A simple tensor network algorithm for 2d steady states

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Motivation

Most systems, in nature, are not isolated! They are usually affected by environment.
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Examples of such systems include heat transfer, quantum decoherence, etc.

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Examples of such systems include heat transfer, quantum decoherence, etc.

Systems interacting with an environment are known as open systems. Such interactions usually lead to dissipation.
In the context of quantum many-body systems, dissipation often leads to many interesting phenomenon such as:

- Decoherence of complex wave functions
- Quantum Thermodynamics
- Engineering Topological order through dissipation
- Driven-dissipative universal quantum computation
- Dissipative phase transitions

References:

- M. Schlosshauer, Rev. Mod. Phys. 76 1267 (2005)
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Despite this, understanding the effects of dissipation in quantum many-body systems is still a very difficult problem.

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Despite this, understanding the effects of dissipation in quantum many-body systems is still a very difficult problem.

Why?

1. Exhibit the same computational complexity class as equilibrium system \( O(d^{2N}) \)
2. Physical constraints of a density matrix that can be used to represent such systems
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1. Exhibit the same computational complexity class as equilibrium system \( O(d^{2N}) \)
2. Physical constraints of a density matrix that can be used to represent such systems

Efficient numerical tools for study of open quantum many-body systems is still lacking and continues to be a challenge!
An important concept: Choi isomorphism
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Finding steady states: parallelism with imaginary time evolution
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Tensors Networks
An important concept: Choi isomorphism

Finding steady states: parallelism with imaginary time evolution

Tensors Networks

Steady States with TNs:
• Recent advances in 1d systems
• Applications in 2d systems
Content of this talk

An important concept: Choi isomorphism

Finding steady states: parallelism with imaginary time evolution

Tensors Networks

Steady States with TNs:
  • Recent advances in 1d systems
  • Applications in 2d systems

Benchmark results: Dissipative spin-1/2 quantum Ising model in 2d
An important concept: Choi isomorphism

Finding steady states: parallelism with imaginary time evolution

Tensors Networks

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• Applications in 2d systems

Benchmark results: Dissipative spin-1/2 quantum Ising model in 2d

Conclusions & Outlook
Choi isomorphism
Simply turning a bra into a ket!

\[ |i\rangle\langle j| \equiv |i\rangle \otimes |j\rangle \]

(Vectorization)
Choi isomorphism

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(Vectorization)

In the context of a reduced density matrix

\[ \rho \rightarrow |\rho\rangle^\# \]

Understanding the coefficients of \( \rho \) as those of a vector \( |\rho\rangle^\# \)
Choi isomorphism

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(Vectorization)

In the context of a reduced density matrix

\[ \rho \to |\rho\rangle_\# \]

Understanding the coefficients of \( \rho \) as those of a vector \( |\rho\rangle_\# \)

Assuming Markovian evolution, the dynamics of an open quantum system can be described by the following Lindbladian Master equation

\[
\frac{d}{dt} \rho = \mathcal{L}[\rho] = -i [H, \rho] + \sum_\mu \left( L_\mu \rho L_\mu^\dagger - \frac{1}{2} L_\mu^\dagger L_\mu \rho - \frac{1}{2} \rho L_\mu^\dagger L_\mu \right)
\]

\( \mathcal{L} \) is the Liouvillian operator, \( H \) the Hamiltonian of the system and \( \{L_\mu, L_\mu^\dagger\} \) the jump/Lindblad operators describing the dissipation.
Using the Choi isomorphism, one can then write the vectorized form of this equation as

\[
\frac{d}{dt} |\rho\rangle_\# = \mathcal{L}_\# |\rho\rangle_\#
\]

where the vectorized Liouvillian operator is given by

\[
\mathcal{L}_\# \equiv -i \left( H \otimes \mathbb{I} - \mathbb{I} \otimes H^T \right) + \sum_\mu \left( L_\mu \otimes L_\mu^* - \frac{1}{2} L_\mu^\dagger L_\mu \otimes \mathbb{I} - \frac{1}{2} \mathbb{I} \otimes L_\mu^* L_\mu^T \right).
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we can now integrate the above equation to get

$$|\rho(T)\rangle_\# = e^{T\mathcal{L}_\#} |\rho(0)\rangle_\#$$
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\]

we can now integrate the above equation to get

\[
|\rho(T)\rangle_\# = e^{T\mathcal{L}_\#} |\rho(0)\rangle_\#
\]

we can then obtain the steady state for very large times

\[
|\rho_s\rangle_\# \equiv \lim_{T \to \infty} |\rho(T)\rangle_\# \quad \text{i.e.} \quad \frac{d}{dt} |\rho_s\rangle_\# = \mathcal{L}_\# |\rho_s\rangle_\# = 0
\]
Assume a nearest-neighbor Liouvillian:

$$\mathcal{L}[\rho] = \sum_{\langle i,j \rangle} \mathcal{L}^{i,j}[\rho]$$

where the sum is over nearest-neighbor terms.
Parallelism with imaginary time evolution

Assume a nearest-neighbor Liouvillian:

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We find a parallelism with finding the ground state of a Hamiltonian using imaginary time evolution

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Imaginary time

Real time
Parallelism with imaginary time evolution

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Imaginary time

Use the usual TN algorithms to obtain the steady states??
A quantum many-body wave function can be written as

\[ |\psi\rangle = \sum_{i_1i_2...i_N} C_{i_1i_2..i_N} |i_1\rangle \otimes |i_2\rangle \otimes ... \otimes |i_N\rangle \]
Steady States with TNs: 1d

Recent advances in 1D systems
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Recent advances in 1D systems

Choi isomorphism + MPOs


Simple & efficient

Issue of positivity!!
Steady States with TNs: 1d

Recent advances in 1D systems

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Positive by construction

Large purification may be required!!

