PRIMAL-DUAL ALGORITHM IN SEMI-IMPLICIT SCHEME AND APPLICATIONS TO NON CONVEX VARIATIONAL PROBLEMS

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VII PDEs, optimal design and numerics

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Motivation

OUR GOAL is to find a numerical algorithm for problem

$$(\mathcal{P}_h) \qquad \inf_{u \in C} \int_\Omega [\varphi(\nabla u) + h(u)] dx$$

where

- Ω is an open bounded subset of $\mathbb{R}^N$,
- $C = \left\{ u \in W^{1,p}(\Omega), u = u_0 \text{ on } \partial\Omega \right\}$, $(p = 1, 2)$,
- φ is a convex, continuous, non differentiable function,
- h is a non convex, non differentiable function.

Example:

- Multiphase problem

$$\inf\left\{\int_\Omega |\nabla u|dx + \lambda\Big|\{u \neq \frac{1}{2}\}\Big| \ : \ u = 0 \text{ on } \Gamma_0, \ u = 1 \text{ on } \partial\Omega \setminus \Gamma_0\right\}$$

$$\varphi(z) = |z|, \qquad h(t) = \begin{cases} 0 & \text{if } \ t = \frac{1}{2} \\ \lambda & \text{if } \ t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ +\infty & \text{otherwise .} \end{cases}$$

- Free boundary problem

$$\inf\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda\Big|\{u > 0\}\Big| : u \in W^{1,2}(\Omega), \ u = 1 \text{ on } \partial\Omega\right\}$$

$$\varphi(z) = \frac{|z|^2}{2}, \qquad h(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \lambda & \text{if } 0 < t \leq 1 \\ +\infty & \text{otherwise .} \end{cases}$$

Min-max formulation

By duality argument, we obtain

$$\inf_{u\in C}\int_\Omega[\varphi(\nabla u)+h(u)]dx = \inf_{u\in C}\sup_{p\in K} L(u,p),$$

with

$$L(u,p)=\int_\Omega[\nabla u\cdot p + h(u)-\varphi^*(p)]dx$$

and $K = L^{p'}(\Omega;\mathbb{R}^N)$.

Aim: To find a saddle point $(\hat{u},\hat{p})$ of $L(u,p)$ in $C\times K$.

We recall that a saddle point $(\hat{u},\hat{p})$ of $L(u,p)$ in $C\times K$ is characterized by the inequalities

$$L(\hat{u},p)\le L(\hat{u},\hat{p})\le L(u,\hat{p}),\ \forall u\in C,\ \forall p\in K.$$

General min-max problem

$$\min_{u\in C}\max_{p\in K} L(u,p) \quad \text{with} \quad L(u,p) = \langle Au, p\rangle + F(u) - G(p),$$

- C and K be closed convex non-empty subsets of Hilbert spaces V and W, respectively,
- $A : V \to W$ a continuous linear operator,
- F, G are convex functions supposed to be differentiable and their derivatives satisfy the Lipschitz condition with constants $L_{F'}$, $L_{G'}$, respectively.

Arrow-Hurwicz Method

$$\inf_{u \in C} \sup_{p \in K} L(u, p)$$

The simplest algorithm [Arrow-Hurwicz] has the form

$$p_{n+1} = \mathrm{Proj}_K \left(p_n + \tau_n \frac{\partial L}{\partial p}(u_n, p_n) \right)$$
$$u_{n+1} = \mathrm{Proj}_C \left(u_n - \tau_n \frac{\partial L}{\partial u}(u_n, p_n) \right)$$

which converges under stringent conditions (like strict convexity-concavity) and special choosing of stepsizes $\tau_n \to 0, \sum_{n=0}^{\infty} \tau_n = \infty$ (cf. M. Kallio and A. Ruszczynski).

