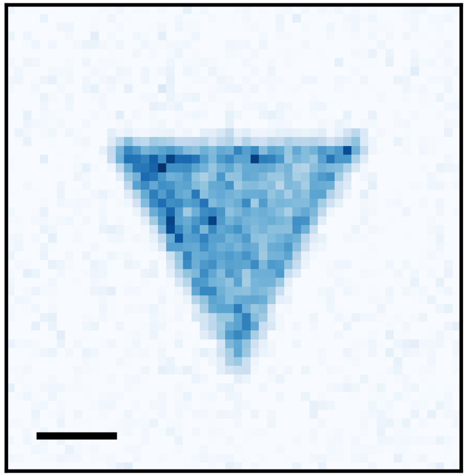
Scale-invariant dynamics and breathers of an interacting 2D Bose gas

Raphaël Saint-Jalm, Patricia C.M. Castilho, Édouard Le Cerf, Brice Bakkali-Hassani,
Sylvain Nascimbène, Jean Dalibard, Jérôme Beugnon



arXiv:1903.04528,
accepted in PRX

Atomtronics meeting, 6th of May 2019, Benasque

Introduction

- Scale invariance: When a physical system and its dilation behave similarly.
- Useful in high energy physics, in statistical physics, etc.
- How do we know a physical system is scale invariant?

$\rightarrow$ Look at its action $$S = \int \mathrm{d}t \, \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$$

$\rightarrow$ rescale space (and time), e.g: $\boldsymbol{q} \rightarrow \boldsymbol{q}/\lambda, \quad \dot{\boldsymbol{q}} \rightarrow \lambda \dot{\boldsymbol{q}}, \quad t \rightarrow t/\lambda^2.$

Is S invariant?

Examples:

1. The ideal gas: $\mathcal{L} = \frac{1}{2} \sum_i m_i \dot{\boldsymbol{q}}_i^2$
2. N particles interacting with $U(\boldsymbol{q}_{ij}) \propto 1/\boldsymbol{q}_{ij}^2$
3. In cold atoms: Unitary Fermi gas

Consequences:

$N = 1$: Kepler problem.

Laplace-Runge-Lenz vector constant, closed orbits ($E < 0$)

Spectrum of the hydrogen atom

No bulk viscosity

Introduction: the 2D Bose gas

- Weakly interacting bosons: description with a classical field $\psi(\boldsymbol{r}, t)$,

  normalization $\int \mathrm{d}^2\boldsymbol{r}\,|\psi(\boldsymbol{r}, t)|^2 = 1,$ density $n(\boldsymbol{r}, t) = N|\psi(\boldsymbol{r}, t)|^2$

- 2D gas: Interaction energy of N particles of mass m:

$$E_{\mathrm{int}} = \frac{N^2\hbar^2}{2m}\,\tilde{g}\int \mathrm{d}^2\boldsymbol{r}\,|\psi(\boldsymbol{r}, t)|^4,$$

  $\tilde{g}$: dimensionless parameter. Description valid if $\tilde{g} \ll 1$

- Full Lagrangian:

$$\mathcal{L}[\psi] = \int \mathrm{d}^2\boldsymbol{r}\left[i\hbar\psi^*\frac{\partial\psi}{\partial t} - \frac{N\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2 - \frac{N^2\hbar^2}{2m}\tilde{g}|\psi|^4\right]$$

- Transformation $\boldsymbol{r} \to \boldsymbol{r}/\lambda$, $t \to t/\lambda^2$. Then $\psi(\boldsymbol{r}) \to \lambda\psi(\lambda\boldsymbol{r})$ (normalization).

  We have $\mathcal{L} \to \lambda^2\mathcal{L}$, and S is unchanged.

- Universal thermodynamics at equilibrium: Hung et al. Nature **140**, 236 (2011)
  Yefsah et al. PRL **107**, 130401 (2011)

Introduction: the 2D Bose gas in a harmonic trap

- Add a harmonic trap of frequency ω:

$$E_\mathrm{p} = \frac{1}{2}\int \mathrm{d}^2\boldsymbol{r}\; m\omega^2\boldsymbol{r}^2\; |\psi|^2$$

- Scaling transformation: $E_\mathrm{p} \to E_\mathrm{p}/\lambda^2$

No scale invariance anymore!

