Benasque
VII Partial differential equations, optimal design and numerics
2017, Aug 20 – Sep 01

Application of boundary triples theory in asymptotic spectral analysis.

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I would like to devote my present talk to the illustration of how useful the theory of boundary triples can be for solving different direct and inverse spectral problems for differential operators.

The theory of boundary triples is in fact an abstract framework for dealing with extensions of symmetric operators and allows to reduce a problem in original Hilbert space to the problem in a boundary space (which in the case of finite deficiency indexes is just a finite-dimensional Euclidian space). The ideology of boundary space triples came out of the ideas of Mark Krein and his mathematical school. Then it was developed mainly by Ukrainian-Soviet mathematicians, among them Gorbachuk, Kochubej, Mihailets, and more recently Derkach and Malamud. At the same time independently similar ideas appeared in the West, in particular in the works of Grubb from Copenhagen. A successful blending of these two approaches is presented in the recent works of joint British-Russian group comprising of Naboko, Marletta, Brown, Wood and in the works of a joint German-Finish-Ukrainian group comprising Derkach, Malamud, Bernt, Hassi, Neidthardt and others.

There are certain classes of problems for the so-called quantum graphs and thin structures for which the classical boundary triples theory works perfectly well. We will consider metric graphs embedded in $\mathbb{R}^d$. A metric graph Γ consists of intervals $e_j = [a_j, b_j]$ (finite or infinite), which are combined at the vertices in some way. Hilbert space $L_2(e_j)$ is associated with each edge, and their orthogonal sum is the Hilbert space associated with the whole graph. This sum $L_2(\Gamma) = \oplus_j L_2(e_j)$ can be either finite or infinite, and this space is the same for all graphs constructed of the same set of edges. On this space consider a second order differential expression (for example, a Laplacian or Shrodinger one). The connectivity of the graph is reflected in the description of the domain of corresponding operator, which is defined by the given differential expression acting on the functions from the Sobolev space $W_2^2(\Gamma)$ satisfying some additional conditions at the junctions of edges – vertices of the graph. There are many different types of such coupling conditions, but in applications two main types are usually singled out (the so-called δ and δ' types of coupling conditions).

δ-type coupling conditions at a vertex V: the function u is assumed to be continuous trough the vertex V and the sum of inward normal derivatives at the endpoints of edges which meet at this vertex must be proportionate to the value of the function u at V:

$$\sum \partial_n u|_{e_j} = \alpha_V u(V),$$

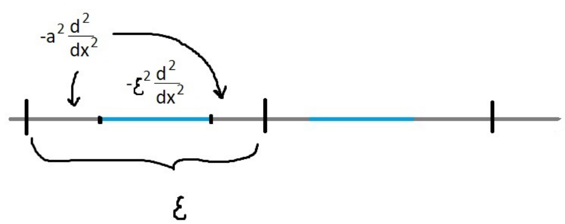
where α_V is the so-called coupling constant. When $\alpha_V = 0$, it is the case of the classical Kirchhoff conditions. When all coupling constant are real, one obtains a self-adjoint operator.

δ' type coupling conditions at a vertex V: This is roughly speaking a “dual” condition to the latter. The normal inward derivative is “continuous” at the vertex V and the sum of values of the function at the endpoints belonging to the vertex V is proportionate to the common value of the normal derivatives at V.

The operator thus defined is usually referred to as a quantum graph. For quantum graphs, different types of inverse problems naturally appear. One of them is the inverse topology problem to reconstruct underlying metric graph based on the given spectrum of the differential operator. It is known due to Kurasov that the graph can be reconstructed in a unique way in the case of rationally independent lengths of edges. However, if this assumption is dropped then only the Euler characteristic and the total length of the graph can be determined. In our joint work with Kiselev and Karpenko we managed to narrow down this gap for arbitrary graph obtaining some additional necessary conditions of isospectrality of graphs.

The second type of inverse problems is the problem of reconstruction of coupling constants at the vertices of the graph, and it was solved in full in [2]. All these results were obtained using boundary triples techniques.

Now let me describe another type of problems connected with a periodic quantum graph. Let the graph be infinite and periodic (the total length of the graph on the periodicity cell is assumed to be equal to ε) and let the differential expression depend on the same parameter ε on some edges of the graph. The simplest example of such graph was considered in the paper of Cherednichenko and Kiselev [3]. It is the following infinite chain-graph: Here the differential expression is $-\varepsilon^2 \frac{d^2}{dx^2}$ on "soft" edges and $-a^2 \frac{d^2}{dx^2}$, $a \in \mathbb{R}$

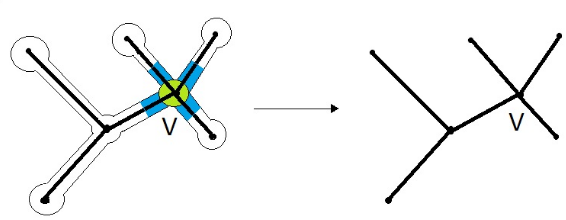


on "stiff" edges. One ends up with the operator family A^ε which depends on ε and this is exactly the case of high-contrast homogenization when ε tends to 0.