An improved version of Arrow-Hurwicz method

$$\inf_{u \in C} \sup_{p \in K} L(u, p)$$

L. D. Popov modified the Arrow-Hurwicz method by introducing "leading" point $(\overline{u}_n, \overline{p}_n)$ with constant stepsize $0 < \tau < \tau_0$,

$$p_{n+1} = \mathrm{Proj}_K \left(p_n + \tau \frac{\partial L}{\partial p}(\overline{u}_n, \overline{p}_n) \right)$$

$$u_{n+1} = \mathrm{Proj}_C \left(u_n - \tau \frac{\partial L}{\partial u}(\overline{u}_n, \overline{p}_n) \right)$$

$$\overline{p}_{n+1} = \mathrm{Proj}_K \left(p_{n+1} + \tau \frac{\partial L}{\partial p}(\overline{u}_n, \overline{p}_n) \right)$$

$$\overline{u}_{n+1} = \mathrm{Proj}_C \left(u_{n+1} - \tau \frac{\partial L}{\partial u}(\overline{u}_n, \overline{p}_n) \right).$$

References

- L. D. Popov, *A modification of the Arrow-Hurwicz method for search of saddle points*, , Mathematical notes of the Academy of Sciences of the USSR, 1980, **28**, 5, 845–848.

Chambolle-Pock et al. dealt with a typical Lagrangian which is linear

$$\inf_{u\in C}\sup_{p\in K}\langle Au,p\rangle+\langle f,u\rangle-\langle g,p\rangle.$$

And it is proved that the iterative process

$$\begin{cases} p_{n+1}=\mathrm{Proj}_K(p_n+\alpha(A\overline{u}_n-g)) \\ u_{n+1}=\mathrm{Proj}_C(u_n-\beta(A^*p_{n+1}+f)) \\ \overline{u}_{n+1}=2u_{n+1}-u_n \end{cases}$$

converges with $\alpha,\beta>0$, $\alpha\beta\|A\|^2<1$. Here, A^* denotes the adjoint of operator A. The steps α,β can be varied to accelerate the convergence (i.e. replaced by α_n,β_n).

References

- T. Pock, D. Cremers, H. Bischof, and A. Chambolle, *An algorithm for minimizing the Mumford-Shah functional*, 2009 IEEE 12th International Conference on Computer Vision, 1133-1140.

General explicit scheme

$$\inf_{u\in C}\sup_{p\in K}\langle Au,p\rangle + F(u) - G(p).$$

We propose the following algorithm:
Initialization: $\alpha,\beta>0$, $(u_0,p_0)\in C\times K$, $\overline{u}_0=u_0$.

$$(E)\qquad \begin{cases} p_{n+1} = \mathrm{Proj}_K(p_n+\alpha(A\overline{u}_n - G'(p_n))) \\ u_{n+1} = \mathrm{Proj}_C(u_n-\beta(A^*p_{n+1}+F'(u_n))) \\ \overline{u}_{n+1} = 2u_{n+1}-u_n \end{cases}$$

Theorem The iterative process (E) converges to a saddle point of $L(u,p)$ if parameters α,β are chosen such that

$$0<\alpha<\frac{1}{L_{G'}},\qquad 0<\beta<\frac{1}{L_{F'}},$$
$$\alpha\beta(\|A\|^2 - L_{F'}L_{G'}) + \alpha L_{G'} + \beta L_{F'} < 1.$$

Comments

- If $F=G=0$ then the problem reduces to $\inf_{u\in C}\sup_{p\in K}\langle Au,p\rangle$.
- What happens if F and G are not differentiable ?

Useful trick for non differentiability

By exploiting the notion of epigraph, we can establish

$$\inf_{u\in C}\sup_{p\in K}\langle Au,p\rangle + F(u) - G(p) = \inf_{\substack{u\in C\\ F(u)\le a}}\ \sup_{\substack{p\in K\\ G(p)\le\zeta}}\ \langle Au,p\rangle + a - \zeta$$

$$= \inf_{\widetilde{u}\in\widetilde{C}}\sup_{\widetilde{p}\in\widetilde{K}}\langle\langle\widetilde{A}\widetilde{u},\widetilde{p}\rangle\rangle,$$

with following notations in the spaces $V\times\mathbb{R}^2$, $W\times\mathbb{R}^2$:

$\widetilde{u} := (u,(a,b))$, $\widetilde{p} := (p,(\zeta,\xi))$, $\widetilde{A}\widetilde{u} := (Au,(-b,a))$, $\left(\widetilde{A}\right)^*\widetilde{p} = (A^*p,(\xi,-\zeta))$,

$\langle\langle\widetilde{A}\widetilde{u},\widetilde{p}\rangle\rangle := \langle Au,p\rangle - b\zeta + a\xi$,

$\widetilde{C} := \{(u,(a,1))\ :\ u\in C, F(u)\le a\}$, $\widetilde{K} := \{(p,(\zeta,1))\ :\ p\in K, G(p)\le\zeta\}$.

