Optimization and control of hyperbolic balance laws on networks for industrial applications

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Benasque VII

Session:

Modeling, simulation and control for PDEs on graphs

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

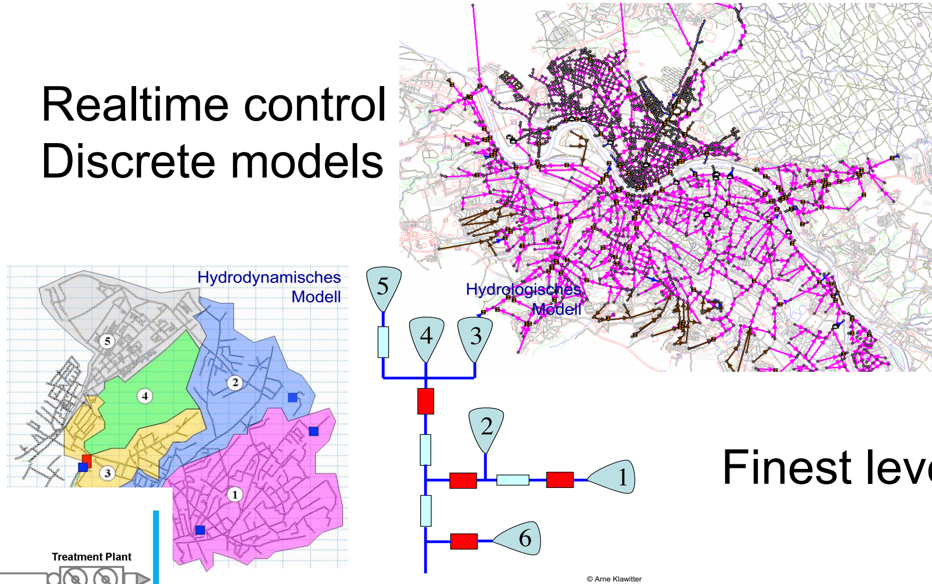
Department MATHEMATIK

CONTINUOUS OPTIMIZATION

Multiscale nature of the problem: waste water

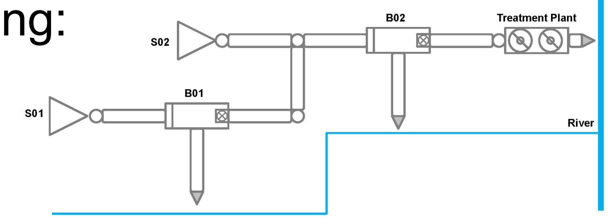
Realtime control
Discrete models

Homogenization
