Schrödinger equation:

from harmonic analysis to IBVP

Chuang Zheng

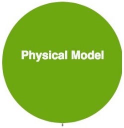
Beijing Normal University

Benasque, Aug 2017

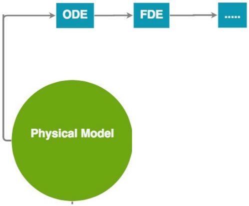
Outline of the talk

Table of contents

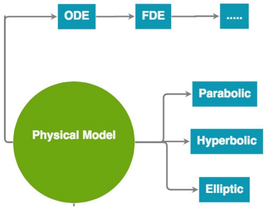
Where am I standing



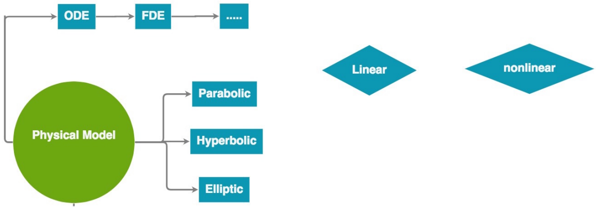
Where am I standing



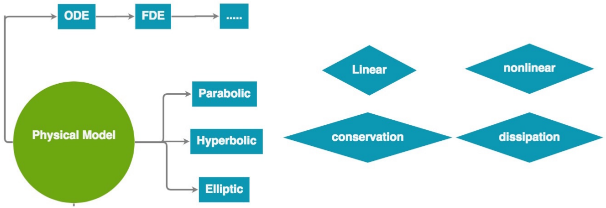
Where am I standing



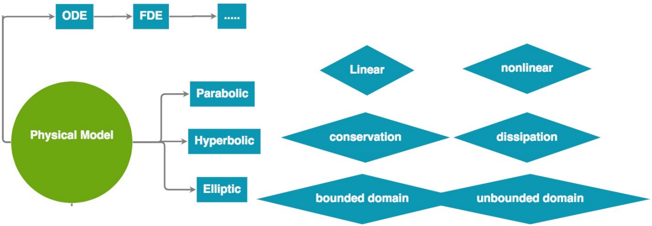
Where am I standing



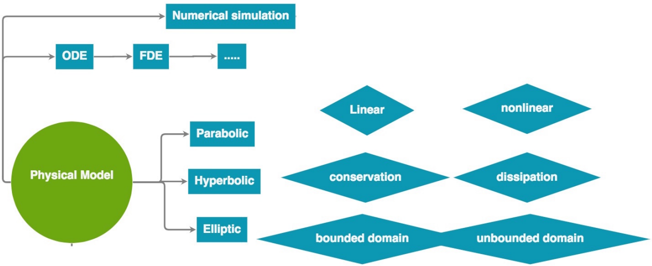
Where am I standing



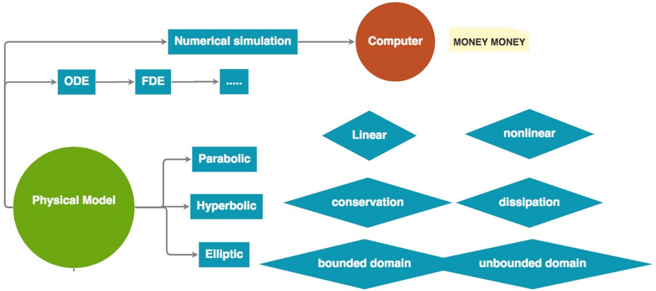
Where am I standing



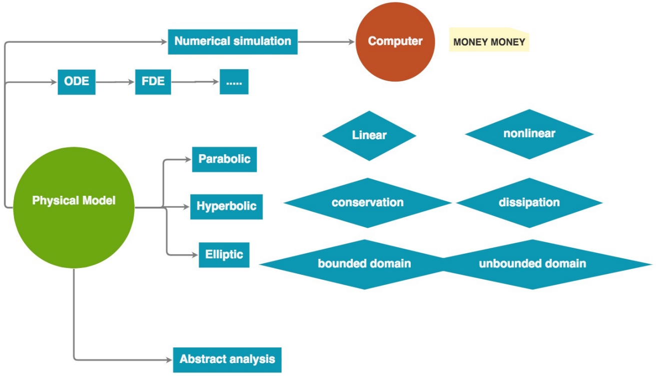
Where am I standing



Where am I standing



Where am I standing



Where am I standing

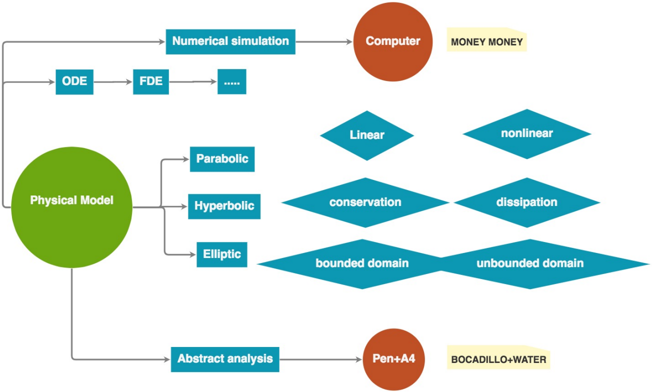
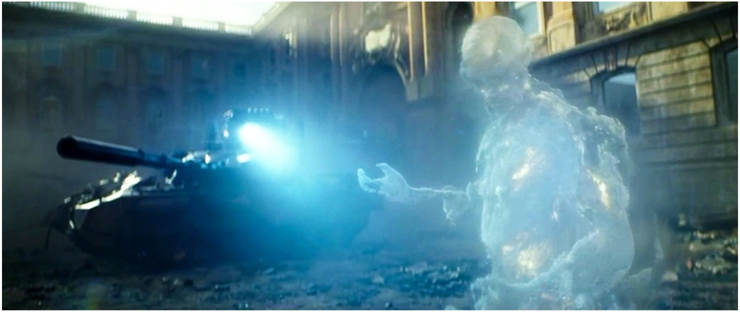


Table of contents

The nonlinear Schrödinger equation arises in many scientific fields such as propagation of light in fiber optics cables, shallow and deep surface water waves, quantum mechanics, Bose-Einstein condensate theory, nonlinear optics and plasma physics. (Avron et al. 78, Karpman 96, Sulem& Sulem 99, Karpman&Shagalov 00, Pausader 07, 09, etc.)