The corresponding algorithm is given by

$$\begin{cases}(p_{n+1},\zeta_{n+1}) = \mathrm{Proj}_{\widetilde{K}}((p_n,\zeta_n)+\alpha(A\overline{u}_n,-1))\\ (u_{n+1},a_{n+1}) = \mathrm{Proj}_{\widetilde{C}}((u_n,a_n)-\beta(A^*p_{n+1},1))\\ \overline{u}_{n+1} = 2u_{n+1}-u_n.\end{cases}$$

Application to 1st example

$$(\mathcal{P}_h) \qquad \inf_{u=u_0 \text{ on } \partial\Omega} \int_\Omega [|\nabla u| + h(u)]dx$$

Theorem [BP] Assume that $u_0 \notin Z := \{h^{**} < h\}$. Then:

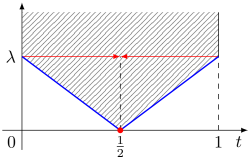
- $\inf(\mathcal{P}_h) = \inf(\mathcal{P}_{h^{**}})$,
- u solves $(\mathcal{P}_h)$ iff u solves $(\mathcal{P}_{h^{**}})$ and $u(x) \notin Z$ for a.e. $x \in \Omega$.

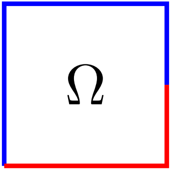
We apply to the 3-phase free boundary problem

$$\inf \left\{ \int_\Omega |\nabla u| dx + \lambda \left| \{u \neq \tfrac{1}{2}\} \right| \;:\; u = 0 \text{ on } \Gamma_0,\ u = 1 \text{ on } \Gamma_1 \right\}$$

where $\Omega = (0,1)^2$, $\Gamma_1 = [0,1] \times \{0\} \cup \{1\} \times [0, \frac{1}{2}]$ and $\Gamma_0 = \partial\Omega \setminus \Gamma_1$.

$$h(t) = \begin{cases} 0 & \text{if } t = \frac{1}{2} \\ \lambda & \text{if } t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ +\infty & \text{otherwise}. \end{cases}$$





Then, the primal solution $\overline{u}$ does not take values in Z i.e. $\overline{u}(x) \in \{0, \frac{1}{2}, 1\}$ (**3 phases**).

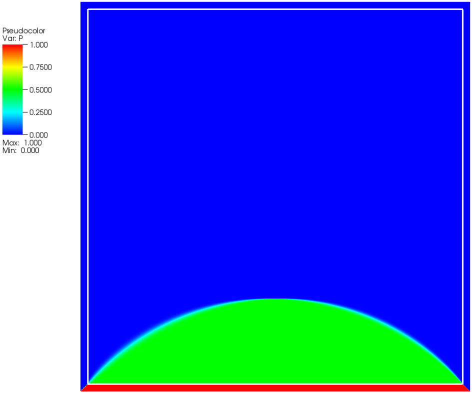
Strategy: We treat the convex problem $(\mathcal{P}_{h^{**}})$ by using the previous min-max algorithm.

Difficulty:

- h^{**} is piecewise affine, not differentiable (need the projection on epigraph of h^{**}),
- To recovery 3-phase solution to $(\mathcal{P}_h)$, we need the uniqueness of solution for $(\mathcal{P}_h^{**})$(no strict convexity)
  (We assume it).

Optimal partitions

The optimal solutions are computed for $\Omega = (0,1)^2$ by using a min-max algorithm for $\mathcal{P}_{h^{**}}$ with $\lambda = \frac{3}{4}$
(h^{**} is not differentiable at $\{0, \frac{1}{2}, 1\}$)



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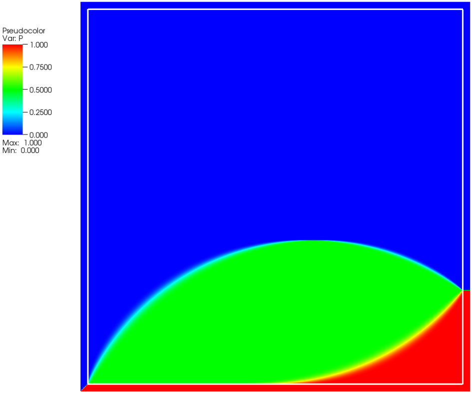

