Boundary feedback for quasilinear wave equations

Benasque PDE VII

Session:
Feedback Control
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(jointwork with Martin Gugat and Ke Wang)
Some relevant results (no friction)


In (2008), they constructed a strict $H^2$-Lyapunov for quasilinear hyperbolic systems with dissipative boundary conditions without source term.

More recently in (2015), Coron and Bastin study the Lyapunov stability of the $C^1$-norm for quasilinear hyperbolic systems of the first order. They consider $W^1_p$–Lyapunov functions for $p < \infty$ and look at the limit for $p \to \infty$.

Controllability and finite time stabilizability (without Friction)


For Systems with source term (Friction):

M. Dick, M. Gugat & G. L. (2010) considered the isothermal Euler equations with friction with Dirichlet boundary feedback at both ends of the system and introduced a strict $H^1$-Lyapunov function.

In (2012), we have defined a strict $H^2$-Lyapunov function for this stabilization problem for Dirichlet boundary feedbacks

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The Euler gas equation

Let a finite time $T > 0$ be given. The system dynamics for the gas flow in a single pipe can be modeled by a hyperbolic system, which is described by the isothermal Euler equations:

\[
\begin{align*}
\rho_t + q_x &= 0, \\
q_t + \left( \frac{q^2}{\rho} + a^2 \rho \right)_x &= -\frac{f_g}{2\delta} \frac{q|q|}{\rho},
\end{align*}
\]

where $\rho = \rho(t, x) > 0$ is the density of the gas, $q = q(t, x)$ is the mass flux, the constant $f_g > 0$ is a friction factor, $\delta > 0$ is the diameter of the pipe and $a > 0$ is the sonic velocity in the gas. We consider the equations on the domain $\Omega := [0, T] \times [0, L]$. 
The corresponding wave equation

We use the notation

\[ \theta = \frac{f_g}{\delta} \]

and consider positive gas flow in subsonic or subcritical states, that is,

\[ 0 < \frac{q}{\rho} < a. \]

The isothermal Euler equations give rise to the second-order equation

\[ \ddot{u} + 2 \dot{u} \dot{u}_x - (a^2 - \ddot{u}^2) \ddot{u}_x = \tilde{F}(\ddot{u}, \dot{u}_x, \dot{u}_t), \]

where \( \ddot{u} \) is the unknown function and satisfies \( \ddot{u} = \frac{q}{\rho} \), that is \( \ddot{u} \) is the velocity of the gas. The lower order term is

\[ \tilde{F}(\ddot{u}, \dot{u}_x, \dot{u}_t) = -2 \dot{u}_t \dot{u}_x - 2 \ddot{u} \ddot{u}_x^2 - \frac{3}{2} \theta \ddot{u} |\ddot{u}| \dot{u}_x - \theta |\ddot{u}| \ddot{u}_t. \]
Density recovery

From the velocity $\tilde{u}$, the density $\rho$ can be obtained from the initial value $\rho(0, \cdot)$ and the differential equation

$$(\ln \rho)_t = \frac{1}{a^2} \left( \tilde{u} \tilde{u}_t + (\tilde{u}^2 - a^2) \tilde{u}_x + \frac{1}{2} \theta |\tilde{u}| \tilde{u}^2 \right).$$

Then $q$ can be obtained from the equation $q = \rho \tilde{u}$. 
Stabilization around steady states

To stabilize the system governed by the quasilinear wave equation (??) locally around a given stationary state \( \bar{u}(x) \), we use the boundary feedback law

\[
\begin{align*}
\tilde{u}_x(t, 0) &= \bar{u}_x(0) + k \tilde{u}_t(t, 0), \\
\tilde{u}(t, L) &= \bar{u}(L),
\end{align*}
\]

with a feedback parameter \( k \in (0, \infty) \).

In terms of the physical variables \((q, \rho)\), the boundary feedback law is

\[
\begin{align*}
\text{at } x &= 0 : \quad q_x - (\ln(\rho))_x q &= \rho \bar{u}_x(0) + k [q_t - (\ln(\rho))_t q] , \\
\text{at } x &= L : \quad q &= \bar{u}(L) \rho .
\end{align*}
\]
Analysis of the 2X2 system

\[ \partial_t \begin{pmatrix} \rho \\ q \end{pmatrix} + \hat{A}(\rho, q) \partial_x \begin{pmatrix} \rho \\ q \end{pmatrix} = \hat{G}(\rho, q) \]

with the matrix

\[ \hat{A}(\rho, q) := \begin{pmatrix} 0 & 1 \\ a^2 - \frac{q^2}{\rho^2} & 2 \frac{q}{\rho} \end{pmatrix} \]

and the source term

\[ \hat{G}(\rho, q) := \begin{pmatrix} 0 \\ -\frac{\theta}{2} \frac{|q|}{\rho} \end{pmatrix}. \]