Application on Sci-Fic movie



Mission of Defense Advanced Research Projects Agency (DARPA, 3 bil US dollars/year) in Moldovan

Mathematically, most existing results concern the well-posedness with IVP

- on whole domain $\mathcal{R}^d$. Cazenave, Ginibre&Velo, Kato, etc.

Mathematically, most existing results concern the well-posedness with IVP

- on whole domain $\mathcal{R}^d$. Cazenave, Ginibre&Velo, Kato, etc.
- on torus $\mathbb{T}$. Bourgain, Tsutsumi, Pausader, etc.

Monograph book by Cazenave 2003.

Mathematically, most existing results concern the well-posedness with IVP

- on whole domain $\mathcal{R}^d$. Cazenave, Ginibre&Velo, Kato, etc.
- on torus $\mathbb{T}$. Bourgain, Tsutsumi, Pausader, etc.

Monograph book by Cazenave 2003.

With non-homogeneous boundary conditions (IBVP):

Mathematically, most existing results concern the well-posedness with IVP

- on whole domain $\mathcal{R}^d$. Cazenave, Ginibre&Velo, Kato, etc.
- on torus $\mathbb{T}$. Bourgain, Tsutsumi, Pausader, etc.

Monograph book by Cazenave 2003.

With non-homogeneous boundary conditions (IBVP):

- on half line $\mathbb{R}^+$. Bona, Sun, Zhang, etc.

