

Greedy optimal control for elliptic equations

Applications to turnpike control

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work in progress with:

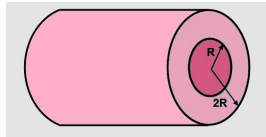
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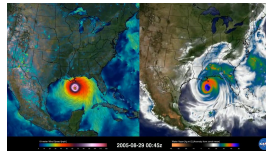
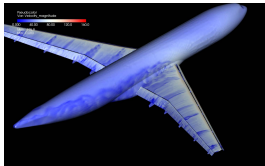
August 29th, 2017

Parameter dependent problems

Real life applications (may) depend on a **large** number of parameters



examples: thickness, conductivity, density, length, humidity, pressure, curvature, . . .



Parameter dependent problems (Cont.)

- When dealing with applications and simulations, we would like to explore within different parameter configurations.
- From the **control point of view**, this implies solving a different problem **for each configuration**.
- Computationally expensive.

OUR GOAL

Apply greedy theory to have a robust and fast numerical solvers.

Parameter dependent control problem

$$\Omega \subset \mathbb{R}^N, \quad \omega \subset \Omega.$$

Consider the system

$$\begin{cases} -\operatorname{div}(a(x, \nu)\nabla y) + c y = \chi_\omega u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

- ν is a parameter
- $u \in L^2(\omega)$ is a control
- $c = c(x) \in L^\infty(\Omega)$

Optimal control problem (OCP_ν)

$$\min_{u \in L^2(\omega)} J_\nu(u) = \frac{1}{2} \|u\|_{L^2(\omega)}^2 + \frac{\beta}{2} \|y - y_d\|_{L^2(\Omega)}^2,$$

Optimal control problem (OCP_ν)

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$\exists!$ optimal solution is well-known (Lions, Tröltzsch, ...)

Characterization: optimal pair (\bar{u}, \bar{y})

$$\bar{u} = -\chi_\omega \bar{q}$$

$$\begin{cases} -\operatorname{div}(a(x, \nu) \nabla \bar{y}) + c \bar{y} = -\chi_\omega \bar{q}, & \text{in } \Omega, \\ -\operatorname{div}(a(x, \nu) \nabla \bar{q}) + c \bar{q} = \beta (\bar{y} - y^d), & \text{in } \Omega, \\ \bar{y} = \bar{q} = 0, & \text{on } \partial\Omega. \end{cases} \quad (2)$$

As the state y depends on ν , also the control u depends on ν .

Parameter dependent control problem (cont.)

$$\begin{cases} -\operatorname{div}(a(x, \nu)\nabla y) + c y = \chi_\omega u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases}$$

From the practical point of view,

- Measure parameter ν and determine u_ν

$$\min_{u \in L^2(\omega)} J_\nu(u) = \frac{1}{2} \|u\|_{L^2(\omega)}^2 + \frac{\beta}{2} \|y - y_d\|_{L^2(\Omega)}^2,$$

using classical methods (iterative methods, ...)

- Repeat the process for each new value of ν .

CAN WE DO IT BETTER?

Greedy control

Assume that ν ranges within a compact set $\mathcal{K} \subset \mathbb{R}^d$ and $a_\nu = a(x, \nu)$ are bounded functions satisfying

$$0 < a_1 \leq a_\nu \leq a_2, \quad \nu \in \mathcal{K}.$$

In this way, we ensure that each control can be uniquely determined by

$$\bar{u}_\nu = -\chi_\omega \bar{q}$$

where (\bar{y}, \bar{q}) solve the optimality system (8). Consider the set of controls \bar{u}_ν for each possible value $\nu \in \mathcal{K}$. That is,

$$\bar{\mathcal{U}} = \{\bar{u}_\nu : \nu \in \mathcal{K}\}$$

THE IDEA

To determine a finite number of values of ν that yield the best possible approximation of the control manifold $\bar{\mathcal{U}}$

Description of the method

We look for a *small* number of parameters $\nu \in \mathcal{K}$ approximating the manifold \bar{u} in the sense of the **Kolmogorov width**. **Roughly**, the **Kolmogorov width** measures how well we can approximate \bar{u} by a finite dimensional space.

With greedy algorithms (Cohen & DeVore, Volkwein, Buffa et. al, ...), we search for the **most representative values of \bar{u}_ν** .

