Homogenization and corrector for the wave equation

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The wave problem.

For a positive number T, we analyze the asymptotic behavior for the problem

$$\begin{cases} \partial_t(\rho_\epsilon \partial_t u_\epsilon) - \mathrm{div}_x(A_\epsilon \nabla_x u_\epsilon) = f_\epsilon \text{ en } Q_T \\ u_{\epsilon|t=0} = u_{0,\epsilon}, \ (\rho_\epsilon \partial_t u_\epsilon)_{|t=0} = \vartheta_\epsilon \ \text{ in } \Omega \\ u_\epsilon \in L^\infty(0,T;H_0^1(\Omega)), \ \partial_t u_\epsilon \in L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{1}$$

where $Q_T = (0,T) \times \Omega$ and ε is a parameter which tends to zero.

Aim of the work

- Computation of an approximation of u_ε in the form

$$u_\epsilon(t,x) \sim u_0(t,x) + \epsilon u_1(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon}) + \epsilon^2 u_2(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon}) + \cdots$$

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- $u_0(t,x) + \epsilon u_1(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon})$ provides of a corrector (approximation in the strong topology) for u_ε,

- Existence of weak solutions. Counter-examples:
  - A_ε constant in the two side of a hyperplane, non-parallel to $\{t = 0\}$ (**[Hurd-Sattinger, 1968]**).
  - $A_\varepsilon \in C^{0,\alpha}(\bar{\Omega} \times [0, T])$, for every $\alpha \in (0, 1)$ (**[Colombini-Spagnolo, 1989]**).
- Homogenization of oscillations in the time variable (**[Colombini-Spagnolo, 1978]**).
- Corrector, well-posed character of the initial data (**[BrahaimOsmane-Francfort-Murat, 1992]**).

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We assume:

- ρ_ε bounded in $BV(0, T; L^\infty(\Omega))$ with $\rho_\varepsilon \geq \alpha > 0$,
- A_ε bounded in $BV(0, T; L^\infty(\Omega; \mathcal{M}_N^H))$, with $A_\varepsilon \xi \cdot \xi \geq \alpha |\xi|^2, \forall \xi \in \mathbb{R}^N$,
- $u_{0,\varepsilon}$ and ϑ_ε are bounded in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively,
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From [CCMM, 2011][1] there exists a unique solution of (5) such that

$$u_\varepsilon \in L^\infty(0, T; H_0^1(\Omega)), \ \partial_t u_\varepsilon \in L^\infty(0, T; L^2(\Omega))$$

and satisfying an energy estimation.

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Theorem (Homogenization)

The unique solution of (5) satisfies

$$\begin{aligned} u_\epsilon &\overset{\star}{\rightharpoonup} u && \textit{in } L^\infty(0,T,H_0^1(\Omega)),\\ \partial_t u_\epsilon &\overset{\star}{\rightharpoonup} \partial_t u && \textit{in } L^\infty(0,T,L^2(\Omega)),\end{aligned}$$

where u is the unique solution of

$$\begin{cases} \partial_t(\rho\partial_t u) - div_x(A\nabla_x u_\epsilon) = f \text{ en } Q_T \\ u_{|t=0} = u_0,\ (\rho\partial_t u)_{|t=0} = \vartheta \ \text{ in } \Omega \\ u \in L^\infty(0,T;H_0^1(\Omega)),\ \partial_t u \in L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{2}$$

where

$$\begin{aligned} &\rho_\varepsilon \overset{\star}{\rightharpoonup} \rho \text{ in } L^\infty((0,T)\times\Omega)\\ &A_\varepsilon(t,\cdot) \overset{H}{\rightharpoonup} A(t,\cdot), \forall t \in (0,T)\setminus\mathcal{N} \end{aligned}$$

Definition (Two-scale convergence)

We say that a sequence $v_\varepsilon \in L^2(Q_T)$ *two-scale converges to a function* $v \in L^2(Q_T; L^2_\sharp(\mathbb{R} \times Y))$ *and we write* $v_\varepsilon \overset{2e}{\rightharpoonup} v$*, if for every* $\psi \in C^0_c(Q_T; C_\sharp(\mathbb{R} \times Y))$ *we have*

$$\lim_{\varepsilon \to 0} \int_{Q_T} v_\varepsilon \psi(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon})\, dtdx = \int_{Q_T} M_{s,y}(v\psi)\, dtdx. \qquad (3)$$