However:

- Some transformations keep the action invariant.
- Interesting consequences on the dynamics of the gas: breathing mode at 2ω

Pitaevskii and Rosch, PRA **55**, R853 (1997)

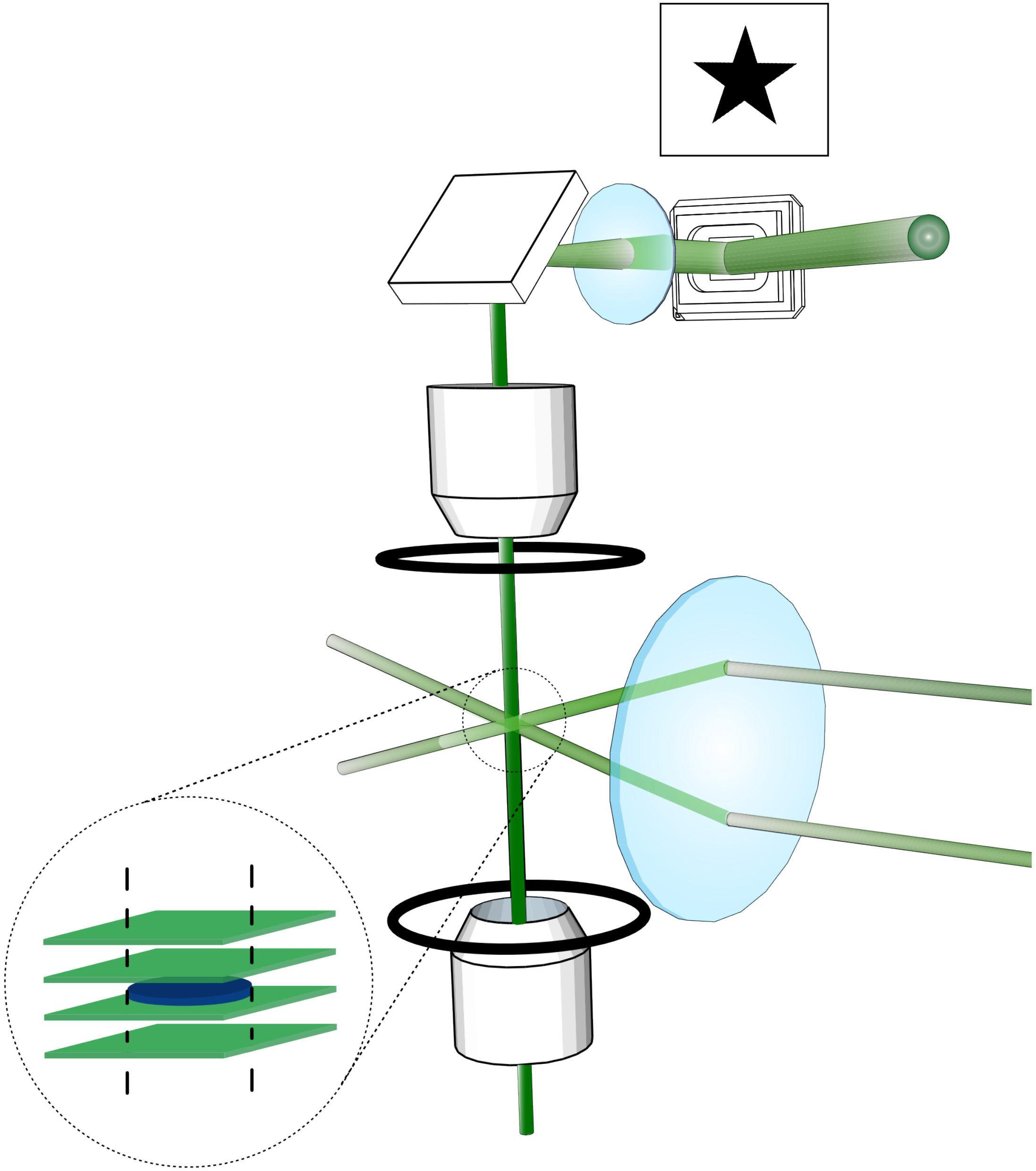
Chevy et al., PRL **88**, 250402 (2001)

Vogt et al., PRL **108**, 070404 (2012)

Experimental setup

- 2D confinement: $\omega_z = 2\pi \cdot 4\,\mathrm{kHz}$
- Repulsive walls:
  - $\rightarrow$ uniform gas
  - $\rightarrow$ arbitrary shape

- Internal state: F=1, m=0.
- Harmonic trap: with a magnetic field
- At $t = 0$, transfer to F=1, m=-1
  - $\rightarrow \omega \approx 2\pi \cdot 20\,\mathrm{Hz}$



Goal of this talk: Investigate the symmetries of this system with clouds far out-of-equilibrium in the harmonic trap.