The system has two eigenvalues \( \tilde{\lambda}_-(\rho, q) \), \( \tilde{\lambda}_+(\rho, q) \) and in the subsonic case we have

\[ \tilde{\lambda}_-(\rho, q) = \frac{q}{\rho} - a < 0 < \tilde{\lambda}_+(\rho, q) = \frac{q}{\rho} + a. \]
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In terms of the Riemann invariants \( R_\pm = R_\pm(\rho, q) = -\frac{q}{\rho} \pm a \ln(\rho) \) our system has the diagonal form

\[
\partial_t \begin{pmatrix} R_+ \\ R_- \end{pmatrix} + \hat{D}(R_+, R_-) \partial_x \begin{pmatrix} R_+ \\ R_- \end{pmatrix} = \hat{S}(R_+, R_-),
\]

where

\[
\hat{D}(R_+, R_-) := \begin{pmatrix} \tilde{\lambda}_+ & 0 \\ 0 & \tilde{\lambda}_- \end{pmatrix} = \begin{pmatrix} -\frac{R_+ + R_-}{2} + a & 0 \\ 0 & -\frac{R_+ + R_-}{2} - a \end{pmatrix},
\]
Riemann invariants

\[ \hat{S}(R_+, R_-) := -\frac{\theta}{8} (R_+ + R_-)|R_+ + R_-| \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]

In terms of \( R_\pm \), for the physical variables \( \rho \) and \( q \) we have

\[
\begin{align*}
\rho &= \exp\left( \frac{R_- - R_+}{2a} \right), \\
q &= -\frac{R_+ + R_-}{2} \exp\left( \frac{R_- - R_+}{2a} \right). 
\end{align*}
\]

A gas flow is positive and subsonic (i.e. \( 0 < q/\rho < a \)) if and only if

\[ -2a < R_+(t, x) + R_-(t, x) < 0 \text{ for all } (t, x) \in \Omega. \]
Riemann invariants... and back

For the velocity \( \tilde{u} = \tilde{u}(\rho, q) \) we have

\[
\tilde{u} = \frac{R_+ + R_-}{-2}, \quad \tilde{u} = \frac{\tilde{\lambda}_+ + \tilde{\lambda}_-}{2}.
\]

The second-order quasilinear equation is hyperbolic with the eigenvalues

\[
\tilde{\lambda}_- = \tilde{u} - a < 0 < \tilde{\lambda}_+ = \tilde{u} + a.
\]

Using the isothermal Euler equations, we obtain the partial derivatives of \( \tilde{u} \) with respect to \( t \) and \( x \), respectively,

\[
\tilde{u}_t = \frac{q_t}{\rho} - \frac{q \rho_t}{\rho^2}
= -\frac{1}{\rho} \left( \frac{q^2}{\rho} + a^2 \rho \right)_x - \frac{q \rho_t}{\rho^2} - \frac{\theta}{2} \frac{q |q|}{\rho^2}
= \tilde{u} \frac{\rho_t}{\rho} + (\tilde{u}^2 - a^2) \frac{\rho_x}{\rho} - \frac{\theta}{2} \tilde{u} |\tilde{u}|.
\]

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Second order system

$$\tilde{u}_x = \frac{q_x}{\rho} - \frac{q \rho_x}{\rho^2} = -\frac{\rho t}{\rho} - \tilde{u} \frac{\rho x}{\rho}.$$ 

Multiplying $\tilde{u}_t$ and $\tilde{u}_x$ by $\tilde{u}$ and $\tilde{u}^2 - a^2$, respectively, and adding the two equations we obtain that $\rho$ and $q$ can be obtained from $\tilde{u}$ and the initial data. Note that since $\tilde{u} = \frac{q}{\rho}$, we have the same value for $\tilde{u}$ for $\lambda q$ and $\lambda \rho$ where $\lambda \in (0, 1]$. So we cannot expect to recover the values of $(q, \rho)$ from $\tilde{u}$ without additional information on $(q, \rho)$. We obtain the equation

$$\ln(\rho)_x = -\frac{1}{a^2} \left( \tilde{u}_t + \tilde{u} \tilde{u}_x + \frac{\theta}{2} |\tilde{u}| \tilde{u} \right).$$