We say that $v_\varepsilon \in L^2(Q_T)$ *two-scale converges strongly to* $v \in L^2(Q_T; L^2_\sharp(\mathbb{R} \times Y))$ *and we write* $v_\varepsilon \overset{2e}{\rightarrow} v$ *if for every bounded sequence* z_ε *in* $L^2(Q_T)$*, which two-scale converges to a function* $z \in L^2(Q_T; L^2_\sharp(\mathbb{R} \times Y))$*, we have*

$$\lim_{\varepsilon \to 0} \int_{Q_T} v_\varepsilon z_\varepsilon\, dtdx = \int_{Q_T} M_{s,y}(vz)\, dtdx. \qquad (4)$$

Theorem (Compactness)

If v_ε is a bounded sequence in $L^2(Q_T)$, then there exists a subsequence of ε, still denoted by ε, and there exists $v \in L^2(Q_T; L^2_\sharp(\mathbb{R} \times Y))$ such that v_ε two-scales converges to v. Moreover, if v_ε is bounded in $H^1(Q_T)$, then, for a subsequence of ε, there exist $v_0 \in H^1(Q_T)$ and $v_1 \in L^2(Q_T; H^1_\sharp(\mathbb{R} \times Y))$, such that $$v_\varepsilon \rightharpoonup v_0 \ \text{ in } H^1(Q_T)$$

$$\nabla_{t,x} v_\varepsilon \overset{2e}{\rightharpoonup} \nabla_{t,x} v_0 + \nabla_{s,y} v_1 .$$

In our case

Proposition

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The wave problem.

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Assumptions

The problem data have the following structure:

$$\rho_\epsilon(t,x) = \rho^0(\frac{x}{\epsilon}) + \epsilon\rho^1(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon}), \quad A_\epsilon(t,x) = A^0(\frac{x}{\epsilon}) + \epsilon A^1(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon})$$

$$B_\epsilon(t,x) = B(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon}), \quad f_\epsilon(t,x) = f(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon})$$

$$u_{0,\epsilon}(x) = u^0(x) + \epsilon u^1(x,\frac{x}{\epsilon}), \quad \theta_\epsilon(x) = \theta(x,\frac{x}{\epsilon}).$$

Assuming that the functions are periodic in the unit cube $Y = (0,1)^N$ with respect to the spatial microscopic variable $y = x/\varepsilon$ and quasi-periodic in the time microscopic variable $s = t/\varepsilon$.

Assumptions

- $\rho^0 \in L^\infty_\sharp(Y)$, $A^0 \in L^\infty_\sharp(Y; \mathcal{M}_N^H)$ Hermitian and coercitive.
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- $B \in L^\infty(Q_T; C_\sharp(\mathbb{R} \times Y)^{N+1})$
- $f \in L^1(0, T; L^2(\mathbb{R}^N; C_\sharp(\mathbb{R} \times Y)))$.
- $u_{0,\varepsilon}$ and ϑ_ε bounded in $H^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ respectively.

Then, the two-scale compactness result provides $u^0 \in H^1(\mathbb{R}^N)$, $u^1 \in L^2(\mathbb{R}^N; H^1_\sharp(Y))$ and $\vartheta \in L^2(\mathbb{R}^N; L^2_\sharp(Y))$ such that, up to a subsequence, we have

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$$\partial_t \rho^1, \partial_s \rho^1 \in L^\infty(Q_T; C_\sharp(\mathbb{R}\times Y)),\ \partial_t A^1, \partial_s A^1 \in L^\infty(Q_T; C_\sharp(\mathbb{R}\times Y; \mathcal{M}^H_N)).$$

- $B \in L^\infty(Q_T; C_\sharp(\mathbb{R} \times Y)^{N+1})$
- $f \in L^1(0, T; L^2(\mathbb{R}^N; C_\sharp(\mathbb{R} \times Y)))$.
- $u_{0,\varepsilon}$ and ϑ_ε bounded in $H^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ respectively.

