Boundary feedback for quasilinear wave equations

Benasque PDE VII

Session:
Feedback Control

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Some relevant results (no friction)

J.-M. Coron, B. d'Andréa-Novel & G. Bastin (2007) constructed a strict H^2-Lyapunov function for the boundary control of hyperbolic systems of conservation laws without source term.

In (2008), they constructed a strict H^2-Lyapunov for quasilinear hyperbolic systems with dissipative boundary conditions without source term.

More recently in (2015), Coron and Bastin study the Lyapunov stability of the C^1-norm for quasilinear hyperbolic systems of the first order. They consider W^1_p–Lyapunov functions for $p < \infty$ and look at the limit for $p \to \infty$.

J.M. Cordo, S.Ervedoza, S.S. Ghoshal, O. Glass and V. Perrollaz: Dissipative boundary conditions for 2X2 hyperbolic systems of conservation laws for entropy solutions in BV. J. Differential Equations 262 (2017), no. 1, 1–30.

Selection of own contributions

Controllability and finite time stabilizability (without Friction)

G.L. and E.J.P.G. Schmidt: On the modelling and stabilization of flows in networks of open canals. SIAM J. Control Optim. 41 (2002), no. 1, 164–180.

For Systems with source term (Friction):

M. Dick, M. Gugat & G. L. (2010) considered the isothermal Euler equations with friction with Dirichlet boundary feedback at both ends of the system and introduced a strict H^1-Lyapunov function.

In (2012), we have defined a strict H^2-Lyapunov function for this stabilization problem for Dirichlet boundary feedbacks

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Gugat, Martin; Leugering, Günter; Wang, Ke; Neumann boundary feedback stabilization for a nonlinear wave equation: A strict H^2-Lyapunov function. Math. Control Relat. Fields 7 (2017), no. 3, 419–448.

The Euler gas equation

Let a finite time $T > 0$ be given. The system dynamics for the gas flow in a single pipe can be modeled by a hyperbolic system, which is described by the isothermal Euler equations:

$$\boxed{\begin{aligned} &\rho_t + q_x = 0, \\ &q_t + \left(\frac{q^2}{\rho} + a^2\,\rho\right)_x = -\frac{f_g}{2\,\delta}\,\frac{q\,|q|}{\rho}, \end{aligned}}$$

where $\rho = \rho(t, x) > 0$ is the density of the gas, $q = q(t, x)$ is the mass flux, the constant $f_g > 0$ is a friction factor, $\delta > 0$ is the diameter of the pipe and $a > 0$ is the sonic velocity in the gas. We consider the equations on the domain $\Omega := [0, T] \times [0, L]$.

The corresponding wave equation

We use the notation

$$\theta = \frac{f_g}{\delta}$$

and consider positive gas flow in subsonic or subcritical states, that is,

$$0 < \frac{q}{\rho} < a.$$

The isothermal Euler equations give rise to the second-order equation

$$\boxed{\tilde{u}_{tt} + 2\,\tilde{u}\,\tilde{u}_{tx} - (a^2 - \tilde{u}^2)\,\tilde{u}_{xx} = \tilde{F}(\tilde{u}, \tilde{u}_x, \tilde{u}_t),}$$

where $\tilde{u}$ is the unknown function and satisfies $\tilde{u} = \frac{q}{\rho}$, that is $\tilde{u}$ is the velocity of the gas. The lower order term is

$$\tilde{F}(\tilde{u}, \tilde{u}_x, \tilde{u}_t) = -2\,\tilde{u}_t\,\tilde{u}_x - 2\,\tilde{u}\,\tilde{u}_x^2 - \frac{3}{2}\,\theta\,\tilde{u}\,|\tilde{u}|\,\tilde{u}_x - \theta\,|\tilde{u}|\,\tilde{u}_t.$$

Density recovery

From the velocity $\tilde{u}$, the density ρ can be obtained from the initial value $\rho(0, \cdot)$ and the differential equation

$$(\ln \rho)_t = \frac{1}{a^2} \left(\tilde{u}\, \tilde{u}_t + (\tilde{u}^2 - a^2)\, \tilde{u}_x + \frac{1}{2}\theta \, |\tilde{u}|\, \tilde{u}^2 \right).$$

Then q can be obtained from the equation $q = \rho\, \tilde{u}$.

