A semi-linear Dirichet problem with a singular term at u=0

Juan Casado-Díaz, University of Seville

Work in collaboration with: **F. Murat, University Paris VI**

We are interested in the semilinear problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = F(x, u) \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

with

$$\Omega \subset \mathbb{R}^N \quad \text{bounded, open}$$

a satisfies usual conditions to have the operator

$$u \in W_0^{1,p}(\Omega) \longmapsto -\operatorname{div} a(x, \nabla u) \in W^{-1,p'}(\Omega)$$

a Leray-Lions monotone operator.

$$\lim_{s \to 0} F(x, s) = +\infty$$

This type of problems has been considered by several authors

M.G. Crandall, P.H. Rabinowitz, L. Tartar (1977).

$$\begin{cases} -\sum_{ij} a_{ij}(x)\partial_{ij}^2 u + \sum_i b_i(x)\partial_i u + c(x)u = F(x,u) \ \text{ in } \ \Omega \\ u = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

Ω smooth, coefficients in $C^0(\overline{\Omega})$.

$$\lim_{s\to 0} F(x,s) = +\infty \ \text{ uniformly in } \overline{\Omega}, \quad F(x,.) \text{ nonincreasing}$$

It is proved the existence of a positive solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$.

For F not necessarily nonincreasing they take $F(x,u) = \lambda\, G(x,u)$.

$\exists$ a set of solutions $(\lambda, u) \in \mathbb{R}\times C^0(\overline{\Omega})$, connected, unbounded and containing (0,0).

Most of the authors consider positive right-hand sides

M.M. Coclite, G. Palmieri (1989).

$$\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} + (\lambda u)^q \ \text{ in } \ \Omega \\ \qquad u = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

$\forall \gamma > 0, q > 1, \exists \tilde{\lambda} > 0$ such that $\exists$ a positive solution if and only if $\lambda < \tilde{\lambda}$

L. Boccardo, L. Orsina (2010).

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = \dfrac{f(x)}{u^\gamma} \ \text{ in } \ \Omega, \\ \qquad u = 0 \ \text{ on } \ \partial\Omega \end{cases} \qquad f \geq 0, \ \ f \in L^1(\Omega).$$

$\exists$ a positive solution in $H^1_{\mathrm{loc}}(\Omega) \cap W^{1,1}_0(\Omega)$.

It is obtained as the limit for the right-hand sides $\frac{f(x)}{u^\gamma + \frac{1}{n}}$.

Integrability results for u and ∇u are also obtained.

L. Boccardo, J. Casado-Díaz (2014). Previous approximation method provides a unique solution. It is also studied the homogenization problem

$$-\operatorname{div}(A_n(x)\nabla u_n) = \frac{f(x)}{u_n^\gamma} \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega$$

D. Giachetti, P.J. Martínez-Aparicio, F. Murat. Several works (2016, 2017, to appear)

$$-\operatorname{div}(A(x)\nabla u) = F(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

$$0 \leq F(x,s) \leq \frac{h(x)}{\Gamma(s)}, \quad h \in L^r(\Omega), \ \Gamma(0) = 0, \Gamma \text{ stric. increasing, Lipschitz}$$

It is given a definition of solution for which there is existence and it is stable when the right-hand side varies.

Solution is unique if $F(x,.)$ nonincreasing.

Strong maximum principle is not used.

Existence of nonnegative solutions

Problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = F(x, u) \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

$\Omega \subset \mathbb{R}^N$ open, bounded

$$p \in (1, \infty), \ \ p' = \frac{p}{p-1}, \quad p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ 1 < p^* < \infty & \text{if } p = N \\ \infty & \text{if } p > N \end{cases}$$

Assumptions on $a = a(x,\xi): \Omega \times \mathbb{R}^N \to \mathbb{R}^N$

a Carathéodory function

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^p, \quad \alpha > 0,$$

$$|a(x,\xi)| \leq \beta\big(|\xi| + b(x)\big)^{p-1}, \quad \beta > 0, b \in L^p(\Omega)$$

$$\big(a(x,\xi) - a(x,\eta)\big)\cdot(\xi - \eta) \geq 0$$

We denote

$$\lambda = \liminf_{R\to\infty} \min\left\{\frac{R}{\int_\Omega |v|^p dx} : v \in W_0^{1,p}(\Omega), \int_\Omega a(x,\nabla v)\cdot\nabla v dx = R\right\}$$