Then, the two-scale compactness result provides $u^0 \in H^1(\mathbb{R}^N)$, $u^1 \in L^2(\mathbb{R}^N; H^1_\sharp(Y))$ and $\vartheta \in L^2(\mathbb{R}^N; L^2_\sharp(Y))$ such that, up to a subsequence, we have

$$u_{0,\varepsilon} \rightharpoonup u^0 \text{ in } H^1(\mathbb{R}^N), \quad \nabla_x u_{0,\varepsilon} \overset{2e}{\rightharpoonup} \nabla_x u^0 + \nabla_y u^1, \quad \vartheta_\varepsilon \overset{2e}{\rightharpoonup} \vartheta. \quad (6)$$

Assumptions

- $\rho^0 \in L^\infty_\sharp(Y)$, $A^0 \in L^\infty_\sharp(Y; \mathcal{M}^H_N)$ Hermitian and coercive.
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Theorem (Part I)

We assume the above hypothesis, then the sequence u_ε satisfies

$$u_\varepsilon \overset{*}{\rightharpoonup} u_0 \ \text{ in } L^\infty(0,T;H^1(\mathbb{R}^N))$$

$$\nabla_{t,x} u_\varepsilon \overset{2e}{\rightharpoonup} \nabla_{t,x} u_0 + \nabla_{s,y} u_1,$$

where u_0, u_1 are the unique solutions of the variational system

$$u_0 \in L^\infty(0,T;H^1(\mathbb{R}^N)), \quad \partial_t u_0 \in L^\infty(0,T;L^2(\mathbb{R}^N)),$$

$$u_1 \in L^\infty(0,T;L^2(\mathbb{R}^N;H^1_\sharp(\mathbb{R}\times Y))), \quad M_{s,y}(u_1) = 0 \text{ a.e. in } Q_T,$$

$$u_{0|t=0} = u^0, \ (M_y(\rho^0)\partial_t u_0)_{|t=0} = M_y(\vartheta),$$

$$\left(\rho^0\left(\partial_t u_0 + \partial_s u_1\right)\right)_{|t=s=0} = \vartheta, \ \nabla_y u_{1|t=s=0} = \nabla_y u^1,$$

Theorem (Part II)

$$\begin{cases} \int_{Q_T} M_{s,y}\big(-\rho^0(\partial_t u_0 \partial_t v_0 + \partial_s u_1 \partial_s v_1) \\ \int_{Q_T} +A^0(\nabla_x u_0 + \nabla_y u_1) \cdot (\nabla_x v_0 + \nabla_y v_1)\big) dtdx \\ + \int_{Q_T} M_{s,y}\big(B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) v_0\big) dtdx \\ = \int_{Q_T} M_{s,y}\big(f v_0 + F^0 \cdot (\nabla_x v_0 + \nabla_y v_1)\big) dtdx \\ \forall\, v_0 \in W_0^{1,1}(0,T; L^2(\mathbb{R}^N)) \cap L^1(0,T; H^1(\mathbb{R}^N)), \\ \forall\, v_1 \in L^1(0,T; L^2(\mathbb{R}^N; H^1_\sharp(\mathbb{R} \times Y))) \end{cases} \tag{7}$$

Theorem (Part III)

$$\begin{cases} \displaystyle\int_{Q_T} M_{s,y}\Big(-2\rho^0 \partial_s u_1 \partial_t \psi + A^0 \nabla_y u_1 \cdot \nabla_x \psi - \mathrm{div}_x(\overline{A^0} \nabla_y \psi)\, u_1 \Big)\, dtdx \\ \displaystyle + \int_{Q_T} M_{s,y}\big(-\rho^1 (\partial_t u_0 + \partial_s u_1) \partial_s \psi + A^1 (\nabla_x u_0 + \nabla_y u_1) \cdot \nabla_y \psi \big)\, dtdx \\ \displaystyle + \int_{Q_T} M_{s,y}\big(B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) \psi \big)\, dtdx \\ \displaystyle = \int_{Q_T} M_{s,y}\big(f\psi + F^1 \cdot \nabla_y \psi \big)\, dtdx, \\ \forall \psi \in H^1(Q_T; \overline{\mathcal{W}}^1), \quad \psi_{|t=0} = \psi_{|t=T} = 0. \end{cases} \tag{8}$$