Effective models

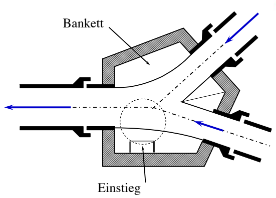


Finest level

PDE-modeling:
St. Venant's
system



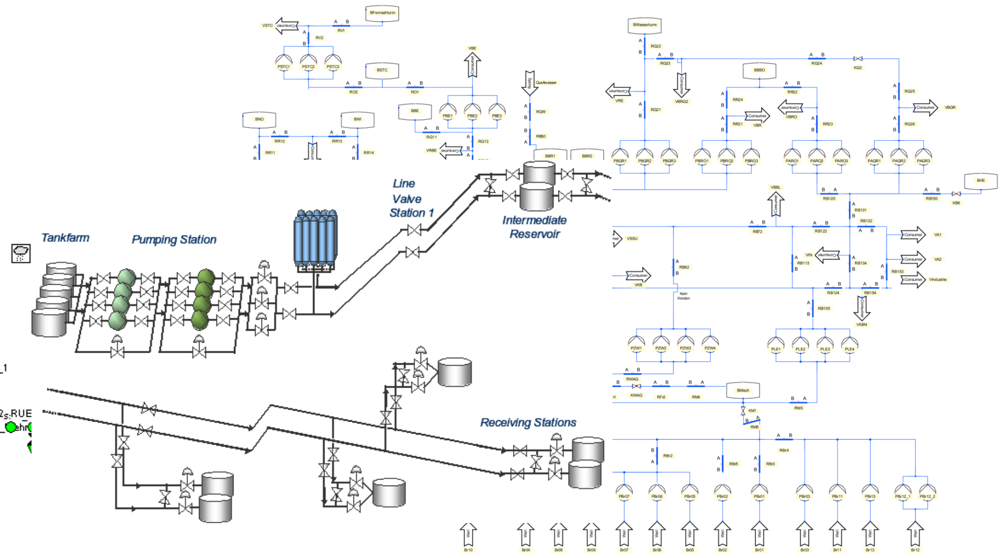
Coarse graining
Graph contraction



Subnet involving
Pumpes and valves

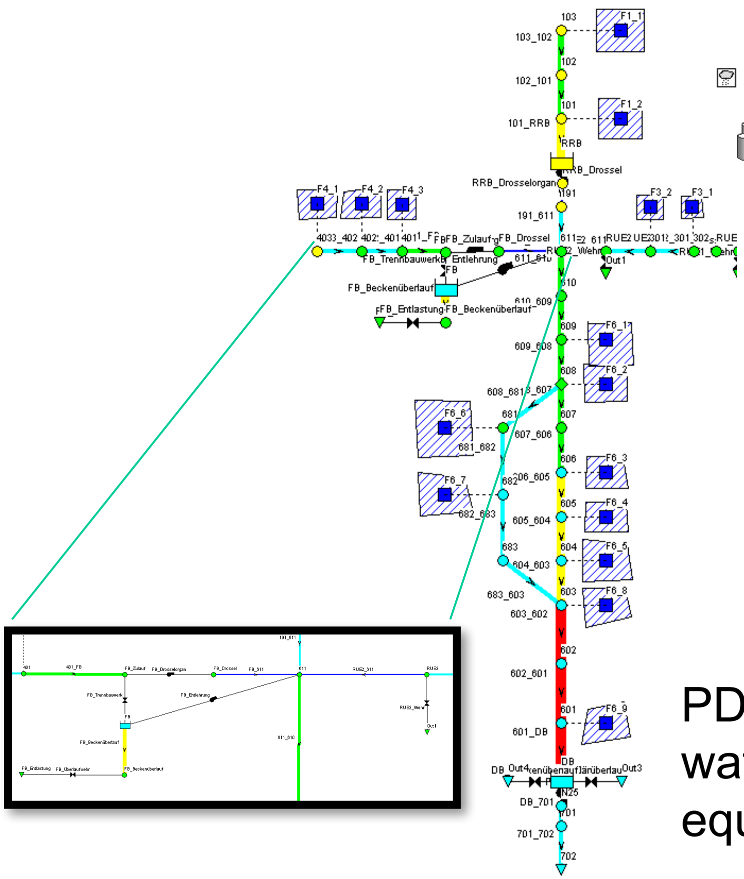
Single pipes
and junctions

Multiscale modeling: fresh water systems



Network flow probem

Hybrid PDE-ODE:
Model switching



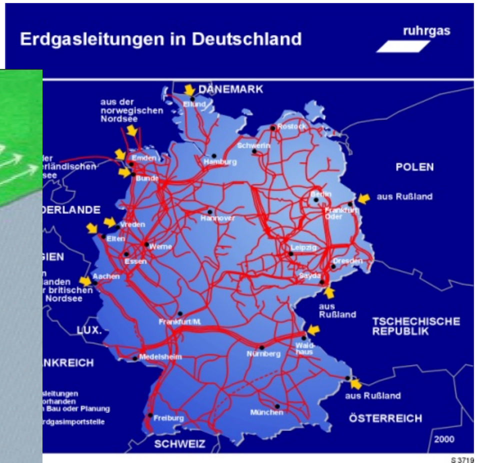
PDE modeling:
water hammer
equations

Gas-networks: Control of gas flow!