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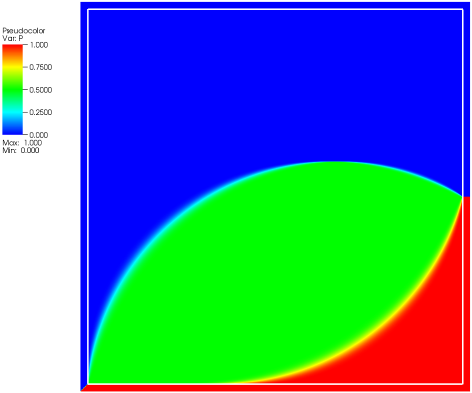

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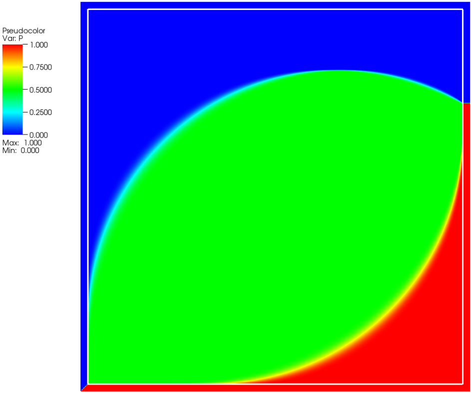

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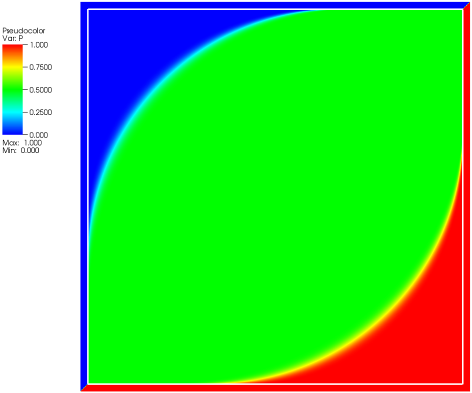


Application to 2nd example

Free boundary problem

$$\inf\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda\Big|\{u>0\}\Big| : u \in W^{1,2}(\Omega),\ u = 1 \text{ on } \partial\Omega\right\}$$

Remark

- We cannot use the previous convexification since Theorem [BP] only works in case $\varphi = |\cdot|$.
- We need another convexification recipe which uses a representation in higher dimension [G. Bouchitte, I. Fragala 2016].

Primal-dual formulation

Convex representation in higher dimension

References

[1] G. Alberti, G. Bouchitté, and G. Dal Maso, *The calibration method for the Mumford-Shah functional and free-discontinuity problems*, Calc. Var. Partial Differential Equations **16** (2003), 299-333.

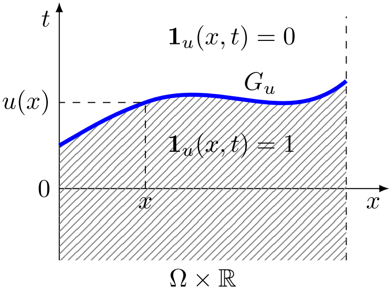
[2] G. Bouchitté, I. Fragalà, *Duality for non-convex variational problems*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 4, 375-379.

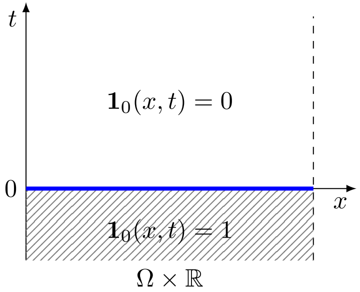
[3] G. Bouchitté, I. Fragalà, *A duality theory for non-convex variational problems*, submitted (arxiv 2016).

Primal-dual formulation

Convex representation in higher dimension

Subgraph functions





$$\mathbf{1}_u(x,t) = \begin{cases} 1 & \text{if } t \leq u(x) \\ 0 & \text{if } t > u(x) \end{cases} \qquad \mathbf{1}_0(x,t) = \begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t > 0 \end{cases}$$