Formally,

$$\rho^0 \partial^2_{ss} u_1 - \mathrm{div}_y(A^0(\nabla_x u_0 + \nabla_y u_1)) = -\mathrm{div}_y F^0 \ \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ \text{a.e. } (t,x) \in (0,T) \times \mathbb{R}^N,$$

$$\rho^0 \partial^2_{tt} u_0 - \mathrm{div}_x(A^0 \nabla_x u_0) + 2\rho^0 \partial^2_{st} u_1 - \mathrm{div}_x(A^0 \nabla_y u_1) - \mathrm{div}_y(A^0 \nabla_x u_1) \\ + \partial_s\big(\rho^1(\partial_t u_0 + \partial_s u_1)\big) - \mathrm{div}_y\big(A^1(\nabla_x u_0 + \nabla_y u_1)\big) \\ + B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) + \rho^0 \partial^2_{ss} u_2 - \mathrm{div}_y(A^0 \nabla_y u_2) \\ = f - \mathrm{div}_x F^0 - \mathrm{div}_y F^1, \ \text{a.e. } (t,x) \in (0,T) \times \mathbb{R}^N.$$

$$u_\epsilon(t,x) \sim u_0(t,x) + \epsilon u_1(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon}) + \epsilon^2 u_2(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon}) + \cdots$$

u_1 non locally depends on u_0.

Formally,

$$\rho^0 \partial^2_{ss} u_1 - \mathrm{div}_y(A^0(\nabla_x u_0 + \nabla_y u_1)) = -\mathrm{div}_y\, F^0 \ \text{ in } \mathbb{R} \times \mathbb{R}^N,$$
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$$\begin{gathered}\rho^0 \partial^2_{tt} u_0 - \mathrm{div}_x(A^0 \nabla_x u_0) + 2\rho^0 \partial^2_{st} u_1 - \mathrm{div}_x(A^0 \nabla_y u_1) - \mathrm{div}_y(A^0 \nabla_x u_1) \\ + \partial_s\big(\rho^1(\partial_t u_0 + \partial_s u_1)\big) - \mathrm{div}_y\big(A^1(\nabla_x u_0 + \nabla_y u_1)\big) \\ + B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) + \rho^0 \partial^2_{ss} u_2 - \mathrm{div}_y(A^0 \nabla_y u_2) \\ = f - \mathrm{div}_x F^0 - \mathrm{div}_y F^1, \text{ a.e. } (t,x) \in (0,T) \times \mathbb{R}^N.\end{gathered}$$

$$u_\epsilon(t,x) \sim u_0(t,x) + \epsilon u_1\Big(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon}\Big) + \epsilon^2 u_2\Big(t,x,\frac{t}{\epsilon},\frac{x}{\epsilon}\Big) + \cdots$$

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$$\text{a.e. } (t, x) \in (0, T) \times \mathbb{R}^N,$$

$$\rho^0 \partial^2_{tt} u_0 - \mathrm{div}_x (A^0 \nabla_x u_0) + 2\rho^0 \partial^2_{st} u_1 - \mathrm{div}_x (A^0 \nabla_y u_1) - \mathrm{div}_y (A^0 \nabla_x u_1)$$
$$+ \partial_s \big(\rho^1 (\partial_t u_0 + \partial_s u_1)\big) - \mathrm{div}_y \big(A^1 (\nabla_x u_0 + \nabla_y u_1)\big)$$
$$+ B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) + \rho^0 \partial^2_{ss} u_2 - \mathrm{div}_y (A^0 \nabla_y u_2)$$
$$= f - \mathrm{div}_x F^0 - \mathrm{div}_y F^1, \ \text{a.e. } (t, x) \in (0, T) \times \mathbb{R}^N.$$

$$u_\epsilon (t, x) \sim u_0 (t, x) + \epsilon u_1 (t, x, \frac{t}{\epsilon}, \frac{x}{\epsilon}) + \epsilon^2 u_2 (t, x, \frac{t}{\epsilon}, \frac{x}{\epsilon}) + \cdots$$

u_1 non locally depends on u_0.