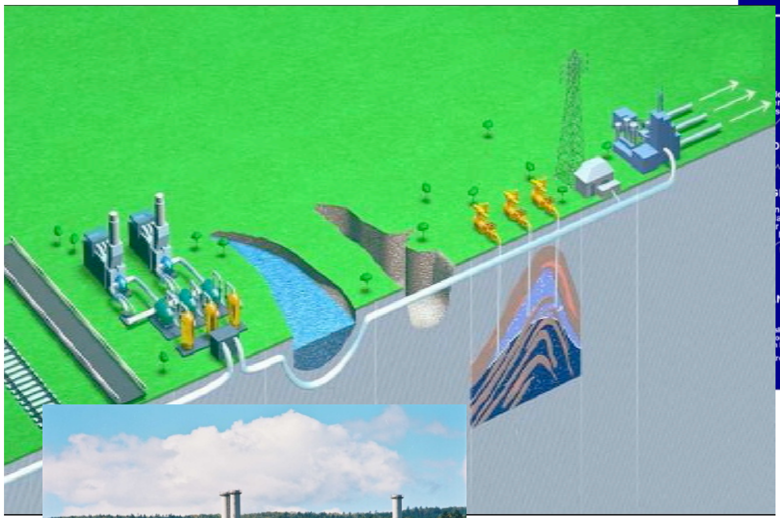
Gas-network in Germany:
DFG-research cluster
TRR154 since 2014

Industry: OGE

Individual pipe:
Parameters
(roughness, thickness,
bends, etc.)



‚Global‘ network:
Effective model,
planning aspects



Subnetwork:
environment

Junctions:
Different pipes,
multijunctions

Compressors,
Valves, releases,
heaters

Control the gas flow from suppliers to customers by operating valves, compressors,etc.
and thereby keep quality and minimze cost!

General picture

$$\min_{(y,u,s)\in\Xi} I(y,u) := \sum_{i\in\mathcal{I}} \int_0^T \int_0^{\ell_i} I_i(y_i)\,dx\,dt + \frac{\nu}{2} \sum_{j\in\mathcal{J}^S\cup\mathcal{J}_c} \int_0^T |u_j(t)|^2 dt$$

$$+ \frac{1}{2} \int_0^T \sum_{j\in\mathcal{J}^M_{v,c}} |s_j^{v,c}(t)|^2 dt$$

$s.t.$

(y,u,s) satisfies

$$\partial_t y + \partial_x F(y) = G(y),$$
$$R(y(0), y(1), u, s) = 0, \;\; y(x,0) = y_0$$

$$\Xi := \{(y,u,s) : \underline{y}_i \le y_i \le \overline{y}_i, \;\underline{u}_i \le u_i \le \overline{u}_i, \;\; i \in \mathcal{I}, s_j^{v,c} \in \{0,1\}\}.$$

Gas networks: A hierarchy of models

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\
\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) &= -\tfrac{\lambda}{2D}\rho v|v| - g\rho h' \\
\frac{\partial}{\partial t}\left(\rho(\frac{1}{2}v^2 + e)\right) + \frac{\partial}{\partial x}\left(\rho v(\frac{1}{2}v^2 + e) + pv\right) &= -\tfrac{k_w}{D}(T - T_w)
\end{aligned}
$$

- Constitutive law $p = R\rho T z(p, T)$.
- ρ density, v velocity, T its temperature and p the pressure.
- g gravitational constant, $h' = h'(x)$ slope of the pipe, λ friction coefficien of the pipe,
- D diameter, k_w heat coefficient, $T_w = T_w(x)$ temperature of the wall,

Reductions to isothermal eqns.

The isothermal Euler equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) &= -\frac{\lambda}{2D}\rho v\,|v| - g\rho h' \end{aligned} \tag{ISO1}$$

together with $p = R\rho T z(p)$ mit $z(p) = 1 + \alpha p$.

We put $q = \rho v$ and assume for simplicity $c^2\rho = p$. $z = h'$ is the slope .