Example (p-Laplacian): $a(x,\xi) = |\xi|^{p-2}\xi$,

$$\lambda = \min_{\substack{v\in W_0^{1,p}(\Omega)\\ v\neq 0}} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega |v|^p dx}$$

Assumptions on $F: \Omega \times [0,+\infty) \to \mathbb{R} \cup \{+\infty\}$

F Carathéodory function

$$F(x,0) \geq 0, \text{ a.e. } x \in \Omega$$

$$\begin{cases} \exists \nu < \lambda, \text{ such that } \forall \delta > 0, \exists k_\delta \in L^{(p^*)'}(\Omega) \\ F(x,s) \leq k_\delta(x) + \nu s^{p-1} \;\; \forall s \geq \delta, \text{ a.e. } x \in \Omega \end{cases}$$

$$\text{If } p \leq N, \quad \begin{cases} \exists \tilde{\nu}, \tilde{k} \in L^{(p^*)'}(\Omega) \\ F(x,s) \geq -\tilde{k}(x) - \tilde{\nu} s^{p^*} \;\; \forall s \geq 0, \text{ a.e. } x \in \Omega \end{cases}$$

$$\text{If } p > N, \quad \begin{cases} \forall m \in \mathbb{N}, \; \exists \tilde{k}_m \in L^1(\Omega) \\ F(x,s) \geq -\tilde{k}_m(x), \; \forall s \in [0,m], \text{ a.e. } x \in \Omega \end{cases}$$

Definition of solution

It is inspired in D. Giachetti, P.J. Martínez-Aparicio and F. Murat

We say that $u\colon \Omega \to \mathbb{R}$ is a nonnegative solution of

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = F(x, u) \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \qquad \text{if}$$

$$u \in L^{p^*}(\Omega), \ u \geq 0 \ \text{ in } \Omega$$

$$(u - \delta)^+ \in W_0^{1,p}(\Omega), \ \ \forall \delta > 0$$

$$\nabla u \phi \in L^p(\Omega)^N, \ \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

$$F(x, u)^+ \phi^p \in L^1(\Omega), \ \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \phi \geq 0$$

$$\begin{cases} \displaystyle\int_\Omega a(x, \nabla u) \nabla[h(u)\phi^p] dx = \int_\Omega F(x, u) h(u) \phi^p dx \\ \forall h \in W^{1,\infty}(\mathbb{R}), \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \phi \geq 0. \end{cases}$$

Remark: Conditions

$$\begin{cases} \exists \tilde{\nu}, \tilde{k} \in L^{(p^*)'}(\Omega) \\ F(x,s) \geq -k(x) - \tilde{\nu} s^{p^*} \ \ \forall s \geq 0, \text{a.e. } x \in \Omega \end{cases} \quad \text{if } p \leq N,$$

$$\begin{cases} \forall m \in \mathbb{N},\ \exists \tilde{k}_m \in L^1(\Omega) \\ F(x,s) \geq -\tilde{k}_m(x),\ \forall s \in [-m,m], \text{a.e. } x \in \Omega \end{cases} \quad \text{if } p > N$$

$\Longrightarrow F(x,u)^- \in L^1(\Omega)$. Combined with

$$F(x,u)^+ \phi^p \in L^1(\Omega),\ \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \phi \geq 0,$$

we get $F(x,u)\phi^p \in L^1(\Omega)$,

Conditions $\quad u \in L^{p^*}(\Omega), \nabla u \phi \in L^p(\Omega)^N,\ \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$

$$\Longrightarrow |a(x,\nabla u)\nabla[h(u)\phi^p]|$$

$$\leq \beta\big(|\nabla u| + b(x)\big)^{p-1} |h'(u)\nabla u \phi^p + p h(u)\phi^{p-1}\nabla\phi| \in L^1(\Omega)$$

Remark. We can take more general test functions. Namely,

$$W := \Big\{ w \in W^{1,p}(\Omega):\ \exists \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \phi \geq 0$$

$$|w| \leq \phi^p, \text{ a.e. in } \Omega,\ \frac{|\nabla w|}{\phi^{p-1}} \chi_{\{w \neq 0\}} \in L^p(\Omega) \Big\}$$

$$W \text{ is a vectorial space, } \quad w \in W, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \Longrightarrow wv \in W$$

$$\{w \in W^{1,p}(\Omega) \cap L^\infty(\Omega),\ \operatorname{supp}(w)\ \text{compact}\} \subset W$$

$$\Big\{ h(u)\phi^p:\ h \in W^{1,\infty}(\mathbb{R}), \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \phi \geq 0 \Big\} \subset W$$

u solution of the singular semilinear problem implies

$$F(x,0) = 0 \text{ a.e. in } \{u = 0\}$$

$$|\nabla u|^{p-1} |\nabla w| \in L^1(\Omega), \quad \forall w \in W$$

$$\int_\Omega a(x, \nabla u) \nabla w dx = \int_\Omega F(x,u) w dx\,, \ \forall w \in W.$$