Very costly!!

Simple & efficient

Issue of positivity!!

Other techniques using MPDOs, disentanglers, etc

F. Verstraete, J. J. García-Ripoll, and J. I. Cirac
Phys. Rev. Lett. 93, 207204 (2004); A. H. Werner et al,
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\[ A \rightarrow A \]


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Perform imaginary time evolution & get direct convergence with TEBD or DMRG

Interactions no more local!!

Bond dimension of the MPOs get squared!!

(Still manageable in 1d)

Target the ground state of \( \mathcal{L}_+^\dagger \mathcal{L}_+ \) (hermitian & positive semi-definite)

E. Mascarenhas, H. Flayac, and V. Savona,
Phys. Rev. A 92, 022116 (2015); J. Cui, J. I. Cirac,
Steady States with TNs: 2d

Applications in 2d systems??

Targeting the ground state of $\mathcal{L}^\dagger \mathcal{L}$ is extremely difficult here

Can (in principle) perform imaginary time evolution & get direct convergence with TEBD

Interactions in $\mathcal{L}^\dagger \mathcal{L}$ no more local!!

Bond dimension of the PEPOs get squared!!

(very difficult in 2d!!)
Steady States with TNs: 2d
Use real time evolution with $L_{\#}$ until steady state is reached.
Steady States with TNs: 2d

Use real time evolution with $\mathcal{L}_\#$ until steady state is reached.

Choi isomorphism + PEPOs
Steady States with TNs: 2d

Use real time evolution with $\mathcal{L}_\#$ until steady state is reached

Choi isomorphism + PEPOs

Simple and efficient
Growth of entanglement??
Not necessarily positive!

Growth of entanglement in 2D may be slow compared to the fixed point attractor
Very good accuracy and small positivity error
Steady States with TNs:2d

Use real time evolution with $\mathcal{L}_\#$ until steady state is reached.

Choi isomorphism + PEPOs

Simple and efficient Growth of entanglement?? Not necessarily positive!

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So, one can apply the usual iPEPS algorithm in the vectorized form.
Benchmark results

The 2d dissipative quantum Ising model

\[ H = \frac{V}{4} \sum_{\langle i, j \rangle} \sigma_{z}^{[i]} \sigma_{z}^{[j]} + \frac{h_x}{2} \sum_{i} \sigma_{x}^{[i]} + \frac{h_z}{2} \sum_{i} \sigma_{z}^{[i]}, \text{ with } L_{\mu} = \sqrt{\gamma} \sigma_{-}^{[\mu]} \]
Benchmark results

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Interesting for a number of reasons

This model is relevant for experiments of ultracold gases of Rydberg atoms

- F. Letscher et al, arXiv:1611.00627
Benchmark results

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Interesting for a number of reasons

This model is relevant for experiments of ultracold gases of Rydberg atoms


Phase diagram of the steady state of this model is still controversial:
Existence of a bistable phase in the steady state??


1st order phase transition?


Antiferromagnetic phase in the steady state?

Benchmark results

\[ V = 5\gamma, \gamma = 0.1, h_z = 0 \]

**spin-up density**

**purity of n-sites**

\[ n^\uparrow = \frac{1}{2N} \sum_{i=1}^{N} \langle 1 + \sigma_z^{(i)} \rangle \]

\[ \Gamma_n = \text{tr}(\rho_n^2) \]

Good agreement with the correlated variational ansatz

1st order transition

No bi-stability

Benchmark results

\[ V = 5\gamma, \gamma = 0.1, h_z = 0 \]

steady-state approximation

\[ \Delta = \# \langle \rho_s | \mathcal{L}_\# | \rho_s \rangle \# \]

positivity error

\[ \epsilon_n = \sum_{i | \nu_i < 0} \nu_i(\rho_n) \]

Very good accuracy

error due to positivity: not very large
Benchmark results

$$H = \frac{V}{4} \sum_{\langle i,j \rangle} \sigma_z^i \sigma_z^j + \frac{h_x}{2} \sum_i \sigma_x^i + \frac{h_z}{2} \sum_i \sigma_z^i,$$
with \( L_\mu = \sqrt{\gamma} \sigma_-^{[\mu]} \)

Turning on the longitudinal field, previous studies have found the existence of AF region in the steady state phase diagram of \( h_x/\gamma \) vs \( h_z/\gamma \)


Using our techniques, the AF region is found to shrink from \( D = 2-5 \) and finally disappear for \( D = 6,7 \). This suggests that the AF region may not be there after all for these parameter regimes.
• Proposed a simple TN algorithm to approximate the steady states of 2d dissipative systems of infinite size.
  - very accurate results and relatively small errors induced.
Conclusions & Outlook

• Proposed a simple TN algorithm to approximate the steady states of 2d dissipative systems of infinite size.
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• Studied the controversial steady state phase diagram of the 2d dissipative quantum Ising model.
  - no bistable region in its steady state.
  - first order phase transition
  - AF region in the presence of longitudinal field seem to disappear.
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• Investigate into dissipative QPTs, Topological order by dissipation, etc?
• Connections to area-laws for rapidly-mixing dissipative systems?
Conclusions

Collaborators

Román Orús, (JGU Mainz)
Hendrik Weimer, (LU Hannover)

Acknowledgements: A. Gangat, Y.-Jer Kao, M. Rizzi
Conclusions

Thank you!

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![Diagram with green and black shapes and arrows, indicating transformations from $\rho$ to $\rho^\sharp$ with $D$ and $d^2$.](image)