Outline of this talk

Introduction

1. Symmetry group of the 2D Bose gas

2. Linking evolutions of different clouds

3. Breathers of the Gross–Pitaevskii equation

Conclusion and outlook

Outline of this talk

SO(2,1) symmetry

Gross–Pitaevskii equation (GPE) of a free cloud:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{m}\tilde{g}N|\psi|^2\psi.$$

Three types of interesting transformations:

- A time translation: $\boldsymbol{r},\ t \quad \rightarrow \quad \boldsymbol{r},\ t+\beta$
- A scaling transformation: $\boldsymbol{r},\ t \quad \rightarrow \quad \boldsymbol{r}/\lambda,\ t/\lambda^2$
- An "expansion": $\boldsymbol{r},\ t \quad \rightarrow \quad \boldsymbol{r}/(\gamma t+1),\ t/(\gamma t+1)$

Combine them: $\begin{cases} \boldsymbol{r} \\ t \end{cases} \rightarrow \begin{cases} \dfrac{\boldsymbol{r}}{\gamma t+\delta} \\ \dfrac{\alpha t+\beta}{\gamma t+\delta} \end{cases}, \qquad \alpha\delta-\beta\gamma=1$

Matrix $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$.

Infinitesimal generators: $L_1 = \frac{i}{2}\left(\begin{smallmatrix}1 & 0 \\ 0 & -1\end{smallmatrix}\right)$, $L_2 = \frac{i}{2}\left(\begin{smallmatrix}0 & 1 \\ 1 & 0\end{smallmatrix}\right)$ and $L_3 = \frac{i}{2}\left(\begin{smallmatrix}0 & -1 \\ 1 & 0\end{smallmatrix}\right)$

$$[L_1, L_2] = -iL_3, \quad [L_2, L_3] = iL_1, \quad [L_3, L_1] = iL_2 \qquad \longrightarrow \quad \mathrm{SO}(2,1)$$

SO(2,1) symmetry: with a harmonic trap

- GPE:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{m}\tilde{g}N|\psi|^2\psi + \frac{1}{2}m\omega^2\boldsymbol{r}^2\psi.$$

- Transformations:

$$\begin{cases} \boldsymbol{r} \\ \eta = \tan(\omega t) \end{cases} \rightarrow \begin{cases} \boldsymbol{r}/\lambda(t) \\ \eta' = (\alpha\eta+\beta)/(\gamma\eta+\delta) \end{cases}, \qquad \alpha\delta-\beta\gamma=1$$

$$\lambda(t)=\Big[(\alpha\sin(\omega t)+\beta\cos(\omega t))^2+(\gamma\sin(\omega t)+\delta\cos(\omega t))^2\Big]^{1/2}$$