Theorem (Stability result). Assume a, a sequence F_n and a function F as above with $v, k_\delta, \tilde{k}, \tilde{k}_m$ indepdent of n.

$$\text{For a.e. } x \in \Omega,\, s_n \to s,\, s_n \geq 0 \Longrightarrow F_n(x, s_n) \to F(x, s).$$

Assume u_n a nonnegative solution of

$$\begin{cases} -\operatorname{div} a(x, \nabla u_n) = F_n(x, u_n) \text{ in } \Omega \\ u_n = 0 \text{ on } \partial\Omega \end{cases}$$

Then, for a subsequence of u_n, $\exists u$ nonegative solution of

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = F(x, u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

such that

$$u_n \to u \text{ in } L^q(\Omega),\; q < p^*$$

$$u_n \rightharpoonup u \text{ in } L^{p^*}(\Omega)$$

$$(u_n - \delta)^+ \rightharpoonup (u - \delta)^+ \text{ in } W_0^{1,p}(\Omega),\;\; \forall \delta > 0.$$

Theorem: Assume $F_1, F_2 : \Omega \times [0,\infty) \to \mathbb{R}$ in the conditions of the singular case,

$$F_1(x,.) \text{ and/or } F_2(x,.) \text{ nonincreasing}, \qquad F_1 \leq F_2$$

$$\big(a(x,\xi_1) - a(x,\xi_2)\big) \cdot (\xi_1 - \xi_2) > 0 \text{ if } \xi_1 \neq \xi_2$$

u_1, u_2 nonnegative solutions of

$$\begin{cases} -\mathrm{div}\, a(x,\nabla u_i) = F_i(x,u_i) \text{ in } \Omega \\ u_i = 0 \text{ on } \partial\Omega. \end{cases}$$

$$\Longrightarrow \qquad u_1 \leq u_2 \text{ a.e. in } \Omega.$$

Corollary: If $F(x,.)$ nonincreasing, a strictly monotone. Then $\exists$ a unique nonnegative solution of the singular problem.

Sketch of the proof. The idea is to take $[(u_1 - u_2)^+]^{2p}$ as test function in the difference of the equations.

Existence of solutions which are nonpositive

A nonexistence result: Assume a as above, $F:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}\cup\{\infty\}$ a Carathéodory function.

$$\text{If } p\le N,\quad \begin{cases} \forall\delta>0,\ \exists\nu_\delta>0, k_\delta\in L^{(p^*)'}(\Omega), k_\delta\ge 0 \\ |F(x,s)|\le k_\delta(x)+\nu_\delta|s|^{p^*}\ \text{ if } |s|\ge\delta, \text{a.e. } x\in\Omega. \end{cases}$$

$$\text{If } p>N,\quad \begin{cases} 0<\delta<m \Longrightarrow\ \exists k_{\delta,m}\in L^1(\Omega) \\ |F(x,s)|\le k_{\delta,m}(x),\ \forall s\in[-m+\delta,m-\delta], \text{a.e. } x\in\Omega. \end{cases}$$

$$\exists\delta_0,\tau_0>0,\ \ F(x,s)\ge-\frac{\tau_0}{s},\ \forall s\in(-\delta_0,0), \text{a.e. } x\in\Omega$$

Assume u a solution of

$$\begin{cases} -\operatorname{div} a(x,\nabla u)=F(x,u) \text{ in } \Omega\backslash\{u=0\} \\ u=0\ \text{ on } \partial\Omega. \end{cases}$$

Then $u\ge 0$ a.e. in Ω.

