Exercise: Noncommutative distance function on Riemann sphere.

We will compute the distance function on the pure state space $S(M_2(\mathbb{C})) \cong \mathbb{CP}^1$, as described by spectral triple $\left(M_2(\mathbb{C}), \mathbb{C}^2, D = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right)$ for $x \neq y$ in $\mathbb{R}$.

Recall that $[z:w] \in \mathbb{CP}^1$ corresponds to the state

$$\varphi_{[z:w]}(M) := \frac{1}{\sqrt{|z|^2+|w|^2}} \left\langle \begin{pmatrix} z \\ w \end{pmatrix}, M \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle \qquad (M \in M_2(\mathbb{C}))$$

The distance function we are after is given by

$$d(\varphi_1, \varphi_2) = \sup_{M \in M_2(\mathbb{C})} \left\{ |\varphi_1(M) - \varphi_2(M)| : \|[D,M]\| \le 1 \right\}$$

ⓐ Show that $\|[D,M]\| = \max\{|b|,|c|\}\,|x-y|$.
for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$.

Let us now consider points $[z:1]$ on a chart of $\mathbb{CP}^1$, $(z \in \mathbb{C})$; denote corresponding states by φ_z.

ⓑ Show that if $|z_1| \neq |z_2|$ then $d(\varphi_{z_1}, \varphi_{z_2}) = \infty$.

ⓒ Now assume $|z_1| = |z_2|$ and show that

$$|\varphi_{z_1}(M) - \varphi_{z_2}(M)| \le \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}} |b+c| \qquad \forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This implies that $d(\varphi_{z_1}, \varphi_{z_2}) \leq 2 \dfrac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}} |x-y|^{-1}$ if $|z_1| = |z_2|$.

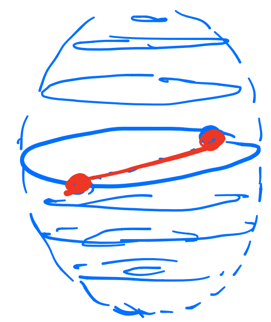
(d) Find an M for which $\|[D,M]\| \leq 1$ and

$$|\varphi_{z_1}(M) - \varphi_{z_2}(M)| = 2 \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}} |x-y|^{-1}$$

Thus, we have shown that if $|z_1| = |z_2|$ then

$$d(\varphi_{z_1}, \varphi_{z_2}) = 2 \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}} |x-y|^{-1}$$

(e) Show that (up to the fact $|x-y|^{-1}$) this distance is the chord distance on the 2-sphere $\mathbb{CP}^1$, say, as embedded in $\mathbb{R}^3$,



(f) Show that $d(\varphi_{[1:0]}, \varphi_{[0:1]}) = \infty$.

Since $d(\varphi_{[z:1]}, \varphi_{[0:1]}) = \infty$ by an argument similar to (b) we have completely determined the distance function on $\mathbb{CP}^1$.