Primal-dual formulation

Convex representation in higher dimension

Theorem $\qquad \inf \mathcal{P}_h = \sup \mathcal{P}_h^* = \inf \mathcal{M}_h$

$$(\mathcal{P}_h) \quad \inf_{u \in C} \int_\Omega [\varphi(\nabla u) + h(u)] dx$$

$$(\mathcal{P}_h^*) \quad \sup \left\{ \int_{G_{u_0}} \sigma \cdot \nu_{u_0} \, d\mathcal{H}^N : \sigma \in \mathcal{K}, \operatorname{div} \sigma = 0 \text{ in } \Omega \times \mathbb{R} \right\}$$

$$(\mathcal{M}_h) \quad \inf \left\{ \widehat{E}(v) : v \in \widehat{C} \right\}$$

where $\widehat{E}$, $\widehat{C}$ are defined later and:

$$\mathcal{K} = \Big\{ \sigma = (\sigma^x, \sigma^t) \in X_1(\Omega \times \mathbb{R}) : \sigma^t(x,t) + h(t) \geq 0, \forall t \in D, \text{ a.e. } x;$$
$$\sigma^t(x,t) + h(t) \geq \varphi^*(\sigma^x(x,t)) \text{ for a.e. } (x,t) \Big\},$$

$$D = \{ t \in \mathbb{R} : h(t) \text{ is discontinuous at } t \}.$$

Convexified problem on $BV_\infty(\Omega \times \mathbb{R}; [0,1])$

- The constraint is:

$$\widehat{C} = \left\{ v \in BV_\infty(\Omega \times \mathbb{R}) : v - \mathbf{1}_0 \in L^1(\Omega \times \mathbb{R}), v = \mathbf{1}_{u_0} \text{ on } \partial\Omega \times \mathbb{R} \right\},$$

- The convex energy is given by

$$\widehat{E}(v) = \sup \left\{ \int_{\Omega \times \mathbb{R}} \sigma \cdot Dv : \sigma \in \mathcal{K} \right\} = \int_{\Omega \times \mathbb{R}} \widetilde{h}(t, Dv),$$

$$\text{with} \quad \widetilde{h}(t, z^x, z^t) = \begin{cases} -z^t[\varphi(\frac{-z^x}{z^t}) + h(t)] & \text{if } z^t < 0, \\ +\infty & \text{otherwise.} \end{cases}$$

It satisfies :

$$\widehat{E}(\mathbf{1}_u) = \int_\Omega [\varphi(\nabla u) + h(u)]dx$$

so that we can rewrite $(\mathcal{P}_h)$ as

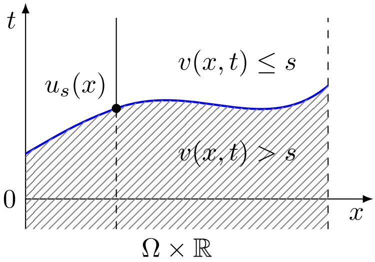
$$\inf\{\widehat{E}(\mathbf{1}_u) : \mathbf{1}_u \in \widehat{C}\} \qquad (\text{ thus } \quad \inf \mathcal{P}_h \geq \inf \mathcal{M}_h)$$

Link with the initial problem in $\mathbb{R}^N$

Theorem It holds:

- if u minimizes $(\mathcal{P}_h)$ then $\mathbf{1}_u$ minimize $(\mathcal{M}_h)$.
- if v solves $(\mathcal{M}_h)$ then $u_s(x) := \inf\{t \in \mathbb{R} : v(x,t) \le s\}$ solves $(\mathcal{P}_h)$ for a.e. $s \in (0,1)$.
- If $(\mathcal{P}_h)$ admits a finite number of solutions $\{u_1, ..., u_k\}$ then

$$\mathrm{argmin}_{\widehat{C}}(\widehat{E}) = \sum_{i=1}^{k} \theta_i \mathbf{1}_{u_i}, \quad \theta_i \in [0,1], \quad \sum_{i=1}^{k} \theta_i = 1.$$



Primal-dual formulation

Convex representation in higher dimension

In summary, the convex representation in higher dimension

$$(\mathcal{P}_h) \quad \inf_{u\in C} \int_\Omega [\varphi(\nabla u) + h(u)]dx = \inf \left\{ \widehat{E}(v) : v \in \widehat{C} \right\} \quad (\mathcal{M}_h)$$

$$= \inf_{v\in\widehat{C}} \sup_{\sigma\in\mathcal{K}} \int_{\Omega\times\mathbb{R}} \sigma \cdot Dv$$

allows to recover solutions of the primal problem $(\mathcal{P}_h)$ by a closed chain relation

v minimizes $(\mathcal{M}_h) \Longrightarrow u_s$ minimizes $(\mathcal{P}_h) \Longrightarrow \mathbf{1}_{u_s}$ minimizes $(\mathcal{M}_h)$.