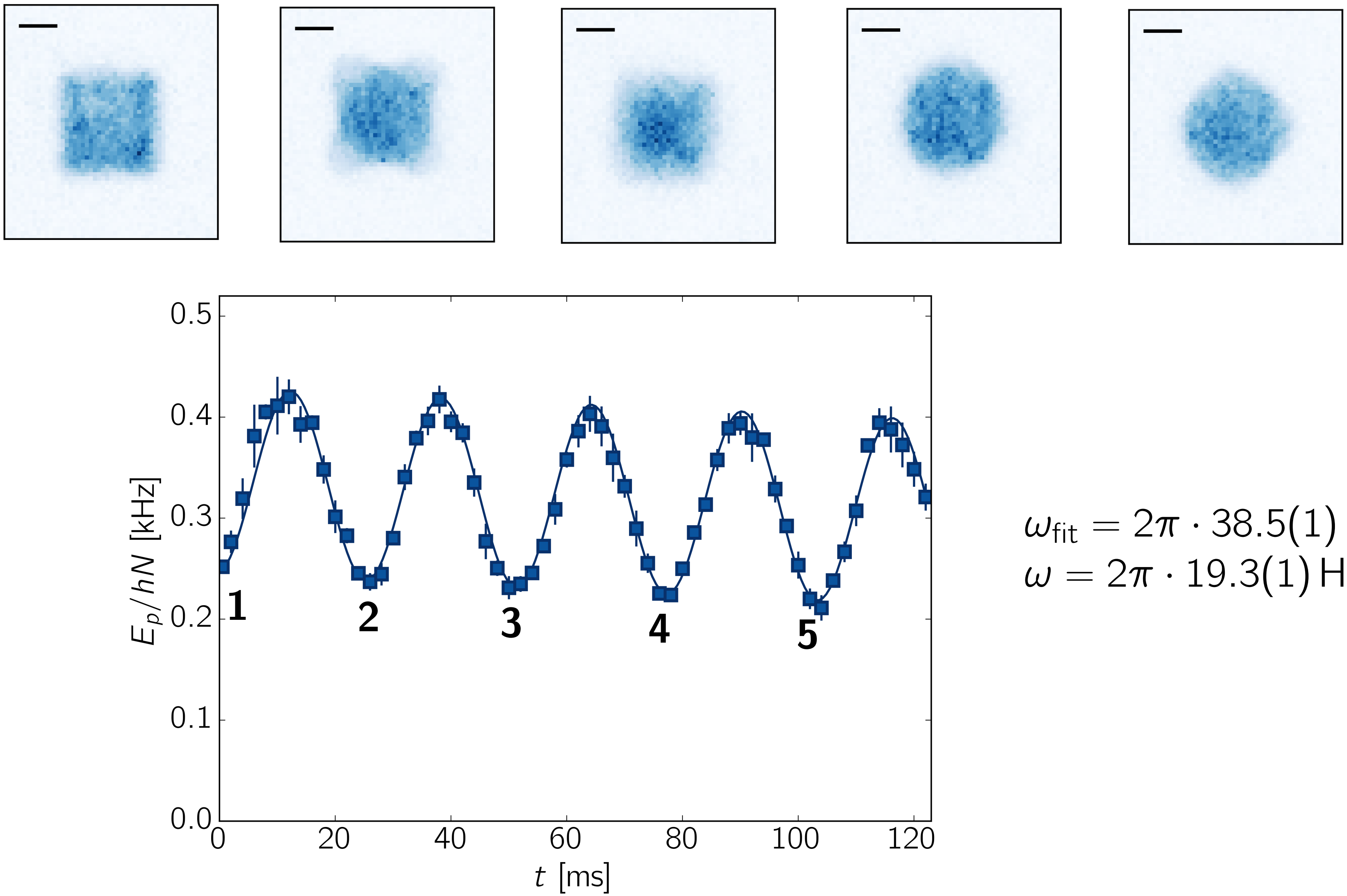
The symmetry group is still SO(2,1).

- A strong consequence: $E_\mathrm{p} \propto \sin(2\omega t)$

$\rightarrow$ Breathing mode at 2ω

We can test this on the experiment.

Periodic potential energy



$\omega_{\text{fit}} = 2\pi \cdot 38.5(1)\,\text{Hz}$
$\omega = 2\pi \cdot 19.3(1)\,\text{Hz}$

Outline of this talk

Introduction

1. Symmetry group of the 2D Bose gas
2. Linking evolutions of different clouds
3. Breathers of the Gross–Pitaevskii equation