That is, given a tolerance ε , the goal is to find

$$\nu_1, \dots, \nu_{n(\varepsilon)}$$

such that, for any other $\nu \in \mathcal{K}$, the corresponding control \bar{u}_ν can be approximated by $u_\nu^* \in \text{span}\{\bar{u}_{\nu_1}, \dots, \bar{u}_{\nu_{n(\varepsilon)}}\}$ and

$$\|u_\nu^* - \bar{u}_\nu\|_{L^2(\omega)} \leq \varepsilon.$$

We also want to minimize n .

The surrogate

In practical implementations, the set $\bar{\mathcal{U}}$ is **unknown**.

Given two parameters ν_1 and ν_2 , how can we measure the distance between \bar{u}_{ν_1} and \bar{u}_{ν_2} ?

Recall that we want to avoid to compute \bar{u}_{ν} .

The surrogate

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Given two parameters ν_1 and ν_2 , how can we measure the distance between \bar{u}_{ν_1} and \bar{u}_{ν_2} ?

Recall that we want to avoid to compute \bar{u}_ν .

Standard residual: Suppose that we have computed u_{ν_1}

$$|u_{\nu_1} - u_{\nu_2}| \sim \nabla J_{\nu_2}(u_{\nu_1}) - \nabla J_{\nu_2}(u_{\nu_2}) = \nabla J_{\nu_2}(u_{\nu_1})$$

Compute $\nabla J_{\nu_2}(u_{\nu_1}) = u_{\nu_1} + \beta S_{\nu_2}^*(S_{\nu_2} u_{\nu_1} - y_d)$, where S_ν is to control-to-state operator. This means

$$\begin{cases} -\operatorname{div}(a_{\nu_2} \nabla y) + c y = \chi_\omega u_{\nu_1}, & \text{in } \Omega, \\ -\operatorname{div}(a_{\nu_2} \nabla q) + c q = \beta (y - y^d), & \text{in } \Omega, \\ y = q = 0, & \text{on } \partial\Omega. \end{cases} \quad \Rightarrow -\chi_\omega q_{\nu_2} = S_{\nu_2}^*(S_{\nu_2} u_{\nu_1} - y_d)$$

A cheap surrogate: Instead of using \bar{u}_ν and approximate the manifold $\bar{\mathcal{U}}$, use the optimal variables $(\bar{q}_\nu, \bar{y}_\nu)$ and approximate the manifold $\bar{\mathcal{Q}} \times \bar{\mathcal{Y}}$.

Denoting $L_\nu z := -\operatorname{div}(a_\nu \nabla z) + c z$, we define

$$R_\nu(q, y) := \begin{pmatrix} L_\nu y + \chi_\omega q \\ L_\nu q - \beta(y - y_d) \end{pmatrix}.$$

With this definition, we are able to compute the following estimates:

$$c_1 \left(\|y - \bar{y}_\nu\|_{H_0^1(\Omega)} + \|q - \bar{q}_\nu\|_{H_0^1(\Omega)} \right) \leq \|R_\nu(q, y)\|_{H^{-1}(\Omega)},$$

$$\|R_\nu(p, y)\|_{H^{-1}\Omega} \leq (1 + \alpha_2) (\|y - \bar{y}_\nu\|_{H_0^1(\Omega)} + \|q - \bar{q}_\nu\|_{H_0^1(\Omega)}).$$

where c_1 and α_2 only depending on a_1 , a_2 and $\|c\|_\infty$.

Upper and lower bounds for $R_\nu(q, y)$ are essential for the proof of greedy algorithms in terms of the Kolmogorov width.

$$R_\nu(q, y) := \begin{pmatrix} L_\nu y + \chi_\omega q \\ L_\nu q - \beta(y - y_d) \end{pmatrix}. \quad (3)$$

Theorem 1 (in progress H-S, Lazar, Zuazua, '17)

The residual (3) provides the approximation estimates for optimal controls and states

- $\|u_\nu^* - \bar{u}_\nu\|_{L^2(\Omega)} \leq \frac{1}{c_1} \|R_\nu(q, y)\|_{[H^{-1}(\Omega)]^2},$
- $\|y_\nu^* - \bar{y}_\nu\|_{H_0^1(\Omega)} \leq \left(\frac{1}{\alpha_1 c_1}\right) \|R_\nu(q, y)\|_{[H^{-1}(\Omega)]^2},$

where c_1 and α_1 only depend on a_1 , a_2 and $\|c\|_\infty$.