Hence, we are interested in the following model for numerical computation:

$$\boxed{\inf_{u\in C} \sup_{p\in K} \langle Au, p\rangle}$$

where A stands for gradient operator ∇.

Semi-implicit scheme

Assume that A^*A is positive definite. We propose a semi-implicit scheme as following

$$(I) \qquad \begin{cases} p_{n+1} = \mathrm{Proj}_K(p_n + \alpha(A\overline{u}_n)) \\ u_{n+1} = \mathrm{Proj}_C(u_n - \beta(A^*A)^{-1}(A^*p_{n+1})) \\ \overline{u}_{n+1} = 2u_{n+1} - u_n \end{cases}$$

Remark

- We just replaced $u_{n+1} = \mathrm{Proj}_C(u_n - \beta(A^*p_{n+1}))$ in explicit scheme by $u_{n+1} = \mathrm{Proj}_C(u_n - \beta(A^*A)^{-1}(A^*p_{n+1}))$
- When A stands for ∇, $A^*A = -\Delta$ and the replacement is equivalent to solve

$$-\Delta w = q, \quad w = 0 \text{ on } \partial\Omega.$$

Theorem The algorithm (I) converges to a saddle point of $L(u,p)$ if the positive stepsizes α, β are chosen such that

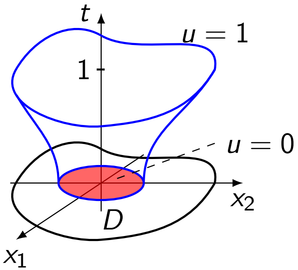
$$\alpha\beta < 1.$$

2nd Example

$$\inf\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda\big|\{u>0\}\big| : u \in W^{1,2}(\Omega),\ u=1 \text{ on } \partial\Omega\right\}.$$

$\rightsquigarrow$ Free boundary Pb in term of $D=\{u>0\}$,
u solves

$$\begin{cases} -\Delta u_D = 0 \text{ in } D \\ u_D = 1 \text{ on } \partial\Omega \\ u_D = 0 \text{ in } \Omega\backslash D. \end{cases}$$



$\rightsquigarrow$ Shape functional

$$J : D \to \lambda|D| + \frac{1}{2}\int_\Omega |\nabla u_D|^2.$$

- In 2-dimension, we take $\Omega=(0,1)^2$ for example.

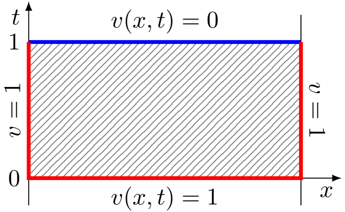
The convex representation in higher dimension gives

$$\inf_{u=1 \text{ on } \partial\Omega} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda|\{u>0\}| = \inf_{v\in\widehat{\mathcal{C}}} \sup_{\sigma\in\mathcal{K}} \int_{\Omega\times[0,1]} \sigma\cdot Dv$$

$$\widehat{\mathcal{C}} = \Big\{ v(x,t)\in BV_\infty(\Omega\times\mathbb{R}) : v=1 \text{ for } t<0,$$
$$v=0 \text{ for } t>1,\ v=1 \text{ on } \partial\Omega\times[0,1] \Big\}$$

$$\mathcal{K} = \Big\{ \sigma = (\sigma^x,\sigma^t)\in X_1(\Omega\times(0,1)) :$$
$$\sigma^t+\lambda \geq \frac{1}{2}|\sigma^x|^2 \text{ a.e. on } \Omega\times(0,1),\ \sigma^t(x,0)\geq 0 \text{ a.e. on } \Omega \Big\}$$

$\Omega\times[0,1] \quad \rightsquigarrow$



Discrete settings in explicit and implicit schemes

$$\inf_{u=1 \text{ on } \partial\Omega} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \lambda |\{u>0\}| = \inf_{v\in\mathcal{C}} \sup_{\sigma\in\mathcal{K}} \int_{\Omega\times[0,1]} \sigma \cdot Dv \,.$$

Explicit:
$$\begin{cases} \sigma^h_{n+1} = \mathrm{Proj}^h_K(\sigma^h_n + \alpha\nabla^h \overline{v}^h_n) \\ v^h_{n+1} = v^h_n + \beta\,\mathrm{div}^h(\sigma^h_{n+1}) \\ \overline{v}^h_{n+1} = 2v^h_{n+1} - v^h_n, \end{cases}$$

where α, β is chosen such that $\alpha\beta c_h^2 < 1$ with $c_h := ||\nabla^h|| = 2\sqrt{N+1}/h$ and h being the mesh size.