Mathematically, most existing results concern the well-posedness with IVP

- on whole domain $\mathcal{R}^d$. Cazenave, Ginibre&Velo, Kato, etc.
- on torus $\mathbb{T}$. Bourgain, Tsutsumi, Pausader, etc.

Monograph book by Cazenave 2003.

With non-homogeneous boundary conditions (IBVP):

- on half line $\mathbb{R}^+$. Bona, Sun, Zhang, etc.
- on bounded domain $(0,1)$. Bona, Zhang, Zheng, etc.

Whole domain V.S. with boundary data

Keywords in common:

- Mixture of the properties: parabolic and hyperbolic.
- Local existence of the solutions, uniqueness, regularity, smoothing effect.
- Global nature: finite time blow-up, global existence, asymptotic behavior of solutions.
- Control theory.

Whole domain V.S. with boundary data

Keywords with differences:

- On $\mathcal{R}^d$ and $\mathbb{T}$. Strichartz estimate, harmonic analysis, etc.
- On $(0, 1)$. PDE techniques: Multipliers, a priori estimate, etc.

Our main target

We consider the well-posedness of the nonlinear Initial Boundary Value Problems (IBVP) of the Nonlinear Schödinger equation on a finite interval $(0, 1)$.

Our main target

We consider the well-posedness of the nonlinear Initial Boundary Value Problems (IBVP) of the Nonlinear Schödinger equation on a finite interval $(0,1)$.

Key ingredient: find a way to describe the boundary condition of the IBVPs on $(0,1)$ as the boundary integral of the solution of the corresponding operator in the periodic domain $\mathbb{T} = \mathcal{R} \setminus \mathbb{Z}$.

Some works have been done on the Schrödinger equation (Bona, Sun, Zhang), KdV equation (Capistrano-Filho, Sun, Zhang), Ginzburg-Landau equation (Li) and Kuramoto-Sivashinsky equation (Zhang) recently.

Table of contents

Second order NLS

Consider $\Omega = (0,1)$ with Dirichlet BC:

$$(*)\begin{cases} iu_t + u_{xx} + \lambda u|u|^{p-2} = 0, & (t,x) \in (0,T) \times \Omega \\ u(t,0) = h_1(t),\ u(t,1) = h_2(t), & t \in (0,T) \\ u(0,x) = \phi(x), & x \in \Omega. \end{cases}$$

Fourth order NLS: Navier BC

Consider $\Omega = (0,1)$ with Navier BC:

$$
(**)\begin{cases}
iu_t + u_{xxxx} + \lambda u|u|^{p-2} = 0, & (t,x) \in (0,T) \times \Omega \\
u(t,0) = h_1(t),\ u(t,1) = h_2(t), & t \in (0,T) \\
u_{xx}(t,0) = h_5(t),\ u_{xx}(t,1) = h_6(t), & t \in (0,T) \\
u(0,x) = \phi(x), & x \in \Omega.
\end{cases}
$$

Fourth order NLS: Dirichlet BC

Consider $\Omega = (0,1)$ with Dirichlet BC:

$$(***)\begin{cases} iu_t + u_{xxxx} + \lambda u|u|^{p-2} = 0, & (t,x) \in (0,T) \times \Omega \\ u(t,0) = h_1(t),\ u(t,1) = h_2(t), & t \in (0,T) \\ u_x(t,0) = h_3(t),\ u_x(t,1) = h_4(t), & t \in (0,T) \\ u(0,x) = \phi(x), & x \in \Omega. \end{cases}$$

Theorem

Define the solutions of NLS on Ω. Under some assumptions

- *(*) is well-posed for appropriate regularity of initial and boundary data. (Bona et al. preprint)*

Theorem

Define the solutions of NLS on Ω. Under some assumptions

- *(*) is well-posed for appropriate regularity of initial and boundary data. (Bona et al. preprint)*
- *(**) is locally well-posed for appropriate regularity of initial and boundary data. (Zheng)*

