Stationary solutions in theories beyond GR

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Numerical Relativity beyond General Relativity workshop
Motivation
   Why theories beyond-GR? How to proceed?

GR
   Brief outline

Theories beyond GR
   What’s out there?
      Scalar-tensor theories
      Quadratic gravity theories

Parametrised deviations from GR
   Deformations of Kerr
   Designer metrics

Conclusions
Why beyond-GR theories:
See morning talk by Thomas (Theoretical motivation for beyond-GR theories).

1 point to remember:
To test GR we also need solutions from theories other than GR in order to form a testbed.

2 approaches:
• Find solutions in specific modifications to GR and work on a case by case basis.
• Construct generic parametric deviations from known GR solutions, like the Kerr solution.

3 ways to proceed:
• Generate an analytic parameterised solution without approximations
• Employ some approximation scheme (slow rotation, small coupling)
• Numerical solution
The Einstein-Hilbert action in GR is

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} R + S_m(g_{\mu
u}, \psi), \]

which results to the field equations

\[ R_{ab} = 8\pi \left( T_{ab} - \frac{1}{2} g_{ab} T \right) \]

Stationary spacetime: symmetry with respect to time translations and rotations (the spacetime admits a timelike, \( \xi^a \), and a spacelike, \( \eta^a \), killing vector).

The line element for such a spacetime can be written as

\[ ds^2 = -e^{2\nu} dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} (d\varphi - \omega dt)^2 + e^{2(\zeta-\nu)} (dr^2 + r^2 d\theta^2) \]

This line element describes the spacetime of a rotating compact object. The field equations can be solved either for vacuum spacetimes (BH solutions) or spacetimes that have matter (NS solutions).

- In vacuum we have the well known BH solutions of the Kerr family (which can be extended to include electromagnetic fields as well).
- For NSs the full field equations can only be solved numerically.

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• The alternative is to employ the slow rotation Hartle-Thorne method.²

\[ ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right] - 2\varepsilon(\Omega - \omega_1)r^2 \sin^2 \theta d\varphi dt. \]

where \( \varepsilon \) is a slow rotation small bookkeeping parameter. The first order correction \( \omega_1 \) is given by the equation

\[ \omega_1'' = \frac{4}{r} \left[ \pi r^2 (\varepsilon + p) e^\lambda - 1 \right] \omega_1' + 16\pi (\varepsilon + p) e^\lambda \omega_1. \]

• Finally there is a general algorithm for constructing any vacuum stationary axisymmetric space-time. Such a spacetime can be described by the Weyl-Papapetrou line element³,

\[ ds^2 = -f (dt - wd\varphi)^2 + f^{-1} \left[ e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right]. \]

By introducing the complex potential \( \mathcal{E}(\rho, z) = f(\rho, z) + i\psi(\rho, z) \)⁴, the Einstein field equations take the form, \((\text{Re} (\mathcal{E})) \nabla^2 \mathcal{E} = \nabla \mathcal{E} \cdot \nabla \mathcal{E}\), where, \( f = \xi^a \xi_a \) and \( \psi \) is the scalar twist, \( \nabla_a \psi = \varepsilon_{abcd} \xi^b \nabla^c \xi^d \). By prescribing an Ernst potential \( \mathcal{E} \) one can calculate a vacuum GR solution.

A brief (incomplete) list of theories beyond GR.

- Scalar-tensor theories
- $f(R)$ theories
- Quadratic gravity theories
  - Einstein-dilaton-Gauss-Bonnet (EdGB)
  - dynamical Chern-Simons (dCS)
- Lorentz-violating theories
- Massive gravity theories
- Theories with non-dynamical fields
The *Bergmann-Wagoner* action for Scalar-Tensor theories is,

\[
S = \int d^4 x \sqrt{-\hat{g}} \left( \varphi \hat{R} - \frac{\omega(\varphi)}{\varphi} \hat{\nabla}^\mu \varphi \hat{\nabla}_\mu \varphi U(\varphi) \right) + S_m(\hat{g}_{\mu\nu}, \psi)
\]

In the Einstein frame it takes the form,

\[
S = \frac{1}{16\pi} \int d^4 x \sqrt{-\tilde{g}} \left( \tilde{R} - 2 \tilde{\nabla}^\mu \phi \tilde{\nabla}_\mu \phi - V(\phi) \right) + S_m(\tilde{g}_{\mu\nu}, \psi)
\]

where \( \varphi \) is redefined to \( \phi \), \( \hat{g}_{\mu\nu} = A^2(\phi)\tilde{g}_{\mu\nu} \), and \( V(\phi) \equiv A^4(\phi)U(\varphi(\phi)) \). Then the field equations take the form,

\[
\tilde{R}_{ab} = 2 \partial_a \phi \partial_b \phi + 8\pi \left( T_{ab} - \frac{1}{2} \tilde{g}_{ab} T \right) + 2V\tilde{g}_{ab}, \quad \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \phi = -4\pi \alpha(\phi) T + \frac{1}{4} \frac{dV}{d\phi}
\]

These equations can be solved as in GR in vacuum or in the presence of matter.\(^5\) Since the actual physics is done in the Jordan frame (the particles follow the geodesics of the Jordan metric), one can return to that frame by the conformal transformation \( \hat{g}_{\mu\nu} = A^2(\phi)\tilde{g}_{\mu\nu} \).