Semi-implicit:
$$\begin{cases} \sigma^h_{n+1} = \mathrm{Proj}^h_K(\sigma^h_n + \alpha\nabla^h \overline{v}^h_n) \\ v^h_{n+1} = v^h_n - \beta(\Delta^h)^{-1}(\mathrm{div}^h(\sigma^h_{n+1})) \\ \overline{v}^h_{n+1} = 2v^h_{n+1} - v^h_n, \end{cases}$$

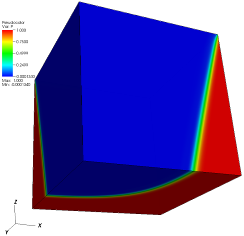
where α, β is chosen such that $\alpha\beta < 1$. The implicitness is realized by solving equation

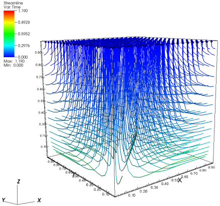
$$-\Delta w = q, \quad w = 0 \text{ on } \partial\Omega.$$

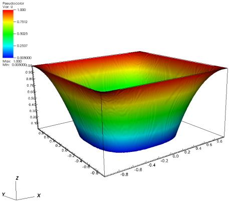
Numerical simulation

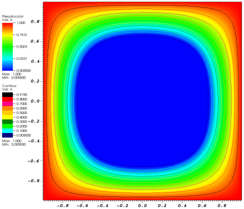
$$\inf_{u=1 \text{ on } \partial\Omega} \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx + \lambda |\{u > 0\}| = \inf_{v\in\mathcal{C}} \sup_{\sigma\in\mathcal{K}} \int_{\Omega\times[0,1]} \sigma \cdot Dv \ .$$

$$u_s(x) := \inf\{t \in \mathbb{R} : v(x,t) \leq s\}$$



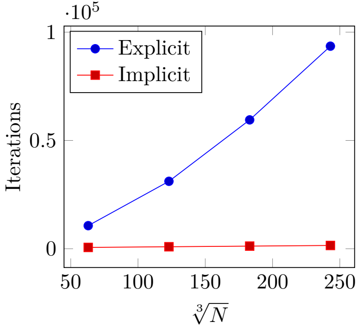


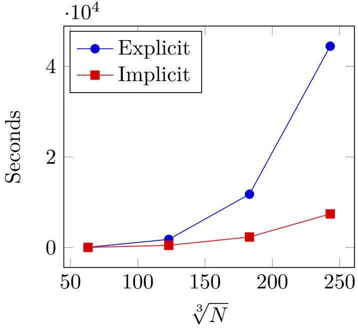




Explicit-implicit computational comparison

$$(E)\begin{cases}\sigma^h_{n+1} = \mathrm{Proj}^h_K(\sigma^h_n + \alpha\nabla^h \overline{v}^h_n)\\ v^h_{n+1} = v^h_n + \beta\,\mathrm{div}^h(\sigma^h_{n+1})\\ \overline{v}^h_{n+1} = 2v^h_{n+1} - v^h_n,\end{cases} \qquad (I)\begin{cases}\sigma^h_{n+1} = \mathrm{Proj}^h_K(\sigma^h_n + \alpha\nabla^h \overline{v}^h_n)\\ v^h_{n+1} = v^h_n - \beta(\Delta^h)^{-1}(\mathrm{div}^h(\sigma^h_{n+1}))\\ \overline{v}^h_{n+1} = 2v^h_{n+1} - v^h_n,\end{cases}$$





- The solver used for the inverse Laplacian operator is AGMGPAR (A parallel version of Algebraic Multigrid method).
- Ours algorithms are implemented with MPI (Massage Passing Interface) providing an effective environment for parallel computation.

References

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[3] I. Dupays, M. Flé, J. Gaidamour, and D. Lecas, *Message Passing Interface (MPI)*, IDRIS (2015).

Thank You for Your Attention !