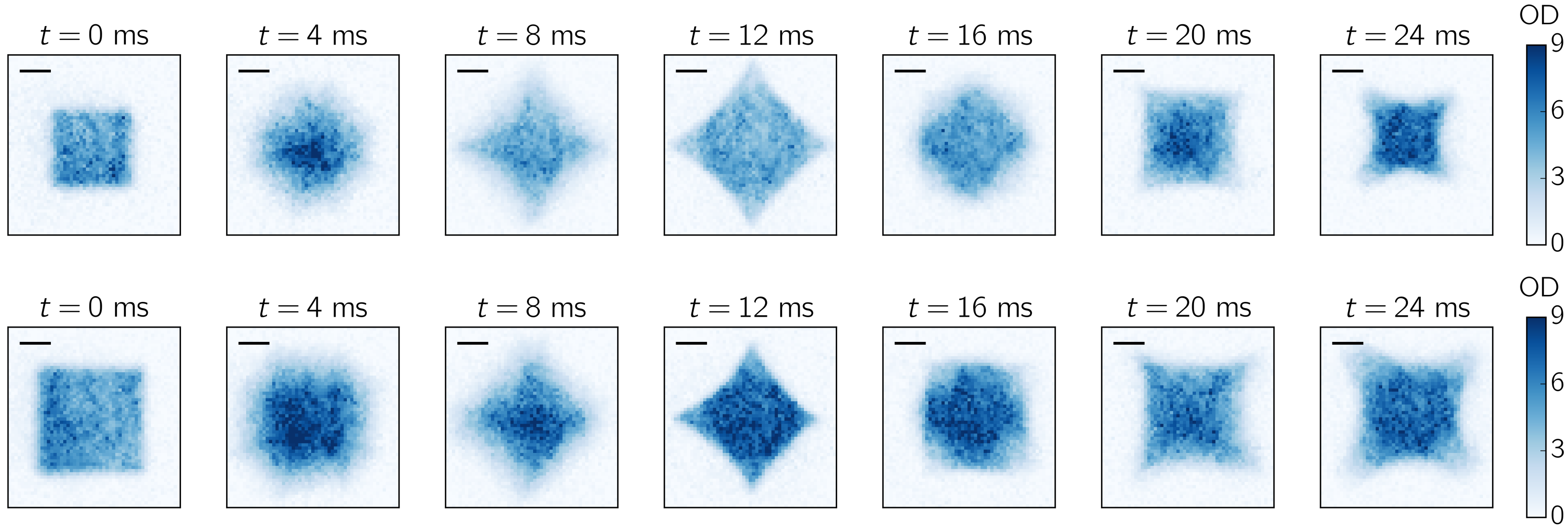
Conclusion and outlook

Goal of this part

$$\omega_1 = \omega_2 = 2\pi \cdot 20\,\mathrm{Hz}$$

$L_1 = 27\,\mu\mathrm{m}$
$N_1 = 3.7 \cdot 10^4$

$L_2 = 37\,\mu\mathrm{m}$
$N_2 = 5.4 \cdot 10^4$



Evolution of ψ_1

$$\begin{cases} L_1 \\ N_1 \\ \omega_1 \\ \tilde{g}_1 \end{cases}$$

same initial shape

$\longleftrightarrow$

Symmetries

Evolution of ψ_2

$$\begin{cases} L_2 \\ N_2 \\ \omega_2 \\ \tilde{g}_2 \end{cases}$$

Three parameters: $\delta = L_2/L_1$,

$$\mu = (\tilde{g}_2 N_2/\tilde{g}_1 N_1)^{1/2},$$

$$\zeta = \omega_2/\omega_1$$

Evolution of clouds with the same $\tilde{g}N$

- GPE:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{m}\tilde{g}N|\psi|^2\psi + \frac{1}{2}m\omega^2\boldsymbol{r}^2\psi.$$

- **Evolution of a dilation:** Apply a SO(2,1) transformation to a solution $\psi(\boldsymbol{r}, t)$:

$$\psi(\boldsymbol{r}, t) \quad \to \quad \lambda(t)\,\psi(\lambda(t)\boldsymbol{r}, \tau)$$

$$\lambda(t) = \left[\delta^2\cos^2(\omega t) + \frac{1}{\delta^2}\sin^2(\omega t)\right]^{-1/2}$$

$$\delta^2\tan(\omega\tau) = \tan(\omega t)$$

$$\begin{array}{ccc} L_1 & & L_2 = \delta L_1 \\ \omega_1 & & \omega_2 = \omega_1 \end{array}$$

Choice $\beta = \gamma = 0$: cloud at rest at $t = 0$.

- **Evolution with a different ω:** Apply a simple scaling transformation to $\psi(\boldsymbol{r}, t)$:

$$\psi(\boldsymbol{r}, t) \quad \to \quad \lambda\,\psi(\lambda\boldsymbol{r}, \lambda^2 t)$$

$$\begin{array}{ccc} L_1 & & L_2 = L_1/\lambda \\ \omega_1 & & \omega_2 = \omega_1/\lambda^2 \end{array}$$

- Know $\psi(\boldsymbol{r}, t)$ $\to$ know the evolution of any dilation in a trap of any frequency