Theorem

Define the solutions of NLS on Ω. Under some assumptions

- *(*) is well-posed for appropriate regularity of initial and boundary data. (Bona et al. preprint)*
- *(**) is locally well-posed for appropriate regularity of initial and boundary data. (Zheng)*
- *(***) is locally well-posed for appropriate regularity of initial and boundary data. (Zheng)*

Key of the proof: L^4 estimate of the Linear system

Solution will be separated by three parts:

1. Nonlinear term: focusing or defocusing;

Key of the proof: L^4 estimate of the Linear system

Solution will be separated by three parts:

1. Nonlinear term: focusing or defocusing;
2. Initial data: $BD = 0$ and $ID = \phi$;

Key of the proof: L^4 estimate of the Linear system

Solution will be separated by three parts:

1. Nonlinear term: focusing or defocusing;
2. Initial data: $BD = 0$ and $ID = \phi$;
3. $BD \neq 0$ and $ID = 0$. NEW.

Table of contents

Example

Consider

$$
(***)\begin{cases}
iu_t + u_{xxxx} = 0, & (t,x) \in (0,T) \times \Omega \\
u(t,0) = h_1(t),\ u(t,1) = h_2(t), & t \in (0,T) \\
u_{xx}(t,0) = h_5(t),\ u_{xx}(t,1) = h_6(t), & t \in (0,T) \\
u(0,x) = 0, & x \in \Omega.
\end{cases}
$$

Boundary integrals

Lemma

u can be expressed as the form

$$u(x,t) = W_{0,h}h_1 + (W_{0,h}h_2)\|_{x\to 1-x} + W_{2,h}h_5 + (W_{2,h}h_6)\|_{x\to 1-x}.$$

with

$$\begin{aligned} W_{0,h}h &= \sum_{k=1}^{\infty} (-2i(k\pi)^3) \int_0^t e^{i(k\pi)^4(t-\tau)} h(\tau) d\tau \sin(k\pi x), \\ W_{2,h}h &= \sum_{k=1}^{\infty} (-\frac{2}{3} ik\pi) \int_0^t e^{i(k\pi)^4(t-\tau)} h(\tau) d\tau \sin(k\pi x) \end{aligned}$$

L^4 estimate of $W_{0,h}(\cdot)h$

Lemma

Let $h \in H^{\frac{3}{4}}(0,T)$. *We have*

$$W_{0,h}(\cdot)h \in L^4(\Omega_T) \cap C([0,T]; L^2(0,1)),$$

which means

$$\|u_{0,h}\|_{L^4(\Omega_T)} \leq C_T \|h\|_{H^{\frac{3}{4}}(0,T)}$$

and

$$\sup_{0 \leq t \leq T} \|u_{0,h}(\cdot,t)\|_{L^2(0,1)} \leq C_T \|h\|_{H^{\frac{3}{4}}(0,T)}.$$

Some remarks:

1. Estimations hold for other boundary integrals

   $W_{0,h}(\cdot)h, i = 1, 2, \cdots$

Some remarks:

1. Estimations hold for other boundary integrals

   $W_{0,h}(\cdot)h, i = 1, 2, \cdots$

2. The regularities of the boundary data are sharp.

Some remarks:

1. Estimations hold for other boundary integrals $W_{0,h}(\cdot)h, i = 1, 2, \cdots$
2. The regularities of the boundary data are sharp.
3. To obtain global existence of the solution, some extra a-priori estimations are needed (under the framework of PDEs). It's done for the second order equation and still open for KdV and 4th order equations.

References

- J. L. Bona, S.M. Sun and B. Y. Zhang, Nonhomogeneous boundary value problems for one-dimensional nonlinear Schrödinger equations, Preprint.
- J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3 (1993), no. 2, 107-156.
- T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 2003.

THANK YOU!