Thus if $\tilde{u}$ is known, the values of $\rho$ can be determined from the value of $\rho$. 
Stationary states

Now we consider the question: Given a constant state $\bar{u} = \lambda \in (0, \infty)$, is there a solution $(q, \rho)$ of the quasilinear system that corresponds to the constant velocity $\bar{u}$? For $\lambda = 0$ we obtain the constant solution of where $q = 0$. For $\lambda > 0$ there is a corresponding solution of travelling wave type (in particular the corresponding solution of is not stationary), namely

$$(q(t, x), \rho(t, x)) = (\lambda \alpha(\lambda t - x), \alpha(\lambda t - x))$$

where the function $\alpha$ is given by

$$\alpha(z) = C \exp \left( \frac{\lambda^2 \theta}{2 a^2} z \right)$$

and $C > 0$ is a positive constant. This can be rewritten in the form

$$\frac{d}{dx} \left( (a^2 - \bar{u}^2(x))\bar{u}_x(x) - \frac{\theta}{2} |\bar{u}(x)| \bar{u}^2(x) \right) = 0.$$
Stationary states

Thus all the stationary solutions must satisfy the equation

\[ \ddot{u}'(0) = \frac{\theta}{2} \frac{|\ddot{u}_0| \dddot{u}_0^2}{a^2 - \dddot{u}_0^2}. \]

**Lemma:**
Let a subsonic stationary state \( \ddot{u}(x) > 0 \) for \( x \in [0, L] \) that is not constant and satisfies the condition above be given. Let \( W_{-1}(x) \) denote the real branch of the Lambert W–function with \( W_{-1}(x) \leq -1 \). Then the following equation holds for all \( x \in [0, L] \):

\[ (\ddot{u}(x))^2 = \frac{a^2}{-W_{-1}(-\exp(\theta x + \bar{C}))}, \]

where \( \bar{C} \) is a real constant such that \( \bar{C} \leq -1 - \theta L \).
Deviation from equilibrium

Now we consider non-stationary solutions locally around a subsonic stationary state \( \bar{u}(x) > 0 \) on \( \Omega \)

\[
 u(t, x) = \tilde{u}(t, x) - \bar{u}(x).
\]

Then we obtain the equation

\[
 u_{tt} + 2(\bar{u} + u)u_{tx} - \left( a^2 - (\bar{u} + u)^2 \right) u_{xx} = F(x, u, u_x, u_t), \tag{1}
\]

where \( F := F(x, u, u_x, u_t) \) satisfies

\[
 F = \tilde{F}(u + \bar{u}, u_x + \bar{u}_x, u_t) + \frac{a^2 - (\bar{u} + u)^2}{a^2 - \bar{u}^2} \bar{u} \left( 2(\bar{u}_x)^2 + \frac{3}{2} \theta |\bar{u}| \bar{u}_x \right)
\]

\[
 = \tilde{F}(u + \bar{u}, u_x + \bar{u}_x, u_t) - \frac{a^2 - (\bar{u} + u)^2}{a^2 - \bar{u}^2} \tilde{F}(\bar{u}, \bar{u}_x, 0).
\]
Lemma

Let a subsonic stationary state \( \bar{u}(x) > 0 \) be given. Choose \( T > 0 \) arbitrarily large. There exist constants \( \varepsilon_0(T) > 0 \) and \( C_T > 0 \), such that if the initial data \( (\varphi(x), \psi(x)) \in C^2([0, L]) \times C^1([0, L]) \) satisfies

\[
\max \left\{ \| \varphi(x) \|_{C^2([0,L])}, \| \psi(x) \|_{C^1([0,L])} \right\} \leq \varepsilon_0(T)
\]

and the \( C^2 \)-compatibility conditions are satisfied at the points \( (t, x) = (0, 0) \) and \( (0, L) \), then the initial-boundary problem (1)-(4) has a unique solution \( u(t, x) \in C^2([0, T] \times [0, L]) \). Moreover the following a priori estimate holds:

\[
\| u \|_{C^2([0,T] \times [0,L])} \leq C_T \max \left\{ \| \varphi(x) \|_{C^2([0,L])}, \| \psi(x) \|_{C^1([0,L])} \right\}.
\]

(2) \hspace{1cm} t = 0 : \quad u = \varphi(x), \quad u_t = \psi(x), \quad x \in [0, L]

(3) \hspace{1cm} x = 0 : \quad u_x = k u_t,

(4) \hspace{1cm} x = L : \quad u = 0,
Liapunov functions

In the sequel we consider

\[
E_1(t) = \int_0^L h_1(x) \left( (a^2 - (\bar{u} + u)^2) u_x^2 + u_t^2 \right) - 2 h_2(x) \left( (\bar{u} + u) u_x^2 + u_t u_x \right) \, dx
\]

since according to the previous considerations, this is a natural candidate to define a Lyapunov function for our system.