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In the case of a massless scalar field \( V(\phi) = 0 \), the vacuum field equations

\[
\tilde{R}_{ab} = 2\partial_a \phi \partial_b \phi, \quad \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \phi = 0
\]
can admit an Ernst formulation as in GR,\(^6\) having the metric,

\[
ds^2 = -f (dt - w d\phi)^2 + f^{-1} \left[ e^{2\gamma} \left( d\rho^2 + dz^2 \right) + \rho^2 d\phi^2 \right]
\]
and the field equations,

\[
(Re(\mathcal{E})) \nabla^2 \mathcal{E} = \nabla \mathcal{E} \cdot \nabla \mathcal{E},
\]
with the addition of a Laplace equation for the scalar field, \( \nabla^2 \phi = 0 \), and a set of equations for the metric function \( \gamma \) of the Weyl-Papapetrou metric,

\[
\frac{\partial \gamma}{\partial \rho} = \left( \frac{\partial \gamma}{\partial \rho} \right)_{GR} + \rho \left[ \left( \frac{\partial \phi}{\partial \rho} \right)^2 - \left( \frac{\partial \phi}{\partial z} \right)^2 \right], \quad \frac{\partial \gamma}{\partial z} = \left( \frac{\partial \gamma}{\partial z} \right)_{GR} + 2\rho \left( \frac{\partial \phi}{\partial \rho} \right) \left( \frac{\partial \phi}{\partial z} \right),
\]

Any vacuum stationary axisymmetric GR solution can be turned into a scalar-tensor solution with a massless scalar field.\(^7\)

In the case of a massive scalar field, one doesn't have the Ernst formulation, but can still do a fully numerical calculation, or employ a HT-like slow rotation approximation, like in GR.\textsuperscript{8}

The TOV equations then become (0th-order in the rotation),

\[
M' = 4\pi r \left( rA^4 \epsilon_0 + \frac{1}{2} (r - 2M) \phi_0'' + rV \right),
\]

\[
\nu' = \frac{2 \left( 4\pi r^3 (A^4 p_0 - V) + M \right)}{r(r - 2M)} + 4\pi r \left( \phi_0' \right)^2,
\]

\[
p' = -(p_0 + \epsilon_0) \frac{1}{2} \nu' - \alpha \phi_0',
\]

\[
\phi_0'' = \frac{2\phi_0' \left( r \left( 2\pi r^2 (A^4 (\epsilon_0 - p_0) + 2V) - 1 \right) + M \right) + r^2 (A^3 A' (\epsilon_0 - 3p_0) + V')}{r(r - 2M)},
\]

and at 1st-order in the rotation,

\[
\omega_1'' = \frac{4 \left( \pi A^4 r^2 (p_0 + \epsilon_0) (r\omega_1' + 4\omega_1) + (r - 2M)\omega_1' \left( \pi r^2 (\phi_0')^2 - 1 \right) \right)}{r(r - 2M)}.
\]

\textsuperscript{8} Pani, Berti, Phys. Rev. D 90, 024025 (2014); Yazadjiev et al., Phys. Rev. D 93, 084038 (2016)
A lot of work has been done in scalar-tensor theories for both BHs and NSs. These results also extend to f(R) theories due to the equivalence between them (although there are subtleties).

▶ Black Holes:
  ▶ Real scalar field:
    No-hair theorems. Same BH solutions as in GR, i.e., Kerr BHs.
  ▶ Complex scalar field:
    Evade the no-hair theorems for $\Psi(t, \varphi, x^i) = e^{-i\omega t} e^{im\varphi} \phi(x^i)$. Stationary BH solutions with scalar clouds when $\omega = m\Omega_H$.

