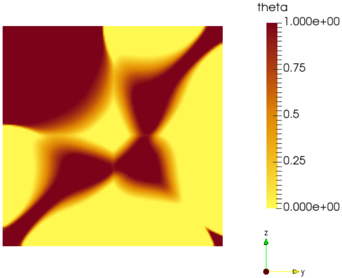
with $\alpha = 1$, $\beta = 2$ and two state equations

$$\begin{cases} -\text{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases}, \quad i = 1, 2.$$

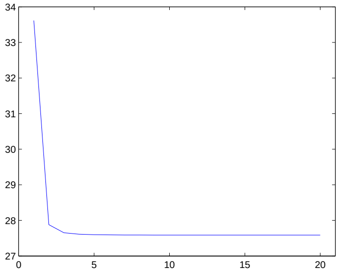
We take cube $\Omega = [-1, 1]^3$ as domain and set function f_1 to be zero on the upper half $(z > 0)$ and 10 on the lower half of the cube, while function f_2 to be zero on the left half $(y < 0)$ and 10 on the right half of the cube.



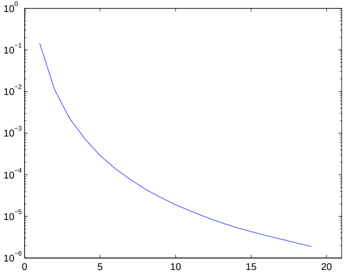
(a) Optimal distribution of materials in a cube.



(b) Intersection of the cube with x=0 plane.



(c) Cost functional J.



(d) $\|\theta^k - \theta^{k+1}\|_{L^2}$ in terms of the iteration number k.

Figure: Convergence history for energy minimization.

Thank you for your attention!