Γ-convergence of non-uniformly bounded energies defined on vector-valued functions

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VII Partial differential equations, optimal design and numerics

Benasque 2017

The problem: To study the Γ-convergence in the strong topology of $L^p(\Omega)^M$ of the sequence

$$F_n(v) := \int_\Omega f_n(x, Dv)\, dx \quad \text{for } v \in W_0^{1,p}(\Omega)^M,$$

with $p \in (1, \infty)$, $M \geq 1$, and Ω a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and $f_n : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty)$ Carathéodory functions, such that

- $\{F_n\}$ is assumed to be equi-coercive,
- $f_n(x, \xi)$ satisfies a Lipschitz condition with respect to ξ,
- $f_n(x, \xi)$ may be not uniformly bounded.

Definition

We consider V a metric space, and $F_n, F : V \to [0, \infty]$, $n \in \mathbb{N}$. Then $\{F_n\}$ is said to Γ-converge to F for the topology of V and we write

$$F_n \xrightarrow{\Gamma} F \quad \text{in } V,$$

if for every $v \in V$ we have

- the Γ-liminf inequality

$$\forall\, v_n \to v \quad \text{in } V, \quad F(v) \le \liminf_{n\to\infty} F_n(v_n), \tag{1}$$

- the Γ-limsup inequality

$$\exists\, w_n \to v \quad \text{in } V, \quad F(v) \ge \limsup_{n\to\infty} F_n(w_n). \tag{2}$$

Any w_n satisfying (2) is called a *recovery sequence* for F_n of limit v.

Remark: If $F_n \xrightarrow{\Gamma} F$ in V, $v \in V$ and w_n is a recovery sequence for F_n of limit v, then

$$F(v) \leq \inf \left\{ \liminf_{n\to\infty} F_n(v_n) \ : \ v_n \to v \quad \text{in} \ V \right\} \leq \liminf_{n\to\infty} F_n(w_n)$$
$$\leq \limsup_{n\to\infty} F_n(w_n) \leq F(v),$$

and therefore

$$F(v) = \lim_{n\to\infty} F_n(w_n) = \min \left\{ \liminf_{n\to\infty} F_n(v_n) \ : \ v_n \to v \quad \text{in} \ V \right\}.$$

In particular, this gives the uniqueness of the Γ-limit. With respect to its existence we have:

Theorem (Compactness result)

We consider V a separable metric space, and $F_n : V \to [0, \infty]$, $n \in \mathbb{N}$. Then, at least for a subsequence, $\exists F : V \to [0, \infty]$ the Γ-limit of F_n.

Motivation: The Γ-convergence is related with the description of the asymptotic behavior of sequences of minimum problems.

Theorem

We consider V a metric space, and $F_n, F : V \to [0, \infty]$, $n \in \mathbb{N}$, such that $F_n \stackrel{\Gamma}{\rightharpoonup} F$ in W. We assume $\exists$ a nonempty compact set $K \subset V$ with

$$\inf_V F_n = \inf_K F_n, \quad \forall n \in \mathbb{N}.$$

Then,

$$\exists \min_V F = \lim_n \inf_V F_n.$$

Moreover, if v_n satisfies $\lim_n F_n(v_n) = \lim_n \inf_V F_n$ and $v_n \to v$, then

$$F(v) = \min_V F.$$

Remark: The Γ-convergence of the sequence

$$F_n(v) := \int_\Omega f_n(x, Dv)\, dx \quad \text{for } v \in W_0^{1,p}(\Omega)^M,$$

is related with the asymptotic behavior of

$$\min \left\{ \int_\Omega f_n\big(x, Dv\big)\, dx : \ v \in W_0^{1,p}(\Omega)^M \right\},$$

and then, assuming smoothness, with the asymptotic behavior of the solutions of

$$-\operatorname{Div}\big(D_\xi f_n(x, Dv)\big) = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

$(f_n = f_n(x, \xi))$ i.e. a high contrast homogenization problem.

Our result extends former results on the topic.

• Linear scalar case (i.e. $M = 1$ and $F_n(x, \xi)$ quadratic in ξ) with uniformly bounded coefficients: S. Spagnolo(1968), F. Murat, L. Tartar(197?), E. De Giorgi(1975)...

• Nonlinear scalar case: L. Carbone, C. Sbordone(1979), G. Buttazzo, G. Dal Maso(1980), A. Braides, M. Briane, J. Casado-Díaz(2009), M. Briane, J. Casado-Díaz (2007, 2013)...

• Elasticity system: G. Duvaut; P. Seppecher, C. Pideri(1997), M. Camar-Eddine, P. Seppecher(2002), M. Briane, J. Casado-Díaz (2014)...

• Nonlocal terms: V.N. Fenchenko, E.Y. Khruslov(1981), M. Bellieud, G. Bouchitté(1998), M. Camar-Eddine, P. Seppecher(2002)...

The vectorial character of our problem and its nonlinearity prevent us from using the maximum principle arguments or div-curl lemma directly.