Stabilization around steady states

To stabilize the system governed by the quasilinear wave equation (**??**) locally around a given stationary state $\bar{u}(x)$, we use the boundary feedback law

$$\begin{aligned}\tilde{u}_x(t,\,0) &= \bar{u}_x(0) + k\,\tilde{u}_t(t,\,0),\\ \tilde{u}(t,\,L) &= \bar{u}(L),\end{aligned}$$

with a feedback parameter $k \in (0, \infty)$.
In terms of the physical variables (q, ρ), the boundary feedback law is

$$\begin{aligned}&\text{at} \quad x = 0: \quad q_x - (\ln(\rho))_x\, q = \rho\, \bar{u}_x(0) + k\left[q_t - (\ln(\rho))_t\, q\right],\\ &\text{at} \quad x = L: \quad q = \bar{u}(L)\,\rho.\end{aligned}$$

Analysis of the 2X2 system

$$\partial_t \begin{pmatrix} \rho \\ q \end{pmatrix} + \hat{A}(\rho, q)\, \partial_x \begin{pmatrix} \rho \\ q \end{pmatrix} = \hat{G}(\rho, q)$$

with the matrix

$$\hat{A}(\rho, q) := \begin{pmatrix} 0 & 1 \\ a^2 - \frac{q^2}{\rho^2} & 2\frac{q}{\rho} \end{pmatrix}$$

and the source term

$$\hat{G}(\rho, q) := \begin{pmatrix} 0 \\ -\frac{\theta}{2} \frac{|q|\, q}{\rho} \end{pmatrix}.$$

The system has two eigenvalues $\tilde{\lambda}_-(\rho, q)$, $\tilde{\lambda}_+(\rho, q)$ and in the subsonic case we have

$$\tilde{\lambda}_-(\rho, q) = \frac{q}{\rho} - a < 0 < \tilde{\lambda}_+(\rho, q) = \frac{q}{\rho} + a.$$

Riemann invariants

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In terms of the Riemann invariants $R_\pm = R_\pm(\rho, q) = -\frac{q}{\rho} \mp a\ln(\rho)$ our system has the diagonal form

$$\partial_t \begin{pmatrix} R_+ \\ R_- \end{pmatrix} + \hat{D}(R_+, R_-)\, \partial_x \begin{pmatrix} R_+ \\ R_- \end{pmatrix} = \hat{S}(R_+, R_-),$$

where

$$\hat{D}(R_+, R_-) := \begin{pmatrix} \tilde{\lambda}_+ & 0 \\ 0 & \tilde{\lambda}_- \end{pmatrix} = \begin{pmatrix} -\frac{R_+ + R_-}{2} + a & 0 \\ 0 & -\frac{R_+ + R_-}{2} - a \end{pmatrix},$$

Riemann invariants

$$\hat{S}(R_+, R_-) := -\frac{\theta}{8}(R_+ + R_-)\,|R_+ + R_-| \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In terms of $R_\pm$, for the physical variables ρ and q we have

$$\begin{aligned} \rho &= \exp\left(\frac{R_- - R_+}{2a}\right), \\ q &= -\frac{R_+ + R_-}{2}\exp\left(\frac{R_- - R_+}{2a}\right). \end{aligned}$$

A gas flow is positive and subsonic (i.e. $0 < q/\rho < a$) if and only if

$$-2a < R_+(t,x) + R_-(t,x) < 0 \text{ for all } (t,x) \in \Omega.$$

Riemann invariants... and back

For the velocity $\tilde{u} = \tilde{u}(\rho, q)$ we have

$$\tilde{u} = \frac{R_+ + R_-}{-2}, \quad \tilde{u} = \frac{\tilde{\lambda}_+ + \tilde{\lambda}_-}{2}.$$

The second-order quasilinear equation is hyperbolic with the eigenvalues

$$\tilde{\lambda}_- = \tilde{u} - a < 0 < \tilde{\lambda}_+ = \tilde{u} + a.$$

Using the isothermal Euler equations, we obtain the partial derivatives of $\tilde{u}$ with respect to t and x, respectively,

$$\begin{aligned}
\tilde{u}_t &= \frac{q_t}{\rho} - \frac{q\rho_t}{\rho^2} \\
&= -\frac{1}{\rho}\left(\frac{q^2}{\rho} + a^2\rho\right)_x - \frac{q\rho_t}{\rho^2} - \frac{\theta}{2}\frac{q\,|q|}{\rho^2} \\
&= \tilde{u}\frac{\rho_t}{\rho} + (\tilde{u}^2 - a^2)\frac{\rho_x}{\rho} - \frac{\theta}{2}\tilde{u}\,|\tilde{u}|
\end{aligned}$$

Second order system

$$\tilde{u}_x = \frac{q_x}{\rho} - \frac{q\rho_x}{\rho^2} = -\frac{\rho_t}{\rho} - \tilde{u}\frac{\rho_x}{\rho}.$$

Multiplying $\tilde{u}_t$ and $\tilde{u}_x$ by $\tilde{u}$ and $\tilde{u}^2 - a^2$, respectively, and adding the two equations we obtain that ρ and q can be obtained from $\tilde{u}$ and the initial data. Note that since $\tilde{u} = \frac{q}{\rho}$, we have the same value for $\tilde{u}$ for λq and $\lambda\rho$ where $\lambda \in (0, 1]$. So we cannot expect to recover the values of (q, ρ) from $\tilde{u}$ without additional information on (q, ρ). We obtain the equation

$$\ln(\rho)_x = -\frac{1}{a^2}\left(\tilde{u}_t + \tilde{u}\,\tilde{u}_x + \frac{\theta}{2}|\tilde{u}|\,\tilde{u}\right).$$

Thus if $\tilde{u}$ is known, the values of ρ can be determined from the value of ρ.