We would like to improve the convergence of the $\nabla_{t,x} u_\varepsilon$

Theorem

Assume that convergence for initial data holds in the two-scale strong convergence sense. Then, for every $\Gamma_\varepsilon \in L^2(Q_T)^{N+1}$ which two-scale converges strongly to $\nabla_{s,y} u_1$, we have

$$\lim_{\varepsilon \to 0} \int_{Q_T} |\nabla_{t,x}(u_\varepsilon - u_0) - \Gamma_\varepsilon|^2 \, dxdt = 0, \tag{9}$$

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Remark

If $\nabla_{s,y} u_1$ *belongs to* $L^2(Q_T; C^0_\sharp(\mathbb{R} \times Y)^{N+1})$, *we can take in (11)*

$$\Gamma_\varepsilon = \nabla_{s,y} u_1(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}).$$

Then Theorem asserts

$$\nabla_{t,x} u_\varepsilon - \nabla_{t,x} u_0 - \nabla_{s,y} u_1(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) \to 0 \quad \text{in } L^2(Q_T)^{N+1},$$

which assuming further regularity in u_1 *implies*

$$u_\varepsilon - u_0 - \varepsilon u_1(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) \to 0 \ \text{ in } H^1(Q_T).$$

We consider $I = (\alpha, \beta)$, $Q_T = (0, T) \times I$

$$\begin{cases} \partial_t(\rho_\epsilon \partial_t u_\epsilon) - \partial_x(a_\epsilon \partial_x u_\epsilon) + B_\epsilon \cdot \nabla_{t,x} u_\epsilon = f_\epsilon \text{ in } Q_T \\ (-c_\alpha a_\varepsilon \partial_x u_\varepsilon + d_\alpha u_e p)_{|x=\alpha} = 0 \text{ in } (0, T) \\ (c_\beta a_\varepsilon \partial_x u_\varepsilon + d_\beta u_e p)_{|x=\beta} = 0 \text{ in } (0, T) \\ u_{\epsilon|t=0} = u_{0,\epsilon}, \ (\rho_\epsilon \partial_t u_\epsilon)_{|t=0} = \vartheta_\epsilon \ \text{ in } \Omega \\ u_\epsilon \in L^\infty(0, T; H^1(I)), \ \partial_t u_\epsilon \in L^\infty(0, T; L^2(I)). \end{cases} \tag{10}$$

where $c_\alpha, d_\alpha, c_\beta, d_\beta$ are non-negative constants such that

$$c_\alpha + d_\alpha > 0, \quad c_\beta + d_\beta > 0.$$

Boundary conditions for u_0 and u_1???

We consider $I = (\alpha, \beta)$, $Q_T = (0, T) \times I$

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Boundary conditions for u_0 and u_1???

Theorem ($\Omega = I$, Part I)

For a subsequence of ε, still denoted by ε, u_ε satisfies

$$u_\varepsilon \overset{*}{\rightharpoonup} u_0 \ \text{ in } L^\infty(0,T;H^1(I))$$

$$\nabla_{t,x} u_\varepsilon \overset{2e}{\rightharpoonup} \nabla_{t,x} u_0 + \nabla_{s,y} u_1,$$

where u_0, u_1 are the unique solutions of the variational system

$$u_0 \in L^\infty(0,T;V), \quad \partial_t u_0 \in L^\infty(0,T;L^2(I))$$

$$u_1 \in L^\infty(0,T;L^2(I;H^1_\sharp(\mathbb{R}\times Y))), \quad M_{s,y}(u_1) = 0 \text{ a.e. in } Q_T$$

$$u_{0|t=0} = u^0, \ \partial_t u_{0|t=0} = \frac{M_y(\rho^0\vartheta)}{M_y(\rho^0)}$$

$$\big(\partial_t u_0 + \partial_s u_1\big)_{|t=s=0} = \vartheta, \ \partial_y u_{1\,|t=s=0} = \partial_y u^1$$

Theorem ($\Omega = I$, Part II)

$$\begin{cases} \displaystyle\int_{Q_T} M_{s,y}\Big(-\rho^0(\partial_t u_0 \partial_t v_0 + \partial_s u_1 \partial_s v_1) \\ +a^0(\partial_x u_0 + \partial_y u_1)(\partial_x v_0 + \partial_y v_1)\Big) dtdx \\ \displaystyle +\frac{d_\alpha}{c_\alpha}\int_{\{x=\alpha\}} M_y(a^0) u_0 v_0 \, dt + \frac{d_\beta}{c_\beta}\int_{\{x=\beta\}} M_y(a^0) u_0 v_0 \, dt \\ \displaystyle + \int_{Q_T} M_{s,y}\Big(B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) v_0\Big) dtdx = \int_{Q_T} M_{s,y}(f v_0) dtdx \\ \forall\, v_0 \in W_0^{1,1}(0,T; L^2(I)) \cap L^1(0,T;V), \quad \forall\, v_1 \in L^2(Q_T; H^1_\sharp(\mathbb{R}\times Y)) \end{cases}$$

