Exercise 1: Gauge invariance of spin-3 Fronsdal equation

Consider the Fronsdal equation in Minkowski space

$$\mathcal{F}_{\mu_1 \dots \mu_s}[\varphi] = \Box \varphi_{\mu_1 \dots \mu_s} - \partial_{(\mu_1} \partial^\rho \varphi_{\mu_2 \dots \mu_s)\rho} + \partial_{(\mu_1} \partial_{\mu_2} \varphi_{\mu_3 \dots \mu_s)\rho}{}^{\rho} = 0$$

and in particular consider spin $s = 3$.

a. Show that the gauge variation of the spin-3 Fronsdal tensor under the linearized spin-3 gauge transformation

$$\delta^{(0)}_{\xi} \varphi_{\mu_1 \mu_2 \mu_3} = \partial_{(\mu_1} \xi_{\mu_2 \mu_3)}$$

Note: in our symmetrization convention:

$\partial_{(\mu_1} \xi_{\mu_2 \mu_3)}$
$= \partial_{\mu_1} \xi_{\mu_2 \mu_3} + \partial_{\mu_2} \xi_{\mu_1 \mu_3} + \partial_{\mu_3} \xi_{\mu_1 \mu_2}$

all necessary permutations without additional numerical factors.

is given by

$$\delta^{(0)}_{\xi} \mathcal{F}_{\mu_1 \mu_2 \mu_3} = 3\, \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \xi^{\sigma}{}_{\sigma}$$

b. Find the general solution to the following equation

$$\partial^x_{\mu_1} \partial^x_{\mu_2} \partial^x_{\mu_3} \xi^{\sigma}{}_{\sigma}(x) = 0$$

as a polynomial in x^μ. Given your answer, show that requiring a gauge transformation that vanishes at infinity implies that $\xi^{\sigma}{}_{\sigma} = 0$ i.e. the gauge parameter is traceless.

c. If you are feeling energetic carry out a. and b. again for generic integer spin s. But we shall do this using a simpler formalism in the next exercises.

Exercise 2 Generating functions for spin-s gauge fields

We can encode totally symmetric rank s tensors in homogeneous polynomials

$$\varphi_{\mu_1\cdots\mu_s}(x) \longrightarrow \varphi_s(x,u) := \frac{1}{s!}\varphi_{\mu_1\cdots\mu_s}u^{\mu_1}\cdots u^{\mu_s}$$

a. Show how we recover $\varphi_{\mu_1\cdots\mu_s}$ by differentiating with respect to ∂_u^μ

b. Show that the trace of $\varphi_{\mu_1\cdots\mu_s}$ is encoded by

$$\varphi_{\mu_1\cdots\mu_{s-2}\sigma}{}^{\sigma} \longrightarrow (\partial_u\cdot\partial_u)^2\varphi_s(x,u)$$

c. Show that the contraction of indices is implemented by replacing u^μ with ∂_u^μ:

$$\varphi_{\mu_1\cdots\mu_s}\varphi^{\mu_1\cdots\mu_s} \longrightarrow s!\,\varphi_s(x,\partial_u)\varphi_s(x,u)$$

d. Show that the divergence is implemented as

$$\partial_x^\rho\varphi_{\mu_1\cdots\mu_{s-1}\rho} \longrightarrow (\partial^x\cdot\partial^u)\varphi_s(x,u)$$

e. Symmetrized gradient:

$$\partial_{(\mu_1}\varphi_{\mu_2\cdots\mu_{s+1})} \longrightarrow (u\cdot\partial^x)\varphi_s(x,u)$$

Exercise 3 Fronsdal's equations in generating function notation

Given exercise 2, show that Fronsdal's equations can be expressed in generating function form as:

$$\mathcal{F}_{\mu_1\cdots\mu_s} \rightarrow \mathcal{F}_s(x,u) = \Box\varphi_s(x,u) - (u\cdot\partial_x)(\partial^x\cdot\partial^u)\varphi_s(x,u) + \frac{1}{2}(u\cdot\partial_x)^2(\partial_u\cdot\partial_u)\varphi_s(x,u)$$

$$\delta_\xi^{(0)}\varphi_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\cdots\mu_s)} \longrightarrow \delta_\xi^{(0)}\varphi_s(x,u) = (u\cdot\partial_x)\xi_{s-1}(x,u)$$

(where: $\xi_{s-1}(x,u) = \frac{1}{(s-1)!}\xi_{\mu_1\cdots\mu_{s-1}}u^{\mu_1}\cdots u^{\mu_{s-1}}$)

