Turnpike property and application to a new shooting method for orbital transfer

Gontran Lance

ENSTA ParisTech

Benasque, 30 Août 2017

- P
- $(e_x, e_y) = e\big(cos(\Omega + \omega), sin(\Omega + \omega)\big)$
- $(h_x, h_y) = tan(\frac{i}{2})\big(cos(\Omega), sin(\Omega)\big)$
- $L = \Omega + \omega + w$

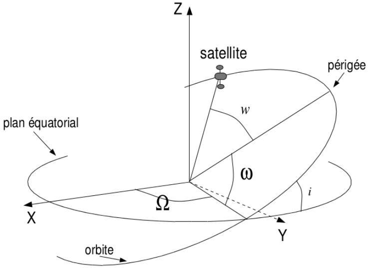

Figure: Orbital parameters

Remark

In 2D :

- $\vec{h} = \vec{0}$ *because* $i = 0$
- $L = \omega + w$

Optimal transfer problem:
let $t_f > 0$, $T_{max} > 0$ and find
$\alpha \in L^\infty([0, t_f], [0, T_{max}])$, $\phi \in L^\infty : \mathbb{R} \to \mathbb{R}$ such that:

$$\min_{\alpha,\phi} -m(t_f) = -m_0 + \beta \int_0^{t_f} \alpha$$

$$\dot{P} = \frac{1}{m}\sqrt{\frac{P}{\mu}}\frac{2P}{W}\alpha \sin(\phi)$$

$$\dot{e}_x = \frac{1}{m}\sqrt{\frac{P}{\mu}}\alpha\Big(\sin(L)\cos(\phi) + \Big(\cos(L) + \frac{\cos(L) + e_x}{W}\Big) \sin(\phi)\Big)$$

$$\dot{e}_y = \frac{1}{m}\sqrt{\frac{P}{\mu}}\alpha\Big(- \cos(L)\cos(\phi) + \Big(\sin(L) + \frac{\sin(L) + e_y}{W}\Big) \sin(\phi)\Big)$$

$$\dot{m} = -\beta\alpha$$

$$\dot{L} = \sqrt{\frac{\mu}{P}}\frac{W^2}{P}$$

$$x(0) = (P_0, e_{0x}, e_{0y}, m_0, L_0), \quad x(t_f) = (P_1, e_{1x}, e_{1y}, -, -)$$

with $W = 1 + e_x \cos(L) + e_y \sin(L)$

Pontryagin's principle

$$\min_u \int_0^{t_f} f^0(s,x(s),u(s))ds + g(t_f,x(t_f))$$
$$\dot{x}(t) = f(t,x(t),u(t))$$
$$x(0) \in M_0, x(t_f) \in M_1$$

- $\begin{array}{rcl} \dot{x}(t) & = & \frac{\partial H}{\partial \lambda}(t,x(t)),\lambda(t),\lambda^0,u(t)) \\ \dot{\lambda}(t) & = & -\frac{\partial H}{\partial x}(t,x(t)),\lambda(t),\lambda^0,u(t)) \end{array}, \forall t \in [0,t_f]$
- $H(t,x(t),\lambda(t),\lambda^0,u(t)) = \max_{v\in\Omega} H(t,x(t),\lambda(t),\lambda^0,v), \ \forall t \in [0,t_f]$
- $\begin{array}{rl} \lambda(0) & \perp T_{x(0)}M_0 \\ \lambda(t_f) - \lambda^0 \frac{\partial g}{\partial x}(t_f,x(t_f)) & \perp T_{x(T)}M_1 \end{array}$
- $H(t_f,x(t_f),\lambda(t_f),\lambda^0,u(t_f)) = -\lambda^0 \frac{\partial g}{\partial t}(t_f,x(t_f))$

with

$$H(t,x,\lambda,\lambda^0,u) = (\lambda|f(t,x,u)) + \lambda^0 f^0(t,x,u)$$

We introduce the state $x = (P, e_x, e_y, m, L)$ and the adjoint-state $\lambda = \left(\lambda_P, \lambda_{e_x}, \lambda_{e_y}, \lambda_m, \lambda_L\right)$ and after the application of the Pontryagin's Principle we get:

$$
\begin{array}{rcl}
\dot{\bar{x}}(t) & = & \dfrac{\partial H}{\partial \lambda}(\bar{x}(t), \bar{\lambda}(t), \bar{\alpha}(t), \bar{\phi(t)}) \\
\dot{\bar{\lambda}}(t) & = & -\dfrac{\partial H}{\partial x}(\bar{x}(t), \bar{\lambda}(t), \bar{\alpha}(t), \bar{\phi(t)}) \\
H\big(\bar{x}(t), \bar{\lambda}(t), \bar{\alpha}(t), \bar{\phi(t)}\big) & \geqslant & H\big(\bar{x}(t), \bar{\lambda}(t), \alpha, \phi\big), \forall \alpha \in [0, T_{max}], \forall \phi \in \mathbb{R} \\
H\big(\bar{x}(t_f), \bar{\lambda}(t_f), \bar{\alpha}(t_f), \bar{\phi}(t_f)\big) & = & 0, \ \text{ free time } t_f \\
\bar{\lambda}(t_f) & = & 0, \ \text{ free } L(t_f) \\
\bar{\lambda}_m(t_f) - 1 & = & 0, \ \text{ free } m(t_f)
\end{array}
$$

Introducing the function Φ as the function of commutation we obtain bang-bang control such that:

$$\begin{aligned}
\bar{\alpha} &= \begin{cases} 0, \text{ if } \Phi(\bar{x}, \bar{\lambda}, \bar{\phi}) < 0 \\ \in [0, T_{max}], \text{ if } \Phi(\bar{x}, \bar{\lambda}, \bar{\phi}) = 0 \\ Tmax, \text{ if } \Phi(\bar{x}, \bar{\lambda}, \bar{\phi}) > 0 \end{cases} \\
\cos(\bar{\phi}) &= \bar{\lambda}_{e_x} * \sin(\bar{L}) - \bar{\lambda}_{e_y} * \cos(\bar{L}) \\
\sin(\bar{\phi}) &= \bar{\lambda}_P \frac{2\bar{P}}{\bar{W}} + \bar{\lambda}_{e_x}\Big(\cos(\bar{L}) + \frac{\cos(\bar{L}) + \bar{e_x}}{\bar{W}}\Big) \\
&\quad + \bar{\lambda}_{e_y}\Big(\sin(\bar{L}) + \frac{\sin(\bar{L}) + \bar{e_y}}{\bar{W}}\Big)
\end{aligned}$$

Denote

$$\bar{z} = \begin{pmatrix} \bar{x} \\ \bar{\lambda} \end{pmatrix}$$

$$\bar{z}(0) \to \bar{z}(t_f) \quad via \quad \forall t \in [0, t_f], \dot{\bar{z}}(t) = \begin{pmatrix} \frac{\partial H}{\partial \lambda}(\bar{x}(t), \bar{\lambda}(t), \bar{\alpha}(t), \phi(\bar{t})) \\ -\frac{\partial H}{\partial x}(\bar{x}(t), \bar{\lambda}(t), \bar{\alpha}(t), \phi(\bar{t})) \end{pmatrix}$$

We get a 2 point boundary-value problem in which the unknown variables necessary to get the optimal trajectory are :

$$X = \left(\lambda_P(0), \lambda_{e_x}(0), \lambda_{e_y}(0), \lambda_m(0), \lambda_L(0), t_1, t_2, t_f\right)$$

where t_1, t_2 are the two times of commutation.
Using a shooting method (written in FORTRAN) with two bigs routines:

- one which lets us to integrate the dynamical system
- the other which is a Newton method for finding a zero of a function

and with the assumptions : Only 2 commutations and transfer in less than one lap

- as if the time t_f was bounded
- most of orbit transfers are made thanks 2 impulses

Principle of the shooting method:

1. we choose initialization for $\lambda(0), t_1, t_2, t_f$
2. $z(0) \underbrace{\rightarrow}_{\alpha=T_{max}} z(t_1) \underbrace{\rightarrow}_{\alpha=0} z(t_2) \underbrace{\rightarrow}_{\alpha=T_{max}} z(t_f)$
3. we compute the function whose we want a zero :

$$F(X) = \begin{pmatrix} P(t_f) - P_f \\ e_x(t_f) - e_{1x} \\ e_y(t_f) - e_{1y} \\ \lambda_L(t_f) \\ \lambda_m(t_f) - 1 \\ \Phi(t_1) \\ \Phi(t_2) \\ H(t_f) \end{pmatrix}$$

4. we search a zero of F (Newton method)

The shooting method is very fast and converges precisely but :

- Due to Newton's method, we need a precise initialization
- It is not easy to find a good initialization
- Not exactly the same results as in an article of **Derek Lawden** called "Impulsive transfer between elliptical orbits"

Ideas for a new shooting method :

- the trajectory on the transfer's ellipse seems more stable
- we can use the Lawden's results to find the optimal transfer ellipse and then initialize a new shooting method

Remark
The trajectory makes us think to the Turnpike property because it's made of three pieces (in the case of 2 impulses)

Optimal control problem (OCP_T)

For $T > 0$ fixed, find $u_T \in L^\infty(0, T, \mathbb{R}^m)$ such that

$$\min \int_0^T f^0\big(x(t), u(t)\big)dt$$
$$\dot{x}(t) = f\big(x(t), u(t)\big)$$
$$R(x(0), x(T)) = 0$$

Static optimal control problem

$$\min_{(x,u)\in\mathbb{R}^n\times\mathbb{R}^m} f^0(x, u)$$
$$f(x, u) = 0$$

Turnpike Property :
The solution of an optimal control problem in large time should spend most of its time near a steady-state (solution of the static optimal control problem)

1^{st} **Idea: Consider the Lawden's Problem**

Advantages :

- solution easy to find (using AMPL+IpOpt)
- gives the same results that Lawden describes

Cons :

- not the same variables as our problem
- it doesn't give all the state and adjoint state

We need to find a good static problem to find good initializations

2^{nd} Idea : New Optimal transfer problem

let $L_f > 0$ free $T_{max} > 0$,
find $\alpha \in L^\infty([L_0, L_f] \to [0, T_{max}])$, $\phi \in L^\infty(\mathbb{R} \to \mathbb{R})$ tel que:

$$\min_{\alpha,\phi} -m(L_f) = -m_0 + \beta \int_{L_0}^{L_f} \sqrt{\frac{P}{\mu}} \frac{P}{W^2} \alpha$$

$$P' = \frac{1}{\mu m} \frac{2P^3}{W^3} \alpha \sin(\phi)$$

$$e_x' = \frac{1}{\mu m} \frac{P^2}{W^2} \alpha \Big(\sin(L) \cos(\phi) + \Big(\cos(L) + \frac{\cos(L) + e_x}{W} \Big) \sin(\phi) \Big)$$

$$e_y' = \frac{1}{\mu m} \frac{P^2}{W^2} \alpha \Big(-\cos(L) \cos(\phi) + \Big(\sin(L) + \frac{\sin(L) + e_y}{W} \Big) \sin(\phi) \Big)$$

$$m' = -\beta \sqrt{\frac{P}{\mu}} \frac{P}{W^2} \alpha$$

$x(0) = (P_0, e_{0x}, e_{0y}, m_0)$, $\quad x(L_f) = (P_1, e_{1x}, e_{1y}, -)$
where $P' = \frac{dP}{dL}$

Computing the associated static problem we have the following difficulty :

- all $(\tilde{P}, \tilde{e_x}, \tilde{e_y}, \tilde{m})$ with $\alpha = 0$ and $\phi \in L^\infty(\mathbb{R}, \mathbb{R})$ are solutions.
- we lose the choice of the possibly best transfert orbit

Idea: consider the previous optimal control problem with a control $\alpha = \alpha_1 \delta_{L_1} + \alpha_2 \delta_{L_2}$ with $\alpha_1, \alpha_2 \in L^\infty([L_0, L_f], [0, T_{max}])$