Offline algorithm

Step 1: Initialization. Fix $\varepsilon > 0$. Choose any $\nu \in \mathcal{K}$, $\nu = \nu_1$ and compute the minimizer of J_{ν_1} . This leads to $\begin{pmatrix} \bar{q}_{\nu_1} \\ \bar{y}_{\nu_1} \end{pmatrix}$.

Step 2: recursive choice of ν .

Assuming we have chosen ν_1, \dots, ν_p , we choose ν_{p+1} as the maximizer of

$$\max_{\nu \in \mathcal{K}} \text{dist} \left(\begin{pmatrix} 0 \\ y^d \end{pmatrix}, R_\nu(\bar{Q}_p, \bar{Y}_p) \right), \quad R_\nu(q, y) = \begin{pmatrix} L_\nu y + \chi_\omega q \\ L_\nu q - \beta(y - y^d) \end{pmatrix}$$

Step 3: Stopping criterion. Stop if the $\max \leq \varepsilon$.

Theorem 1 (in progress H-S, Lazar, Zuazua, '17)

The offline algorithm stops after $n_0(\varepsilon)$ iterations, and fullfills the requirements of the greedy theory.

After choosing the most representative values of ν , we can construct an approximated optimal control u_ν^* for any arbitrary given value $\nu \in \mathcal{K}$ by taking

$$u_\nu^* = \sum_{i=1}^k \lambda_i \bar{q}_{\nu_i} |_\omega$$

where λ_i are determined by the projection of the vector $\begin{pmatrix} 0 \\ y_d \end{pmatrix}$ to the space

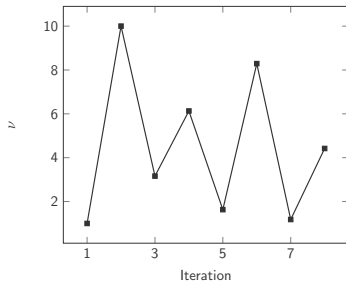
$$\text{span}\{R_\nu(\bar{q}_{\nu_1}, \bar{y}_{\nu_1}), \dots, R_\nu(\bar{q}_{\nu_k}, \bar{y}_{\nu_k})\}$$

NUMERICAL RESULTS

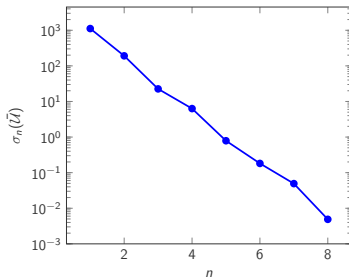
- $\Omega = (0, 1)^2$ in 2-D or $\Omega = (0, 1)$ in 1-D.
- Uniform meshes, i.e., meshes with constant discretization steps in each direction, $N = 400$.
- We will approximate the operator $\mathcal{A} = -\operatorname{div}(a(x, \nu)\nabla \cdot)$ by using the standard 5-point discretization.
- Discretize-then-optimize.
- $\nu \in \mathcal{K} = [1, 10]$.
- \mathcal{K} sampled in 100 equidistant points.

Greedy test # 1

- $a(x, \nu) = 1 + \nu(x_1^2 + x_2^2)$, ○ $c(x) = \sin(2\pi x_1) \sin(2\pi x_2)$,
- $y_d = \sin(\pi x_1)$, ○ $\beta = 10^4$, ○ $\varepsilon = 0.005$
- $t_{\text{cheap}} = 304s$, ○ $t_{\text{std}} = 384s$