Here, we say that u is a solution of

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = F(x,u) \ \text{ in } \Omega\backslash\{u=0\} \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

if

$$(u-\delta)^+, (u+\delta)^- \in W_0^{1,p}(\Omega),\ \forall\delta > 0$$

$$\int_\Omega a(x,\nabla u)\cdot\nabla v dx = \int_\Omega F(x,u) v dx$$

$$\forall v \in W_0^{1,p}(\Omega),\ \text{such that } \exists\delta > 0 \text{ with } v = 0 \text{ q.e. in } \{|u| < \delta\}.$$

The result is a Corollary of

Lemma: a as above, $H, u\colon \Omega \longrightarrow \mathbb{R}$

$$(u+\delta)^- \in W_0^{1,p}(\Omega), \ \ \forall \delta > 0$$

$$H \in L^1(\{u < -\delta\}), \ \ \forall \delta > 0$$

$$\exists \delta_0, \tau_0 > 0, \ \ H \geq -\frac{\tau_0}{u}, \ \ \text{a.e. in } \{-\delta_0 < u < 0\}$$

$$\int_\Omega a(x, \nabla u) \cdot \nabla v dx = \int_\Omega H v dx$$

$$\forall v \in W_0^{1,p}(\Omega), \ \text{such that } \exists \delta > 0 \text{ with } v = 0 \text{ q.e. in } \{u > -\delta\}.$$

Then $u \geq 0$ a.e. in Ω.

An example in dimension one

We consider the problem

$$\begin{cases} -u'' = \dfrac{f}{|u|^\gamma} + g \text{ in } (0,l) \\ u(0) = u(l) = 0, \end{cases}$$

with $f, g, \gamma \in \mathbb{R}$, $f, \gamma > 0$.

We know, there exists a unique nonnegative solution

If $g \geq 0$ and or $\gamma \geq 1$, only the nonnegative solution is possible.

Take $g < 0, \gamma < 1$. By a change of variables we can assume

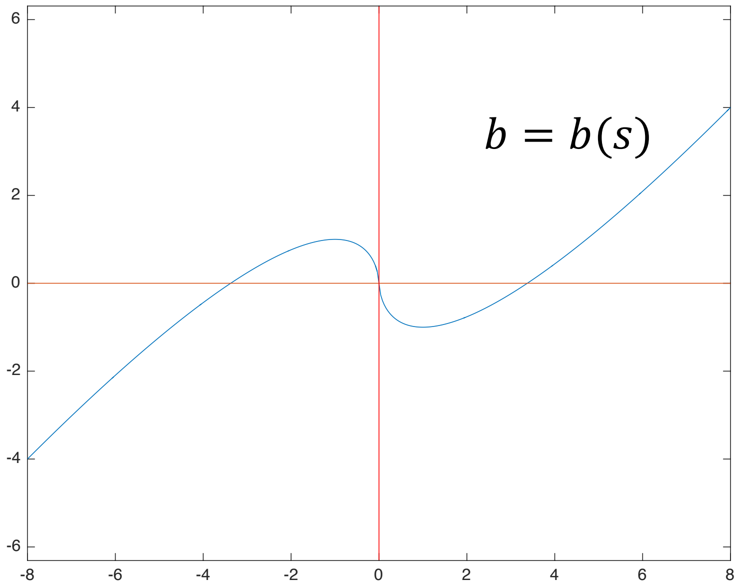
$$f = 1,\ g = -1$$

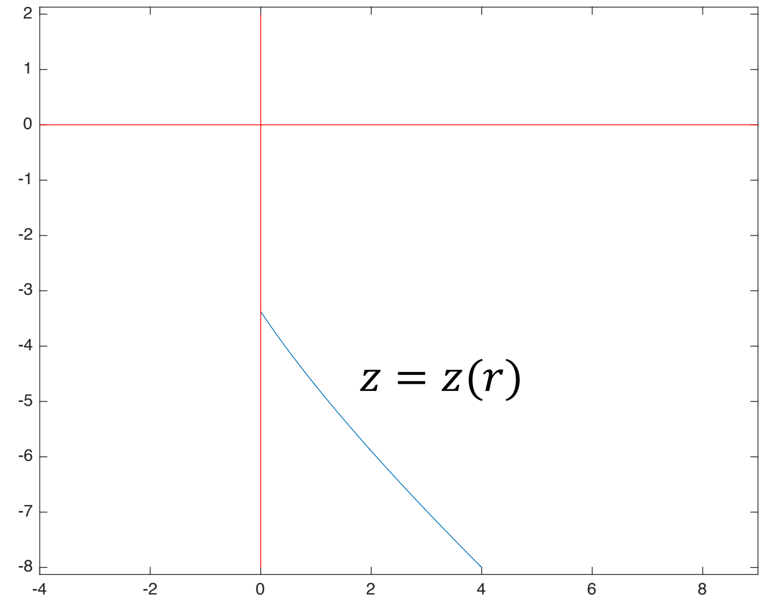
Are there nonpositive solutions and/or solutions changing the sign?