In fact, the boundary triple theory is the simplest way to compute the limiting spectrum of the family. It was done in [3] by an explicit construction of the corresponding Weyl-Titchmarsh matrix-function.

Obviously, quantum graphs are ideal objects which do not appear in nature, but they can be shown to describe thin enough networks provided that such network converges to a graph.

For example, let us consider a network surrounding the graph where the green area is the vertex part for



vertex V and the blue area is the corresponding edge part for V.

Where does this network converge to when the volume of tubes and junctions tends to zero is the main question for such thin structures. Rigorously this problem was considered by a number of researchers including Exner, Post, Kuchment, Zeng about ten years ago. They investigated the Laplacian with Neumann boundary conditions on the boundary of such domains. They proved the spectral convergence of the spectra of these Laplacians to the spectra of limiting quantum graphs with some special conditions at the vertices. Depending on the ratio of the volume of vertex part and the volume of the corresponding edge part of the network, there can be three different types of coupling conditions at the vertex in the corresponding limiting graph. In the most interesting case when the volumes decay at the same rate, they ended up with the so-called energy-dependant δ-type coupling condition. The coupling constant here is a function of the spectral parameter λ. More precisely, the coupling constant is $\alpha_V\lambda$. We can go much further and prove on the basis of boundary triple techniques that this limiting quantum graph is in fact unitary equivalent to the quantum graph of the same structure but with proper δ'-type coupling condition at the corresponding vertex with $\alpha_V = \lim_{\varepsilon\to 0}\frac{Vol(V)}{Vol(E)}$. Moreover, we can prove not only spectral but also a version of norm-resolvent convergence of the thin structure to this graph.

All these results can be obtained thanks to a correct choice of the boundary triple for the differential operator considered.

Let me finish this talk by outlining the main concepts of the boundary triples theory. For any closed densely defined symmetric operator $A_{\min}$ with equal deficiency indices $n_+ = n_- = n$ one can construct a triple which consists of an auxiliary boundary space $\mathfrak{H}$ (which can be chosen as $\mathbb{C}^n$ for finite n) and two boundary operators Γ_0 and Γ_1 acting from $Dom(A^*_{\min})$ to $\mathfrak{H}$. Denote $A_{\max} = A^*_{\min}$

The operators Γ_0, Γ_1 here must be surjective and satisfy the Green identity:

$$\langle A_{max}u, v\rangle_H - \langle u, A_{max}v\rangle_H = \langle \Gamma_1 u, \Gamma_0 v\rangle_{\mathfrak{H}} - \langle \Gamma_0 u, \Gamma_1 v\rangle_{\mathfrak{H}}\,.$$

In this triple the differential operator A considered is a proper almost solvable extension of the suitably chosen minimal symmetric operator $A_{\min}$ ($A_{\min} \subset A \subset A_{\max}$) and can be parameterized by a bounded operator $B \in \mathcal{B}(\mathfrak{H})$ such that $Dom(A) \equiv Dom(A_B) = \{u \in Dom(A_{\max}) | \Gamma_1 u = B\Gamma_0 u\}$.

Let us also introduce the main object of the theory of boundary triples which is the Weyl-Titchmarsh function. It is defined by the following relation:

$$M(z)\Gamma_0 u_z = \Gamma_1 u_z, \quad \forall u_z \in Ker(A_{\max} - z)$$

which is the natural generalization of the classical Weyl function for Sturm-Liuville operators on the half line. In many cases of interest, $M(z)$ can be constructed explicitly and contains all the information about the structure of the underlying metric graph. The operator B for a given self-adjoint operator A can be chosen as a diagonal matrix comprising the coupling constants. The M-function has lots of nice properties. First of all, it is a double-sided R-function and the spectrum of the operator A_B is the set of zeroes of the matrix $M(z) - B$. By zeroes of $M(z) - B$ we understand such points z for which one of the eigenvalues of matrix vanishes. The consideration of $M(z) - B$ allows us to pass over from the study of an unbounded operator to the study of a finite-dimensional object which makes solving spectral problems for differential operators much easier. *References:*

[1] *Ershova Yu., Karpenko I. I., Kiselev A.V.* Isospectrality for graph Laplacians under the change of coupling at graph vertices: necessary and sufficient conditions. *Mathematika* **62** (2016), no. 1, 210–242.

[2] *Ershova Yu., Karpenko I. I., Kiselev A.V.* On Inverse Topology Problem for Laplace Operators on Graphs. *Carpathian Math.Pub.* **6**(2) (2014), 230-236.

[3] *Cherednichenko K., Kiselev A.* Norm-Resolvent Convergence of One-Dimensional High-Contrast Periodic Problems to a KronigPenney Dipole-Type Model. *Comm. Math. Phys.* **349** (2017), no. 2, 441–480.