The main result: To study the Γ-convergence in the strong topology of $L^p(\Omega)^M$ of the sequence

$$F_n(v) := \int_\Omega f_n(x, Dv)\, dx \quad \text{for } v \in W_0^{1,p}(\Omega)^M,$$

with $p \in (1, \infty)$, $M \geq 1$, Ω a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and $f_n : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty)$ Carathéodory functions such that:

i) There exist two constants $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$\int_\Omega f_n(x, Dv)\, dx \geq \alpha \int_\Omega |Dv|^p\, dx + \beta, \quad \forall\, v \in W_0^{1,p}(\Omega)^M, \tag{3}$$

and

$$f_n(\cdot, 0) = 0 \;\text{ a.e. in } \Omega. \tag{4}$$

$ii)$ There exist two sequences of measurable functions $h_n, a_n \geq 0$, and a constant $\gamma > 0$ such that

$$h_n \text{ is bounded in } L^1(\Omega),$$

$$a_n \text{ is bounded in } L^r(\Omega) \text{ with } \begin{cases} r > \dfrac{N-1}{p}, & \text{if } p \leq N-1 \\ r = 1, & \text{if } p > N-1, \end{cases}$$

$$\begin{cases} \big|f_n(x,\xi) - f_n(x,\eta)\big| \\ \quad \leq \big(h_n(x) + f_n(x,\xi) + f_n(x,\eta) + |\xi|^p + |\eta|^p\big)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}} \, |\xi - \eta| \\ \qquad \forall\, \xi, \eta \in \mathbb{R}^{M\times N}, \text{ a.e. } x \in \Omega, \end{cases}$$

and

$$f_n(x, \lambda\xi) \leq h_n(x) + \gamma\, f_n(x,\xi), \quad \forall\, \lambda \in [0,1], \ \forall\, \xi \in \mathbb{R}^{M\times N}, \text{ a.e. } x \in \Omega.$$

From the hypothesis

$$h_n \text{ is bounded in } L^1(\Omega),$$

$$a_n \text{ is bounded in } L^r(\Omega) \text{ with } \begin{cases} r > \dfrac{N-1}{p}, & \text{if } p \leq N-1 \\ r = 1, & \text{if } p > N-1, \end{cases}$$

we have, at least for a subsequence, that

$$a_n^r \stackrel{*}{\rightharpoonup} \mathrm{A} \text{ in } \mathcal{M}(\Omega) \quad \text{and} \quad h_n \stackrel{*}{\rightharpoonup} h \text{ in } \mathcal{M}(\Omega).$$

where $\mathcal{M}(\Omega)$ denotes the space of the Radon measures on Ω with bounded total variation.

Theorem

There exist a function $f : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$, *a subspace* W *of* $W_0^{1,p}(\Omega)^M$, *and a subsequence of* n, *still denoted by* n, *satisfying*

$$\left\{\begin{array}{lll} f(\cdot,\xi) \ \text{is Lebesgue measurable}, & \text{if } p \le N-1, \\ f(\cdot,\xi) \ \text{is } \mathrm{A}\text{-measurable}, & \text{if } p > N-1, \end{array}\right. \qquad \forall \xi \in \mathbb{R}^N,$$

$$\big|f(x,\xi) - f(x,\eta)\big| \le$$

$$\left\{\begin{array}{l} C\big(h^L + f(x,\xi) + f(x,\eta) + (1+(\mathrm{A}^L)^{\frac{1}{r}})(|\xi|^p + |\eta|^p)\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|\xi-\eta| \\ \qquad\qquad \text{a.e. in } \Omega, \qquad\qquad \text{if } p \le N-1 \\ C\big(1 + h^{\mathrm{A}} + f(x,\xi) + f(x,\eta) + |\xi|^p + |\eta|^p\big)^{\frac{p-1}{p}}|\xi-\eta| \\ \qquad\qquad \mathrm{A}\text{-a.e. in } \Omega, \qquad\qquad \text{if } p > N-1 \end{array}\right.$$

for every $\xi, \eta \in \mathbb{R}^N$, *where* $h^L := \frac{dh}{dx}$, $\mathrm{A}^L := \frac{d\mathrm{A}}{dx}$, $h^{\mathrm{A}} := \frac{dh}{d\mathrm{A}}$, *and*

$$\left\{\begin{array}{ll} f(\cdot,0) = 0 \ \text{ a.e. in } \Omega, & \text{if } p \le N-1 \\ f(\cdot,0) = 0 \ \mathrm{A}\text{-a.e. in } \Omega, & \text{if } p > N-1, \end{array}\right. \tag{5}$$

Theorem (continuation)

$$W \subset \begin{cases} W^{1,\frac{pr}{r-1}}(\Omega)^M, & \text{if } p \leq N-1 \\ C^1(\overline{\Omega})^M, & \text{if } p > N-1, \end{cases}$$

and such that the Γ*-limit* F *of* F_n *for the strong topology of* $L^p(\Omega)^M$ *satisfies*

$$F(v) = \int_\Omega f(x, Dv)\, d\nu, \quad \forall v \in W,$$

where

$$\nu = \begin{cases} \text{Lebesgue measure}, & \text{if } p \leq N-1 \\ \mathrm{A} & \text{if } p > N-1. \end{cases}$$

Remark: The limit density f is strongly local, i.e., if $\omega \subset \Omega$ is an open set, the values of f in $\omega \times \mathbb{R}^{M\times N}$ only depends on the value of f_n in $\omega \times \mathbb{R}^{M\times N}$.