Let us write the **optimal control static problem :**

$$\max_{\alpha_1,\alpha_2,\phi_1,\phi_2,L_1,L_2,P,e_x,e_y,m_2} m_1$$

$$P - P_0 = \frac{1}{\mu m_0}\frac{2P_0^3}{W_0^3}\alpha_1 \sin(\phi_1)$$

$$e_x - e_{0x} = \frac{1}{\mu m_0}\frac{P_1^2}{W_0^2}\Big(\sin(L_1)\alpha_1\cos(\phi_1) + \Big(\cos(L_1) + \frac{\cos(L_1)+e_{0x}}{W_0}\Big)\alpha_1\sin(\phi_1)\Big)$$

$$e_y - e_{0y} = \frac{1}{\mu m_0}\frac{P_0^2}{W_0^2}\Big(-\cos(L_1)\alpha_1\cos(\phi_1) + \Big(\sin(L_1) + \frac{\sin(L_1)+e_{0y}}{W_0}\Big)\alpha_1\sin(\phi_1)\Big)$$

$$m - m_0 = -\beta\sqrt{\frac{P_0}{\mu}}\frac{P_0}{W_0^2}\alpha_1$$

$$P_1 - P = \frac{1}{\mu m_1}\frac{2P_1^3}{W_1^3}\alpha_2 \sin(\phi_2)$$

$$e_{1x} - e_x = \frac{1}{\mu m_1}\frac{P_1^2}{W_1^2}\Big(\sin(L_2)\alpha_2\cos(\phi_2) + \Big(\cos(L_2) + \frac{\cos(L_2)+e_{1x}}{W_1}\Big)\alpha_2\sin(\phi_2)\Big)$$

$$e_{1y} - e_y = \frac{1}{\mu m_1}\frac{P_1^2}{W_1^2}\Big(-\cos(L_2)\alpha_2\cos(\phi_2) + \Big(\sin(L_2) + \frac{\sin(L_2)+e_{1y}}{W_1}\Big)\alpha_2\sin(\phi_2)\Big)$$

$$m_1 - m = -\beta\sqrt{\frac{P_1}{\mu}}\frac{P_1}{W_1^2}\alpha_2$$

From this problem we can have (by resolution with AMPL+IpOpt):

- the intermediate state (P, e_x, e_y, m)
- the final mass m_1
- the longitude of the two impulses L_1 and L_2
- the intermediate state adjoint (AMPL can give us the Lagrange multipliers for the 8 equations)

So this time : $x = (P, e_x, e_y, m)$ and $\lambda = (\lambda_P, \lambda_{e_x}, \lambda_{e_y}, \lambda_m)$
By application of Pontryagin's principle we get :

- α and ϕ
- the extremal dynamic system
- transversality conditions

Always denote $z = \begin{pmatrix} x \\ \lambda \end{pmatrix}$ We can now write the new shooting method:
Unknown variables are :

$$X = \Big(P(L_f/2), e_x(L_f/2), e_y(L_f/2), m(L_f/2)...$$
$$\lambda_P(L_f/2), \lambda_{e_x}(L_f/2), \lambda_{e_y}(L_f/2), \lambda_m(L_f/2)...$$
$$\mathrm{L}_0, L_1, L_2, L_f \Big)$$

Principle of the new shooting method:

1. we choose initialization for X
2. $z(L_0) \underbrace{\leftarrow}_{\alpha=T_{max}} z(L_1) \underbrace{\leftarrow}_{\alpha=0} z(\frac{L_f}{2}) \underbrace{\rightarrow}_{\alpha=0} z(L_2) \underbrace{\rightarrow}_{\alpha=T_{max}} z(L_f)$
3. we compute the function whose we want a zero :

$$F(X) = \begin{pmatrix} P(L_0) - P_0 \\ e_x(L_0) - e_{0x} \\ e_y(L_0) - e_{0y} \\ m(L_0) - m_0 \\ P(L_f) - P_f \\ e_x(L_f) - e_{1x} \\ e_y(L_f) - e_{1y} \\ \lambda_m(L_f) - 1 \\ \Phi(L_1) \\ \Phi(L_2) \\ H(L_0) \\ H(L_f) \end{pmatrix}$$

4. we search a zero of F

THANK YOU FOR YOUR ATTENTION