We define $b\colon \mathbb{R} \to \mathbb{R}$ by

$$b(s) = -\frac{2}{1-\gamma}\frac{s}{|s|^{\gamma}} + 2s$$

and $z\colon [0,\infty[\to \left]-\infty, -\left(\frac{1}{1-\gamma}\right)^{\frac{1}{\gamma}}\right]$ by $b\big(z(r)\big) + r = 0$





Lemma: Define

$$T(r) = 2\int_{z(r)}^{0} \frac{dt}{\sqrt{b(t) + r}}$$

$$\Longrightarrow \qquad T \in C^0([0,\infty)),\ \ T_0 := \min_{r\geq 0} T(r) > 0,\ \ \lim_{r\to\infty} T(r) = \infty$$

Theorem: We consider the problem

$$\begin{cases} -u'' = \dfrac{1}{|u|^{\gamma}} - 1 \ \text{ in } \ (0,l) \\ \qquad u(0) = u(l) = 0, \end{cases}$$

- If $0 < l < T_0$, $\exists$ a unique solution. It is positive in $(0,l)$.
- If $l = T_0$, $\exists$ two solutions, one is positive and the other one is negative in $(0,l)$.

- If $kT_0 < l < (k+1)T_0, k \geq 1$. $\exists$ at least $4k+1$ solutions. One is positive, one is negative and for $1 < m \leq k$, there are 3 solutions if $m = 1$, and 4 if $m \geq 2$, which are the restrictions of periodic functions to $(0, l)$. They are negative in m subintervals and positive in $m-1, m$ or $m+1$.
- If $l = kT_0, k \geq 2$, often the $4k-3$ solutions corresponding to the case $(k-1)T_0 < l < kT_0$, there exists another solution which is the restriction of a periodic solutions. It vanishes at jl.

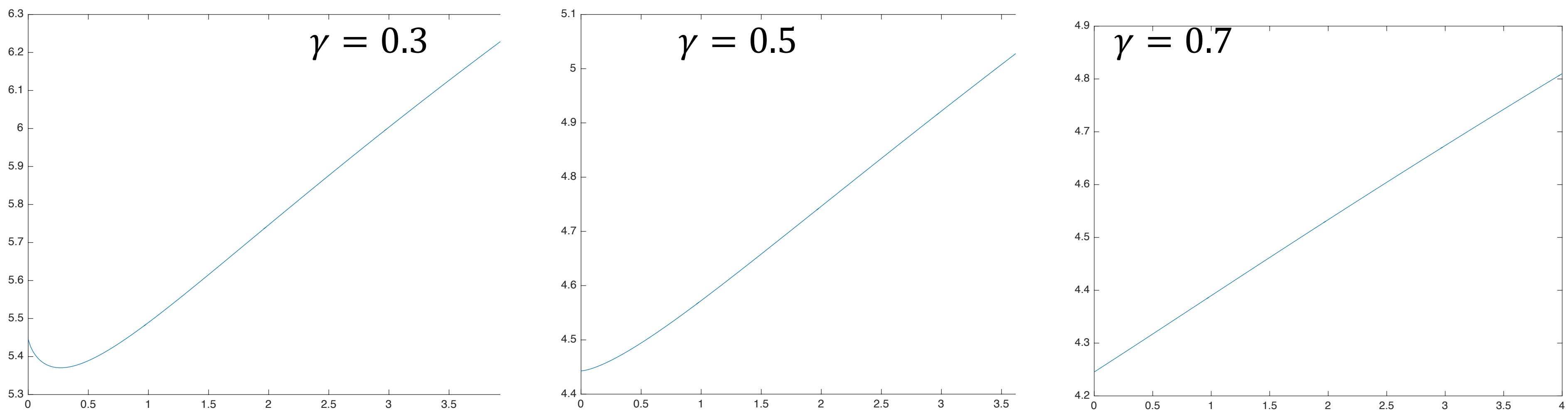
Remark: If

$$T(r) = 2\int_{z(r)}^{0} \frac{dt}{\sqrt{b(t)+r}}$$

is increasing, the solutions given by Theorem are the unique solutions.

For $\gamma = 0.5$, T is increasing.

Numerically, it seems that T is increasing for $\gamma \geq \gamma_0$, with $\gamma_0 \leq 0.5$.



Use $\qquad -u'' = \frac{1}{|u|^\gamma} - 1 \Longrightarrow |u'|^2 = -\frac{2}{1-\gamma}\frac{u}{|u|^\gamma} + 2u + c \;\; (= b(u) + c),$

with $c(= |u'(0)|^2) \in \mathbb{R}$, and then a shooting method.