Semilinear approximations

For constant $z(p) = z_0$, the speed of sound is $c = \sqrt{\frac{p}{\rho}}$. Hence,

$$p + \rho v^2 = p\left(1 + \frac{v^2}{c^2}\right)$$

For small velocities $|v| \ll c$ we have $\frac{\partial}{\partial x}(\rho v^2)$ small and, hence,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial p}{\partial x} &= -\frac{\lambda}{2D}\rho v\,|v| - g\rho h' \end{aligned} \tag{ISO2}$$

If also $\frac{\partial}{\partial t}(\rho v)$ is small, then

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial p}{\partial x} &= -\frac{\lambda}{2D}\rho v\,|v| - g\rho h'. \end{aligned} \tag{ISO3}$$

Further reductions up to algebraic equations

With $h' = 0$ follows even

$$\frac{\partial p}{\partial x} = -\frac{\lambda}{2D}\rho v\,|v|$$

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sqrt{\frac{DRTz_0}{\lambda}}\,\frac{\frac{\partial^2}{\partial x^2}p^2}{\sqrt{\left|\frac{\partial}{\partial x}p^2\right|}} \qquad \text{(ISO3P)}$$

For stationary preocesses and $z = z_0$ we obtain the algebraic system

$$\frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial p}{\partial x} = -\frac{\lambda}{2D}\rho v\,|v| \qquad \text{(ISO4)}$$

or

$$p(x) = \sqrt{p_{in}^2 - \frac{\lambda c^2 x}{2D}\rho v\,|\rho v|} \qquad (1)$$

Model adaptivity needed

Given the flow state $y(x,t)$ for some (sampled, measured) (x,t)

- decide which model is to be used on each edge of the graph (slow flow $\rightarrow$ algebraic, linear or semi-linear model, otherwise the full set of equations by a discrete decition process
- choose the appropriate coupling conditions
- generate a 'multiscale' model
- obtain a stationary state
- and solve the corresponding forward simulation problem on the entire network
- this is then used as warm-start for the optimal control problems

Network notation

Let $G = (V, E)$ denote the graph of the gas network with vertices (nodes) $V = \{n_1, n_2, \ldots, n_{|V|}\}$ an edges $E = \{e_1, e_2, \ldots, e_{|E|}\}$. Node indices are denoted $j \in \mathcal{J}, |\mathcal{J}| = |V|$, while edges are labelled $i \in \mathcal{I}, |\mathcal{I}| = |E|$. For the sake of uniqueness, we associate to each edge a direction. Accordingly, we introduce the edge-node incidence matrix

$$d_{ij} = \begin{cases} -1, & \text{if node } n_j \text{ is the left node of the edge } e_i, \\ +1, & \text{if node } n_j \text{ is the right node of the edge } e_i, \\ 0, & \text{else.} \end{cases}$$

The semi-linear network model

$$
\begin{aligned}
\partial_t p_i(x,t) + \frac{c_i^2}{a_i}\partial_x q_i(x,t) &= 0 \\
\partial_t q_i(x,t) + \partial_x p_i(x,t) &= -\frac{\lambda c_i^2}{2Da_i^2}\,\frac{q_i(x,t)\mathrm{abs} q_i(x,t)}{p_i(x,t)}, \\
g_j(p_i(n_j,t), q_i(n_j,t)) &= u_j(t),\ d_j = 1,\ i \in \mathcal{I}_j, \\
p_i(n_j,t) &= p_k(n_j,t),\ \forall i,k \in \mathcal{I}_j,\ d_j > 1, \\
\sum_{i\in\mathcal{I}_j} d_{ij} q_i(n_j,t) &= 0,\ d_j > 1, \\
s_j^v(t)\left[p_i(n_j,t)_{i\in\mathcal{I}_j^-} - p_k(n_j,t)_{k\in\mathcal{I}_j^+}\right] + (1-s_j^v(t)) q_i(n_j,t)_{i\in\mathcal{I}_j^-} &= 0, j \in \mathcal{J}_v \\
s_j^c(t)\left[u_j - C\left(\left(\frac{p_k(n_j,t)}{p_i(n_j,t)}\right)^{\text{sign }(q_k(n_j,t))\kappa} - 1\right)\right] & \\
+ (1-s_j^c(t))\left[p_i(n_j,t)_{i\in\mathcal{I}_j^-} - p_k(n_j,t)_{k\in\mathcal{I}_j^+}\right. & \\
p_i(x,0) = p_{i,0}(x),\ q_i(x,0) &= q_{ij}(x), \\
i \in \mathcal{I},\ x \in (0,\ell_i),\ t &\in (0,T).
\end{aligned}
\tag{1}
$$