In other words, show:

a. $\Box\varphi_{\mu_1\cdots\mu_s} \rightarrow \Box\varphi_s(x,u)$ (trivial)

b. $\partial_{(\mu_1}\partial_{\mu_2}\varphi_{\mu_3\cdots\mu_s)\sigma}{}^{\sigma} \longrightarrow \frac{1}{2}(u\cdot\partial_x)^2(\partial_u\cdot\partial_u)\varphi_s(x,u)$

c. $\partial_{(\mu_1}\partial^\sigma\varphi_{\mu_2\cdots\mu_s)\sigma} \longrightarrow (u\cdot\partial_x)(\partial^x\cdot\partial^u)\varphi_s(x,u)$

Exercise 4 Using the generating function form of Fronsdal's equations, show that:

$$\delta_\xi^{(0)}\mathcal{F}_s(x,u) = \frac{1}{2}(u\cdot\partial_x)^3(\partial_u\cdot\partial_u)\xi_{s-1}(x,u)$$

Show that this is equivalent to:

$$\delta_\xi^{(0)}\mathcal{F}_{\mu_1\cdots\mu_s} = 3\,\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\xi_{\mu_4\cdots\mu_s)\sigma}{}^{\sigma}$$

Exercise 5 : Fronsdal's system in de Donder gauge

Consider the de Donder gauge condition:

$$D_{\mu_1\cdots\mu_{s-1}} = \partial^\sigma \varphi_{\mu_1\cdots\mu_{s-1}\sigma} - \tfrac{1}{2}\,\partial_{(\mu_1}\varphi_{\mu_2\cdots\mu_{s-1})\sigma}{}^{\sigma} = 0$$

a. Show that in this gauge the Fronsdal equation becomes: (use either generating functions or explicit indices)

$$\Box\,\varphi_{\mu_1\cdots\mu_s} = 0$$

b. Show also that : $\delta^{(0)}_\xi D_{\mu_1\cdots\mu_{s-1}} = \Box\,\xi_{\mu_1\cdots\mu_{s-1}}$

Therefore, Fronsdal's system in de Donder gauge reads:

$$\Box\,\varphi_{\mu_1\cdots\mu_s} = 0 \qquad (*)_1$$
$$\varphi_{\mu_1\cdots\mu_{s-4}}{}^{\rho}{}_{\rho}{}^{\sigma}{}_{\sigma} = 0$$

with residual gauge symmetry: $\delta^{(0)}_\xi \varphi_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}$,

where
$$\Box\,\xi_{\mu_1\cdots\mu_{s-1}} = 0 \qquad (*)_2$$
$$\xi_{\mu_1\cdots\mu_{s-3}}{}^{\sigma}{}_{\sigma} = 0 \,.$$

c. What are the solutions to equations $(*)_1$ and $(*)_2$?

d. Show that the de Donder tensor is traceless : $D_{\mu_1\cdots\mu_{s-3}}{}^{\sigma}{}_{\sigma} = 0$

e. Use your results to parts c & d to compute the number of physical polarizations of a spin-s gauge field

(answer: $(d-4+2s)\,\dfrac{(d+s-5)!}{s!\,(d-4)!}$)

Hint: number of independent components of a symmetric rank s tensor in d-dims is: $\begin{pmatrix} d+s-1 \\ s \end{pmatrix}$

Exercise 6: Fronsdal action

$$S^{(2)} = \frac{1}{2}\int d^dx\, \varphi_{\mu_1 \ldots \mu_s} G^{\mu_1 \ldots \mu_s}[\varphi]$$

$$G_{\mu_1 \ldots \mu_s} = \mathcal{F}_{\mu_1 \ldots \mu_s} - \frac{1}{2}\eta_{(\mu_1 \mu_2} \mathcal{F}_{\mu_3 \ldots \mu_s)\sigma}{}^{\sigma}$$

a. Show that the Fronsdal equation follows from the equation of motion $G_{\mu_1 \ldots \mu_s} = 0$

(Hint: consider the trace)

b. Show that in generating function notation the action reads:

$$S^{(2)} = \frac{1}{2}\int d^dx\, \varphi_s(x, \partial_u)\, G_s(x, u)$$

$$G_s(x, u) = \mathcal{F}_s(x, u) - \frac{1}{4}u^2 \partial_u^2 \mathcal{F}_s(x, u)$$

c. Show that the action is gauge invariant: $\delta_\xi^{(0)} S^{(2)}[\varphi] = 0$

(Hint: recall $\xi_{\mu_1 \ldots \mu_{s-3}\sigma}{}^{\sigma} = 0$)

This exercise is very messy

warning!