Stationary states

Now we consider the question: Given a constant state $\bar{u} = \lambda \in (0, \infty)$, is there a solution (q, ρ) of the quasilinear system that corresponds to the constant velocity $\bar{u}$? For $\lambda = 0$ we obtain the constant solution of where $q = 0$. For $\lambda > 0$ there is a corresponding solution of travelling wave type (in particular the corresponding solution of is not stationary), namely

$$(q(t,x), \rho(t,x)) = (\lambda\, \alpha(\lambda\, t - x),\ \alpha(\lambda t - x))$$

where the function α is given by

$$\alpha(z) = C \exp\left(\frac{\lambda^2\theta}{2\, a^2} z\right)$$

and $C > 0$ is a positive constant. This can be rewritten in the form

$$\frac{d}{dx}\left((a^2 - \bar{u}^2(x))\bar{u}_x(x) - \frac{\theta}{2}|\bar{u}(x)|\, \bar{u}^2(x)\right) = 0.$$

Stationary states

Thus all the stationary solutions must satisfy the equation

$$\bar{u}'(0) = \frac{\theta}{2}\frac{|\bar{u}_0|\,\bar{u}_0^2}{a^2 - \bar{u}_0^2}.$$

Lemma:
Let a subsonic stationary state $\bar{u}(x) > 0$ for $x \in [0, L]$ that is not constant and satisfies the condition above be given. Let $W_{-1}(x)$ denote the real branch of the Lambert W–function with $W_{-1}(x) \leq -1$. Then the following equation holds for all $x \in [0, L]$:

$$(\bar{u}(x))^2 = \frac{a^2}{-W_{-1}(-\exp(\theta\, x + \bar{C}))},$$

where $\bar{C}$ is a real constant such that $\bar{C} \leq -1 - \theta L$.

Deviation from equlibrium

Now we consider non-stationary solutions locally around a subsonic stationary state $\bar{u}(x) > 0$ on Ω

$$u(t,x) = \tilde{u}(t,x) - \bar{u}(x).$$

Then we obtain the equation

$$\boxed{u_{tt} + 2\,(\bar{u} + u)\,u_{tx} - \left(a^2 - (\bar{u}+u)^2\right) u_{xx} = F(x,u,u_x,u_t), \qquad (1)}$$

where $F := F(x,u,u_x,u_t)$ satisfies

$$\begin{aligned} F &= \tilde{F}(u+\bar{u},\ u_x + \bar{u}_x\,,u_t) + \frac{a^2 - (\bar{u}+u)^2}{a^2 - \bar{u}^2}\,\bar{u}\left(2(\bar{u}_x)^2 + \frac{3}{2}\,\theta\,|\bar{u}|\,\bar{u}_x\right) \\ &= \tilde{F}(u+\bar{u},\ u_x + \bar{u}_x\,,u_t) - \frac{a^2 - (\bar{u}+u)^2}{a^2 - \bar{u}^2}\,\tilde{F}(\bar{u},\ \bar{u}_x\,,0). \end{aligned}$$

Semi-global classical solutions

Lemma

Let a subsonic stationary state $\bar{u}(x) > 0$ *be given. Choose* $T > 0$ *arbitrarily large. There exist constants* $\varepsilon_0(T) > 0$ *and* $C_T > 0$*, such that if the initial data* $(\varphi(x), \psi(x)) \in C^2([0, L]) \times C^1([0, L])$ *satisfies*

$$\max\left\{\|\varphi(x)\|_{C^2([0,L])},\ \|\psi(x)\|_{C^1([0,L])}\right\} \leq \varepsilon_0(T)$$

and the C^2*-compatibility conditions are satisfied at the points* $(t, x) = (0, 0)$ *and* $(0, L)$*, then the initial-boundary problem (1)-(4) has a unique solution* $u(t, x) \in C^2([0, T] \times [0, L])$*. Moreover the following a priori estimate holds:*

$$\|u\|_{C^2([0,T]\times[0,L])} \leq C_T \max\left\{\|\varphi(x)\|_{C^2([0,L])},\ \|\psi(x)\|_{C^1([0,L])}\right\}.$$