Idea of the proof: Based on a careful analysis of the asymptotic minimizers associated with the functional F_n.

For any open set $\omega \subset \Omega$, and any sequence $\{u_n\}$ in $W^{1,p}(\omega)^M$ which converges weakly in $W^{1,p}(\omega)^M$ to a function u satisfying

$$u \in \begin{cases} W^{1,\frac{pr}{r-1}}(\omega)^M, & \text{if } p \leq N-1 \\ C^1(\omega)^M, & \text{if } p > N-1, \end{cases}$$

and such that

$$\begin{aligned} &\exists \lim_{n\to\infty} \int_\omega f_n(x, Du_n)\, dx \\ &= \min\left\{ \liminf_{n\to\infty} \int_\omega f_n(x, Dw_n)\, dx : \ w_n - u_n \rightharpoonup 0 \text{ in } W_0^{1,p}(\omega)^M \right\} < \infty, \end{aligned}$$

we prove

$$f_n(\cdot, Du_n) \overset{*}{\rightharpoonup} \begin{cases} f(\cdot, Du), & \text{if } p \leq N-1 \\ f(\cdot, Du)\,\mathrm{A}, & \text{if } p > N-1 \end{cases} \quad \text{in } \mathcal{M}(\omega). \tag{6}$$

Instead dealing with sequences of the form

$$F_n\left(\varphi(u_n - u)\right),$$

with φ a fixed cut-off function, we deal with sequences of the form

$$F_n\left(\varphi_n(u_n - u)\right),$$

with φ_n radial cut-off functions whose gradient has support in n-dependent sets on which $u_n - u$ satisfies some uniform estimate with respect to the radial coordinate. This allows us to control the zero-order term $\nabla\varphi_n(u_n - u)$.

M. Briane, J. Casado-Díaz (2016): adapted div-curl lemmas

Example 1

$$F_n(u) := \int_\Omega \left|A_n e(u) : e(u)\right|^{\frac{p}{2}} dx \quad \text{for } u \in W_0^{1,p}(\Omega)^N,$$

where $e(u) := \frac{1}{2}\left(Du + Du^T\right)$, $p \in (1,\infty)$, and $A_n \in L^\infty\big(\Omega; \mathcal{L}(\mathbb{R}_s^{N\times N})\big)$ such that

$$\exists \alpha > 0 \;:\; A_n(x)\xi : \xi \geq \alpha\,|\xi|^2, \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}_s^{N\times N},$$

$$|A_n|^{\frac{p}{2}} \text{ is bounded in } L^r(\Omega) \text{ with } r \text{ defined before.}$$

For $p = 2$, it corresponds to the linear elasticity. Functionals of this type also appear with non-Newtonian fluids.

Example 2 For $N = 3$, the Saint Venant-Kirchhoff hyperelastic model

$$F_n(u) := \int_\Omega \left(\frac{\lambda_n}{2} \left[\mathrm{tr}\big(E(u)\big)\right]^2 + \mu_n \left|E(u)\right|^2 \right) dx \quad \text{for } u \in W_0^{1,4}(\Omega)^3,$$

with $E(u) := \frac{1}{2}\left(Du + Du^T + Du^T Du\right)$ and $\exists C > 1$ such that

$$\lambda_n, \mu_n \geq 0 \text{ a.e. in } \Omega, \quad \operatorname*{ess\text{-}inf}_\Omega (\lambda_n + \mu_n) > C^{-1}, \quad \int_\Omega (\lambda_n + \mu_n)\, dx \leq C.$$

Example 3 For $N = 3$, $p \in [2, \infty)$, the Ogden's type hyperelastic model

$$F_n(u) := \int_\Omega a_n(x) \Big[\mathrm{tr}\big(C(u)^{\frac{p}{2}} - I_3\big) \Big]^+ dx \quad \text{for } u \in W_0^{1,p}(\Omega)^3,$$

with $C(u) := (I_3 + Du)^T (I_3 + Du)$, $t^+ := \max(t, 0)$, and $\exists C > 1$ such that

$$\operatorname*{ess\text{-}inf}_\Omega a_n > C^{-1} \quad \text{and} \quad \int_\Omega a_n^r\, dx \leq C \quad \text{with} \begin{cases} r > 1, & \text{if } p = 2 \\ r = 1, & \text{if } p > 2. \end{cases}$$

For the details:

Γ*-convergence of equi-coercive nonlinear energies defined on vector-valued functions, with non-uniformly bounded coefficients*, (Nonlinear Analysis 2017)

Open problems:

- Optimality of the hypothesis on the energy densities f_n.
- Optimal design of hyperelastic materials.