The optimal control problem

$$\min_{(p,q,u,s)\in\Xi} I(p,q,u) := \sum_{i\in\mathcal{I}} \int_0^T \int_0^{\ell_i} I_i(p_i,q_i)dxdt + \frac{\nu}{2} \sum_{j\in\mathcal{J}^S\cup\mathcal{J}_c} \int_0^T |u_j(t)|^2 dt$$

$$+ \frac{1}{2}\int_0^T \sum_{j\in\mathcal{J}^M_{v,c}} |s_j^{v,c}(t)|^2 dt$$

$s.t.$

(p,q,u,s) satisfies (1),

$$\Xi := \{(p,q,u,s) : \underline{p}_i \leq p_i \leq \overline{p}_i,\ \underline{q}_i \leq q_i \leq \overline{q}_i,\ \underline{u}_i \leq u_i \leq \overline{u}_i,\ i\in\mathcal{I}, s_j^{v,c} \in \{0,1\}\}$$

We now consider the time discretization such that $[0,T]$ is decomposed into break points $t_0 = 0 < t_1 < \cdots < t_N = T$ with widths ${}^{\Delta}t_n := t_{n+1} - t_n, n = 0,\dots,N-1$. Accordingly, we denote $p_i(x,t_n) := p_{i,n}(x), q_i(x,t_n) := q_{i,n}(x), n = 0,\dots,N-1$. We consider a semi-implicit Euler scheme which takes p_i in the friction term in an explicit manner.

The time discrete model

$$\begin{aligned}
\frac{1}{\Delta t} p_{i,n+1}(x) + \frac{c_i^2}{a_i}\partial_x q_{i,n+1}(x) &= \frac{1}{\Delta t} p_{i,n}(x),\ x \in (0,\ell_i), i \in \mathcal{I} \\
\frac{1}{\Delta t} q_{i,n+1}(x) + \partial_x p_{i,n+1}(x) &= -\frac{\lambda c_i^2}{2D_i a_i^2}\frac{q_{i,n+1}(x)\mathrm{abs} q_{i,n+1}(x)}{p_{i,n}(x)} \\
&\quad + \frac{1}{\Delta t} q_{i,n}(x),\ x \in (0,\ell_i), i \in \mathcal{I} \\
g_j(p_{i,n+1}(n_j), q_{i,n+1}(n_j)) &= u_{j,n+1},\quad i \in \mathcal{I}_j,\ j \in \mathcal{J}^S \\
p_{i,n+1}(n_j) &= p_{k,n+1}(n_j),\ \forall i,k \in \mathcal{I}_j,\ ,j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij} q_{i,n+1}(n_j) &= 0,\ j \in \mathcal{J}^M, \\
s^v_{j,n+1}\left[p_{i,n+1}(n_j)_{i\in\mathcal{I}_j^-} - p_{k,n}(n_j)_{k\in\mathcal{I}_j^+}\right] &+ (1 - s^v_{j,n+1}) q_{i,n+1}(n_j)_{i\in\mathcal{I}_j^-} = 0, j \in \\
s^c_{j,n+1}\left[p_{i,n+1}(n_j)_{i\in\mathcal{I}_j^-} u_j - p_{k,n+1}(n_j)_{k\in\mathcal{I}_j^+}\right] &+ (1 - s^c_{j,n+1})\left[p_{i,n+1}(n_j)_{i\in\mathcal{I}_j^-} - p_{k,n+}\right. \\
p_{i,0}(x) = p_{i,0}(x),\ q_{i,0}(x) &= q_{i0}(x),\ x \in (0,\ell_i), i \in \mathcal{I}.
\end{aligned} \tag{2}$$