▶ Neutron Stars:
  Systematic studies in slow rotation and rapid rotation for massless and massive scalar fields. Spontaneously scalarised solutions.
  f(R) theories such as $f(R) = R + aR^2$ studied in their scalar-tensor formulation.
The most general action for quadratic gravity with a scalar field is

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2\nabla^\mu \phi \nabla_\mu \phi - V(\phi) + f_1(\phi)R^2 + f_2(\phi)R_{\mu\nu}R^{\mu\nu} + f_3(\phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + f_4(\phi)^* RR \right] + S_m(\gamma(\phi)g_{\mu\nu}, \psi)
\]

Special cases are the EdGB gravity (Gauss-Bonnet scalar)

\[
R_{GB}^2 \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},
\]

and dCS gravity (Pontryagin scalar),

\[
*RR \equiv \frac{1}{2} R_{\mu\nu\rho\sigma}\epsilon^{\nu\mu\lambda\kappa} R_{\lambda\kappa}^{\rho\sigma}.
\]

Quite some work has been done in quadratic gravity for both BHs and NSs.

- **Black Holes:**
  Solutions have been found in the perturbative slow-rotation limit. The BH solutions are scalarised. In EdGB solutions have been also found for rapidly rotating BHs. EdGB also has spontaneously scalarised BHs.

- **Neutron Stars:**
  - **EdGB:** Studies in slow rotation and rapid rotation. Spontaneously scalarised solutions. There exists a critical \( p_c \).
  - **dCS:** Studies in 1st and 2nd order in rotation. Scalar dipole. No-scalar-monopole theorem for both theories at the perturbative level.
It is probably impossible to study all solutions of all the theories beyond GR. Still, without committing to a specific theory, one could construct parameterised spacetimes that can be used to test GR. These spacetimes do not necessarily satisfy some specific field equations.

- One such example is the Cardoso, Pani, Rico extension of the Johannsen-Psaltis non-Kerr metric:\(^9\)

\[
ds^2 = -f dt^2 + \frac{\Sigma(1 + h^r)}{\Delta + a^2 \sin^2 \theta h^r} dr^2 + \Sigma d\theta^2 - 2a \sin^2 \theta (H - f) d\varphi dt \\
+ \sin^2 \theta \left[ \Sigma + a^2 \sin^2 \theta (2H - f) \right] d\varphi^2
\]

where \(f = (1 - \frac{2mr}{\Sigma})(1 + h^t)\), \(H = \sqrt{(1 + h^r)(1 + h^t)}\), \(\Sigma = r^2 + a^2 \cos^2 \theta\), \(\Delta = r^2 + a^2 - 2mr\), and \(h^{r,t} = \sum_{k=0}^{\infty} (\varepsilon_{2k}^{r,t} + \varepsilon_{2k+1}^{r,t} \frac{mr}{\Sigma}) \left( \frac{m^2}{\Sigma} \right)^k\).

Asymptotic flatness imposes \(\varepsilon_0^{r,t} = 0\), while the mass is \(M = m(1 - \varepsilon_1^t/2)\). Caveat: seems to be mapped to known static solutions but not stationary.

There are other more successful approaches in that respect.\(^10\)


Another approach: metrics with special properties, i.e., a **Carter constant**. Such examples are the bumpy Kerr and the Johannsen metric,\(^\text{11}\)

\[
\begin{align*}
    ds^2 &= -\frac{2a [(r^2 + a^2)A_1(r)A_2(r) - \Delta]}{[(r^2 + a^2)A_1(r) - a^2A_2(r)\sin^2 \theta]^2} \tilde{\Sigma} \sin^2 \theta d\varphi dt + \frac{\tilde{\Sigma} dr^2}{\Delta A_5(r)} + \tilde{\Sigma} d\theta^2 \\
    &= -\frac{\tilde{\Sigma} \left[\Delta - a^2A_2(r)^2 \sin^2 \theta\right]}{[(r^2 + a^2)A_1(r) - a^2A_2(r)\sin^2 \theta]^2} dt^2 + \frac{\tilde{\Sigma} \sin^2 \theta \left[(r^2 + a^2)A_1(r)^2 - a^2 \Delta \sin^2 \theta\right]}{[(r^2 + a^2)A_1(r) - a^2A_2(r)\sin^2 \theta]^2} d\varphi^2
\end{align*}
\]

where \(\tilde{\Sigma} = \Sigma + f(r)\), and \(A_1, A_2, A_5,\) and \(f\) are expansions in powers of \(1/r\). The Johannsen metric can be related to the bumpy Kerr and these two can describe slowly rotating dCS, and some static EdGB BHs.

**History note:** Johannsen’s metric is of the form of Carter’s canonical metric\(^\text{12}\),

\[
    ds^2 = \frac{Z}{\Delta_r} dr^2 + \frac{Z}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta}{Z} \left(P_r d\varphi - Q_r dt\right)^2 + \frac{\Delta_r}{Z} \left(Q_\theta dt - P_\theta d\varphi\right)^2 ,
\]

where \(Z = P_r Q_\theta - Q_r P_\theta\), and the \(P, Q,\) and \(\Delta\) are functions of either \(r\) or \(\theta\).


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