To show the exponential decay with respect to the \( H^2 \)-norm, it is necessary to deal with the second order derivatives. Therefore we also introduce \( E_2(t) \) which is defined analogously to \( E_1 \) to show the decay of the partial derivatives of second order. We define

\[
E_2(t) = \int_0^L h_1(x) \left( (a^2 - (\bar{u} + u)^2) u_{xx}^2 + u_{tx}^2 \right) - 2 h_2(x) \left( (\bar{u} + u) u_{xx}^2 + u_{tx} u_{xx} \right) \, dx.
\]
The perturbed system

We define the Lyapunov function $E(t)$ as

$$E(t) = E_1(t) + E_2(t).$$

We show that $E(t)$ as defined in is bounded above and below by the product of appropriate constants and the square of the $H^2$-norm of $u$.

Consider the system

$$
\begin{aligned}
\tilde{u}_{tt} + 2 \tilde{u} \tilde{u}_{tx} - (a^2 - \tilde{u}^2) \tilde{u}_{xx} &= \tilde{F}(\tilde{u}, \tilde{u}_x, \tilde{u}_t), \\
\tilde{u}_x(t, 0) &= \bar{u}_x(0) + k \tilde{u}_t(t, 0), \; t \in [0, T], \\
\tilde{u}(t, L) &= \bar{u}(L), \; t \in [0, T], \\
t = 0 : \; \tilde{u} = \varphi(x) + \bar{u}(x), \; \tilde{u}_t = \psi(x), \; x \in [0, L]
\end{aligned}
$$

(6)
Exponential stability

**Theorem:** (Exponential Decay of the $H^2$-Lyapunov Function). Let a real number $\gamma \in (0, \frac{1}{2}]$ be given. Choose a real number $k > 0$ such that

$$a (1 - \gamma) k > 1.$$ 

Let a stationary subsonic state $\bar{u}(x) \in C^2(0, L)$ be given that satisfies

$$\bar{u}'(0) = \frac{\theta \, |\bar{u}_0| \, \bar{u}_0^2}{2 \, a^2 - \bar{u}_0^2}.$$ 

Assume that for all $x \in L$ we have $\bar{u}(x) \in (0, \gamma a)$. Assume that for $K_{\partial}(k, \bar{u}_0)$ as given by

$$K_{\partial}(k, \bar{u}_0) = 2 \left[ \frac{4}{k^2} + \frac{2 \bar{u}_0}{k} + \theta \, \frac{\bar{u}_0^4 + 3a^2 \bar{u}_0^2 + \frac{2}{k} a^2 \bar{u}_0}{2(a^2 - \bar{u}_0^2)} + \frac{5}{2} \, \frac{\theta}{k^2} + \frac{\theta}{k} \, \frac{3a^2 \bar{u}_0 - \bar{u}_0^3}{a^2 - \bar{u}_0^2} \right]^2.$$ 

we have
2 k^2 K_\Theta(k, \bar{u}(0)) \leq a^2 - \left( \bar{u}(0) + \frac{2}{k} \right)^2.

Assume that \|\bar{u}\|_{C^2([0,L])} is sufficiently small such that
\|\bar{u}\|_{C([0,L])} < \varepsilon_1(2k^2) holds.
Let \(T > 0\) be given. If the initial data satisfies

\[\|(\varphi(x), \psi(x))\|_{C^2([0,L]) \times C^1([0,L])} \leq \varepsilon_0(T)\]

and the \(C^2\)-compatibility conditions at the points \((t, x) = (0, 0)\) and \((t, x) = (0, L)\).
Then the initial-boundary value problem (6) for $\tilde{u}$ has a unique classical solution $\tilde{u} \in C^2([0, T] \times [0, L])$. Define the number

$$\mu = \frac{1}{2 e L k} - \kappa \geq \frac{1}{4 e L k}.$$ 

Then we have

$$E_1(t) \leq E_1(0) \exp(-\mu t) \text{ for all } t \in [0, T],$$

$$E(t) \leq E(0) \exp(-\mu t) \text{ for all } t \in [0, T]$$

that is $E_1(t)$ and $E(t)$ are strict Lyapunov functions for our control system (6).
References


References


References