$$t = 0: \quad u = \varphi(x),\ \ u_t = \psi(x), \quad x \in [0, L] \tag{2}$$

$$x = 0: \quad u_x = k\, u_t, \tag{3}$$

$$x = L: \quad u = 0, \tag{4}$$

Liapunov functions

In the sequel we consider

$$E_1(t) = \int_0^L h_1(x)\left((a^2-(\bar{u}+u)^2)\, u_x^2+u_t^2\right)-2\, h_2(x)\left((\bar{u}+u)u_x^2+u_t\, u_x\right) dx$$

since according to the previous considerations, this is a natural candidate to define a Lyapunov function for our system.
To show the exponential decay with respect to the H^2-norm, it is necessary to deal with the second order derivatives. Therefore we also introduce $E_2(t)$ which is defined analogously to E_1 to show the decay of the partial derivatives of second order. We define

$$E_2(t) = \int_0^L h_1(x)\left((a^2-(\bar{u}+u)^2)u_{xx}^2+u_{tx}^2\right)-2\, h_2(x)\left((\bar{u}+u)u_{xx}^2+u_{tx}\, u_{xx}\right) dx.$$

The perturbed system

We define the Lyapunov function $E(t)$ as

$$E(t) = E_1(t) + E_2(t). \tag{5}$$

We show that $E(t)$ as defined in is bounded above and below by the product of appropriate constants and the square of the H^2-norm of u.

Consider the system

$$\begin{cases} \tilde{u}_{tt} + 2\,\tilde{u}\,\tilde{u}_{tx} - (a^2 - \tilde{u}^2)\,\tilde{u}_{xx} = \tilde{F}(\tilde{u}, \tilde{u}_x, \tilde{u}_t), \\ \tilde{u}_x(t, 0) = \bar{u}_x(0) + k\,\tilde{u}_t(t, 0),\ t \in [0, T], \\ \tilde{u}(t, L) = \bar{u}(L),\ t \in [0, T], \\ t = 0:\ \ \tilde{u} = \varphi(x) + \bar{u}(x),\ \ \tilde{u}_t = \psi(x), \quad x \in [0, L] \end{cases} \tag{6}$$

Exponential stability

Theorem: (Exponential Decay of the H^2-Lyapunov Function). Let a real number $\gamma \in (0, \frac{1}{2}]$ be given. Choose a real number $k > 0$ such that

$$a\,(1-\gamma)\,k > 1.$$

Let a stationary subsonic state $\bar{u}(x) \in C^2(0, L)$ be given that satisfies

$$\bar{u}'(0) = \frac{\theta}{2} \frac{|\bar{u}_0|\,\bar{u}_0^2}{a^2 - \bar{u}_0^2}.$$

Assume that for all $x \in L$ we have $\bar{u}(x) \in (0, \gamma\, a)$. Assume that for $K_\partial(k,\ \bar{u}_0)$ as given by

$$K_\partial(k,\ \bar{u}_0) = 2\left[\frac{4}{k^2} + \frac{2\,\bar{u}_0}{k} + \theta\,\frac{\bar{u}_0^4 + 3a^2\,\bar{u}_0^2 + \frac{2}{k}\,a^2\,\bar{u}_0}{2(a^2 - \bar{u}_0^2)} + \frac{5}{2}\,\frac{\theta}{k^2} + \frac{\theta}{k}\,\frac{3a^2\,\bar{u}_0 - \bar{u}_0^3}{a^2 - \bar{u}_0^2}\right]^2.$$

we have

Continued...

$$2\,k^2\,K_\partial(k,\ \bar{u}(0)) \leq a^2 - \left(\bar{u}(0) + \frac{2}{k}\right)^2.$$

Assume that $\|\bar{u}\|_{C^2([0,L])}$ is sufficiently small such that $\|\bar{u}\|_{C([0,L])} < \varepsilon_1(2\,k^2)$ holds.
Let $T > 0$ be given. If the initial data satisfies

$$\|(\varphi(x), \psi(x))\|_{C^2([0,L])\times C^1([0,L])} \leq \varepsilon_0(T)$$

and the C^2-compatibility conditions at the points $(t,x) = (0,0)$ and $(t,x) = (0,L)$.

....continued.

Then the initial-boundary value problem (6) for $\tilde{u}$ has a unique classical solution $\tilde{u} \in C^2([0, T] \times [0, L])$. Define the number

$$\mu = \frac{1}{2\,\mathrm{e}\,L\,k} - \kappa \geq \frac{1}{4\,\mathrm{e}Lk}.$$

Then we have

$$E_1(t) \leq E_1(0)\,\exp\left(-\mu\,t\right) \text{ for all } \; t \in [0, T],$$

$$E(t) \leq E(0)\,\exp\left(-\mu\,t\right) \text{ for all } \; t \in [0, T]$$

that is $E_1(t)$ and $E(t)$ are strict Lyapunov functions for our control system (6).

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