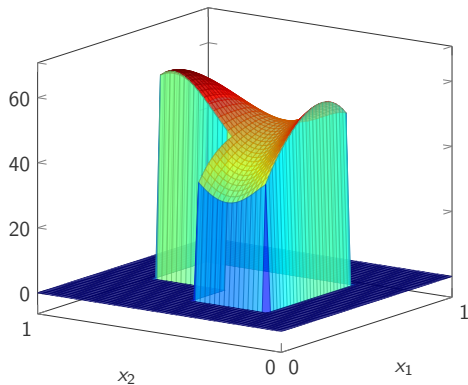


(a) Selected ν

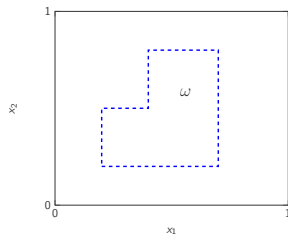


(b) Approximation error

Approximation for $\nu = \pi/2$



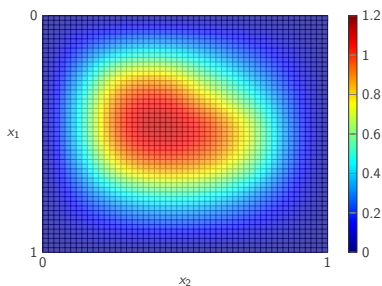
(c) The approximated control



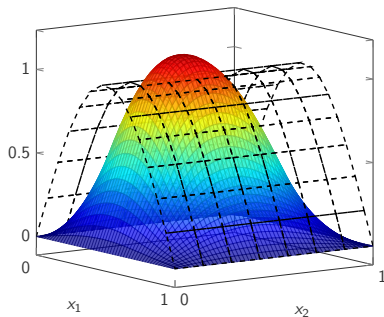
(d) The control set ω

$$\circ |u_{\pi/2}^* - \bar{u}_{\pi/2}|_{L^2(\omega)} \approx 1.45 \times 10^{-5}, \quad t_{\text{online}} = 0.45\text{s}, \quad t_{\text{iterative}} = 6.01\text{s}.$$

Approximation for $\nu = \pi/2$ (cont.)



(e) The controlled state

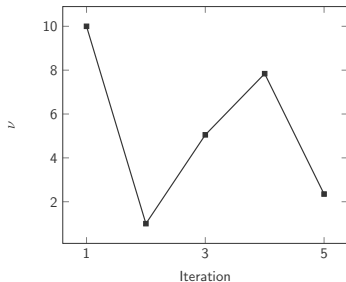


(f) The state $y_{\pi/2}^*$ and the target function y^d (dashed)

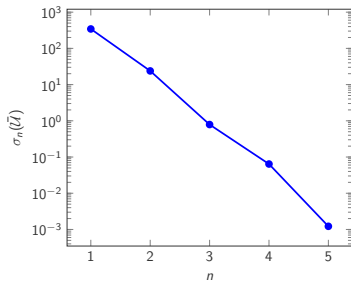
$$\circ |y_{\pi/2}^* - \bar{y}_{\pi/2}|_{L^2(\Omega)} \approx 1.15 \times 10^{-7}$$

Greedy test # 2

- $a(x, \nu) = 1 + \nu x^2(1 - x)^2$, $c(x) = -15 \sin(\pi x)$, $y_d = \chi_{(0.5, 0.8)}$,
- $\beta = 10^4$, $\omega = (0.3, 0.9)$, $\varepsilon = 0.005$
- $t_{\text{cheap}} = 0.68s$, $t_{\text{std}} = 0.809s$

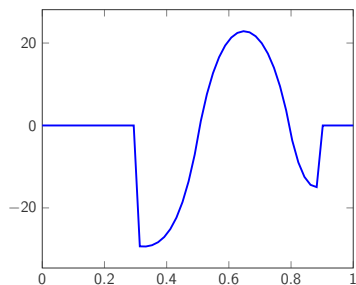


(g) Selected ν

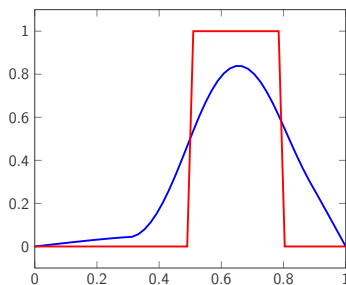


(h) Approximation error

Greedy test # 2



(i) The approximated control



(j) The state $y_{\pi/2}^*$ (blue) and the target y^d (red)

$$\circ |u_{\pi/2}^* - \bar{u}_{\pi/2}|_{L^2(\omega)} \approx 1.17 \times 10^{-5}, \quad \circ \|y_{\pi/2}^* - \bar{y}_{\pi/2}\|_{L^2(\Omega)} \approx 5.09 \times 10^{-7},$$