Theorem ($\Omega = I$, Part III)

$$\begin{cases} \displaystyle\int_{Q_T} M_{s,y}\Big(-2\rho^0 \partial_s u_1 \partial_t \psi - \rho^1 (\partial_t u_0 + \partial_s u_1)\partial_s \psi \\ +a^1 (\partial_x u_0 + \partial_y u_1)\partial_y \psi\Big) dtdx \\ \displaystyle + \int_{Q_T} M_{s,y}\Big(B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1)\psi\Big) dtdx = \int_{Q_T} M_{s,y}(f\psi) dtdx \\ \forall \psi \in H_0^1(0, T; L^2(I; P_1(\mathcal{W}^1))) \end{cases}$$

Theorem ($\Omega = I$, Part IV)

$$\begin{cases} \displaystyle\int_{Q_T} M_{s,y}\big(-2\rho^0 \partial_s u_1 \partial_t \psi + a^0 \partial_y u_1 \partial_x \psi - a^0 \partial^2_{x,y} \psi\, u_1 \big)\, dtdx \\ \displaystyle + \int_{Q_T} M_{s,y}\big(-\rho^1 (\partial_t u_0 + \partial_s u_1) \partial_s \psi + a^1 (\partial_x u_0 + \partial_y u_1) \partial_y \psi\big)\, dtdx \\ \displaystyle + \int_{Q_T} M_{s,y}\big(B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1)\psi\big)\, dtdx = \int_{Q_T} M_{s,y}(f\psi) dtdx, \\ \forall \psi \in H^1(Q_T; P_2(\mathcal{W}^1)), \quad \psi_{|t=0} = \psi_{|t=T} = 0 \\ (a^0 \partial_y \psi)_{|x=\alpha, y=\bar{\alpha}} = 0 \ \ if\ c_\alpha \neq 0, \quad \psi_{|x=\alpha, y=\bar{\alpha}} = 0 \ \ if\ c_\alpha = 0 \\ (a^0 \partial_y \psi)_{|x=\beta, y=\bar{\beta}} = 0 \ \ if\ c_\beta \neq 0, \quad \psi_{|x=\beta, y=\bar{\beta}} = 0 \ \ if\ c_\beta = 0. \end{cases}$$

Boundary conditions

Proposition

In the conditions of the above Theorem ($\Omega = I$), one has

$$(a^0 \partial_y P_2 u_1)_{|x=\alpha, y=\bar{\alpha}} = 0 \;\; \textit{if } c_\alpha \neq 0, \quad P_2 u_1{}_{|x=\alpha, y=\bar{\alpha}} = 0 \;\; \textit{if } c_\alpha = 0,$$

$$(a^0 \partial_y P_2 u_1)_{|x=\beta, y=\bar{\beta}} = 0 \;\; \textit{if } c_\beta \neq 0, \quad P_2 u_1{}_{|x=\beta, y=\bar{\beta}} = 0 \;\; \textit{if } c_\beta = 0.$$

1. Introduction
   - Introduction
   - Difficulties
2. Framework and tools
   - Regularity
   - Two-scale convergence
3. Periodic Case
4. Case: $\Omega = \mathbb{R}^N$
   - Limit system
   - Corrector
5. Case: $\Omega = (\alpha, \beta)$
   - The problem
   - The boundary conditions
   - Corrector

Theorem

We assume that in previous Theorem (Ω=I), the sequences $u_{0,\varepsilon}$ and ϑ_ε converge in two-scale in strong sense, then, for every sequence $\Gamma_\varepsilon \in L^2(Q_T)^2$ which two-scale strongly converges to $\nabla_{s,y} u_1$, we have

$$\lim_{\varepsilon \to 0} \int_{Q_T} |\nabla_{t,x}(u_\varepsilon - u_0) - \Gamma_\varepsilon|^2 \, dxdt = 0.$$

Assuming regularity enough for u_1 the above Theorem asserts

$$u_\varepsilon - u_0 - \varepsilon u_1(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) \to 0 \text{ in } H^1(Q_T).$$

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THANK YOU VERY MUCH,

FOR YOUR

ATTENTION!!!