Time-discrete optimal control problem

We then obtain the optimal control problem on the time-discrete level:

$$\min_{(p,q,u,s)\in\Xi} I(p,q,u,s) := \sum_{i\in\mathcal{I}} \sum_{n=1}^{N} \int_0^{\ell_i} I_i(p_{i,n}, q_{i,n})dx + \frac{\nu}{2} \sum_{j\in\mathcal{J}^S\cup\mathcal{J}_c} \sum_{n=1}^{N} |u_j(n)|^2$$

$$+ \frac{1}{2} \sum_{n=1}^{N} \sum_{j\in\mathcal{J}_{v,c}^M} |s_{j,n}^{v,c}|^2$$

$$s.t.$$

(p, q, u, s) satisfies (2).

$$I_i((p_{i,n}, q_{i,n})(x) := \frac{1}{2}|p_{i,n}(x) - p_{i,n}^d(x)|^2 + |q_{i,n}(x) - q_{i,n}^d(x)|^2, x \in (0, \ell_i), \ i \in \mathcal{I}.$$

Instantaneous control

$$\min_{(p,q,u,s)\in\Xi} I(p,q,u) := \sum_{i\in\mathcal{I}} \int_0^{\ell_i} I_i(p_i,q_i)dx + \frac{\nu}{2}\sum_{j\in\mathcal{J}^S\cup\mathcal{J}_c} |u_j|^2 + \frac{1}{2}\sum_{j\in\mathcal{J}^M_{v,c}} |s^{v,c}_{j,n}|^2$$

$$s.t.$$

$$\beta_i p_i(x) + \partial_x q_i(x) = f_i^1,\ x\in(0,\ell_i),\ i\in\mathcal{I}$$

$$\alpha_i q_i(x) + \partial_x p_i(x) + g_i(x;q_i(x)) = f_i^2,\ x\in(0,\ell_i),\ i\in\mathcal{I}$$

$$g_j(p_i(n_j),q_i(n_j)) = u_j,\ i\in\mathcal{I}_j,\ j\in\mathcal{J}^S$$

$$p_i(n_j) = p_k(n_j),\ \forall i,k\in\mathcal{I}_j,\ j\in\mathcal{J}^M$$

$$\sum_{i\in\mathcal{I}_j} d_{ij}q_i(n_j) = 0,\ j\in\mathcal{J}^M$$

$$s_j^v\left[p_i(n_j)_{i\in\mathcal{I}_j^-} - p_k(n_j)_{k\in\mathcal{I}_j^+}\right] + (1-s_j^v)q_i(n_j)_{i\in\mathcal{I}_j^-} = 0, j\in\mathcal{J}_v$$

$$s_j^c\left[p_i(n_j)_{i\in\mathcal{I}_j^-}u_j - p_k(n_j)_{k\in\mathcal{I}_j^+}\right] + (1-s_j^c)\left[p_i(n_j)_{i\in\mathcal{I}_j^-} - p_k(n_j)_{k\in\mathcal{I}_j^+}\right] = 0, j\in\mathcal{J}_c$$

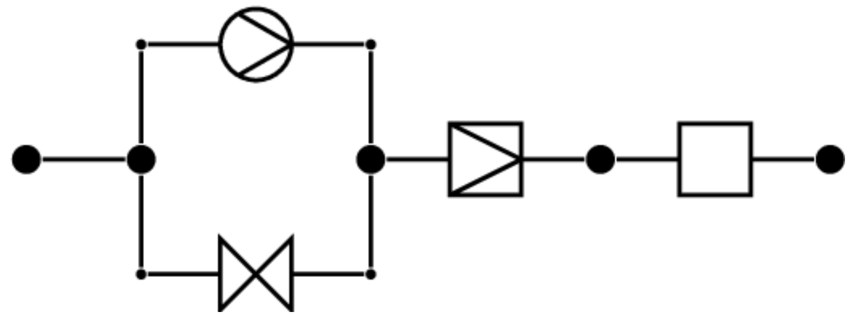
Master-slave decomposition

Given the flow state $y(x, t)$ for some (sampled, measured) (x, t) and given an appropriate multi-scale model simulation

- solve the discrete optimal control problem andobtain a switching structure s (master)
- with this s, solve the continuous optimal control problem (slave) (using e.g. domain decomposition)
- check feasibility of the solutions (need measure for mismatch)
- convey the result to the master
- prove convergence

Consortium in Germany

http://trr154.fau.de/index.php/de/



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