CONNECTION WITH THE TURNPIKE PROBLEMS

Time dependent control problem

Consider

$$\begin{cases} \partial_t y - \operatorname{div}(a(x, \nu) \nabla y) + c y = \chi_\omega u & \text{in } Q = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(x, 0) = y^0(x) & \text{in } \Omega. \end{cases} \quad (4)$$

and the control problem

$$\min_u J_\nu^T(u) = \frac{1}{2} \int_0^T \|u(t)\|_{L^2(\omega)}^2 dt + \frac{\beta}{2} \int_0^T \|y(t) - y^d\|_{L^2(\Omega)}^2 dt.$$

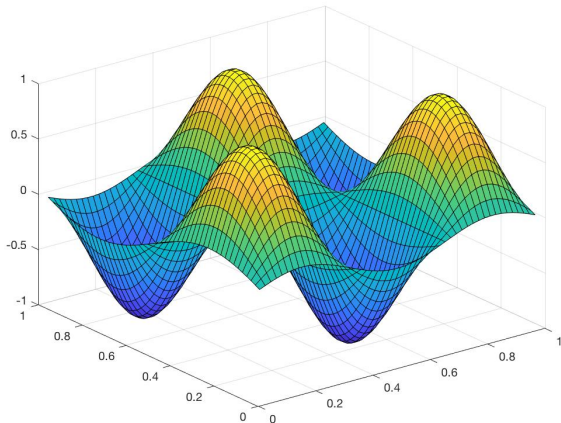
The optimal solution (u^T, y^T) satisfies

$$\|y^T(t) - \bar{y}\|_{L^2(\Omega)} + \|u^T(t) - \bar{u}\|_{L^2(\Omega)} \leq K \left(e^{-\mu t} + e^{-\mu(T-t)} \right), \quad \forall t \in [0, T]$$

- Exponential convergence of the finite-time horizon control problem to the steady one as $T \rightarrow \infty$.

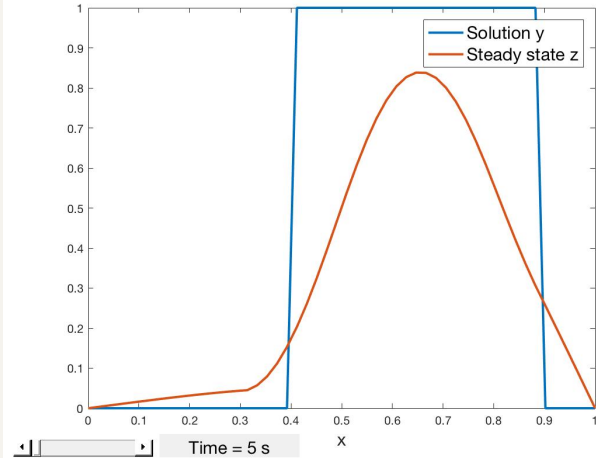
The case $c(x) \geq 0$ (greedy test #1)

$$u(x, t) = u_{\pi/2}^*(x), \quad y_0(x) = \sin(3\pi x_1) \sin(2\pi x_2)$$



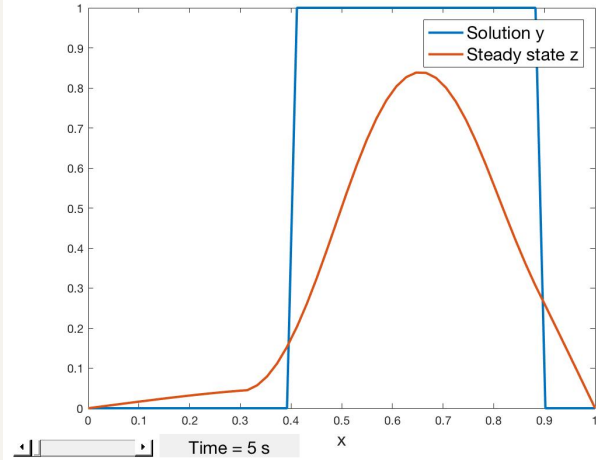
The case $c(x) \leq -\lambda_1$ (greedy test #2)

$$u(x, t) = u_{\pi/2}^*(x), \quad y_0(x) = \chi_{(0.4, 0.9)}$$



The case $c(x) \leq -\lambda_1$ (greedy test #2)

$$u = u_{\pi/2}^T(x, t), \quad y_0(x) = \chi_{(0.4, 0.9)}$$



THANK YOU!