Evolution of clouds with different $\tilde{g}N$'s

- Only in the hydrodynamic regime: Translate GPE in terms of

$$n(\boldsymbol{r}) = N|\psi(\boldsymbol{r})|^2 \qquad \boldsymbol{v}(\boldsymbol{r}) = \frac{\hbar}{m}\frac{\mathrm{Im}(\psi^*(\boldsymbol{r})\nabla\psi(\boldsymbol{r}))}{n(\boldsymbol{r})}:$$

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\boldsymbol{v}) = 0$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\frac{1}{2}m\boldsymbol{v}^2 + \frac{\hbar^2}{m}\tilde{g}n + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \cancel{\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}}\right) = 0$$

Healing length
$\xi \ll L$

- Invariant under $\boldsymbol{r} \to \boldsymbol{r}, \quad t \to t/\mu, \quad \tilde{g}n \to \mu^2\tilde{g}n, \quad \boldsymbol{v} \to \mu\boldsymbol{v}, \quad \omega \to \mu\omega.$

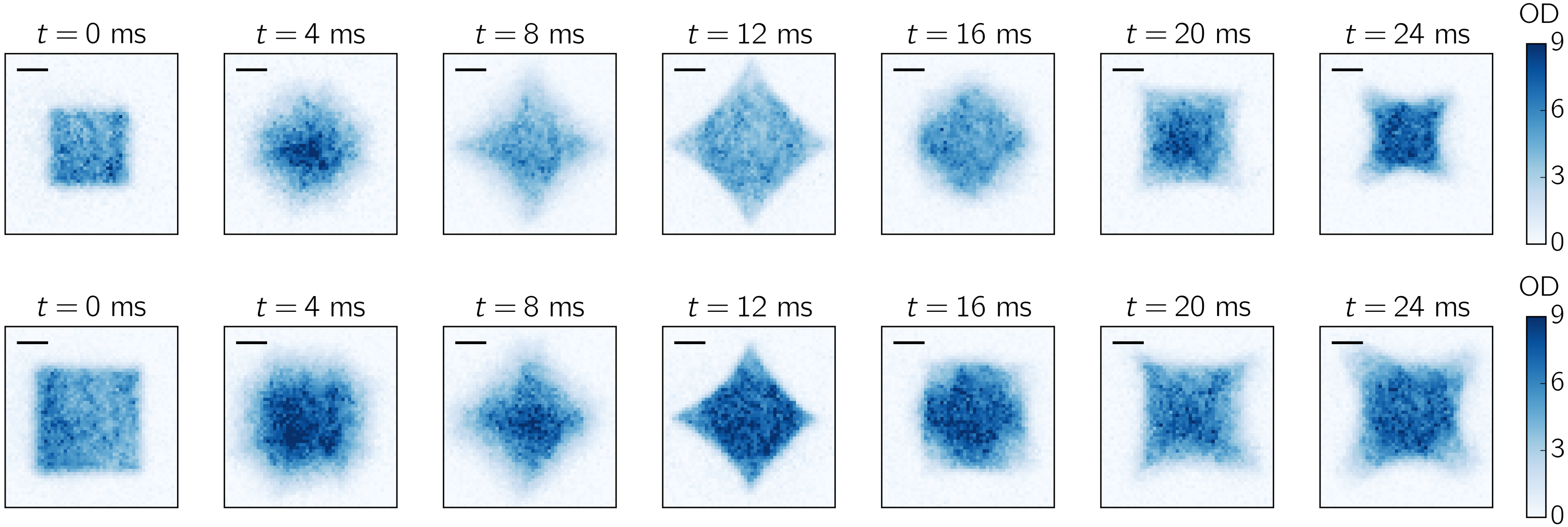
Summary: Link evolution $\psi(\boldsymbol{r}, t)$ (size L_1, N_1, ω_1, $\tilde{g}_1$) with evolution of:
(1) any dilation of ψ with size $L_2 = \delta L_1$
(2) in any trap frequency $\omega_2 = \zeta\omega_1$
(3) with any atom number N_2 and parameter $\tilde{g}_2$, with $\tilde{g}_2 N_2 = \mu^2\tilde{g}_1 N_1$

$$\lambda(t) = \left[\delta^2\cos^2(\omega_2 t) + \left(\frac{\mu}{\delta\zeta}\right)^2\sin^2(\omega_2 t)\right]^{-1/2} \qquad \tan(\omega_1\tau) = \frac{\mu}{\zeta\delta^2}\tan(\omega_2 t)$$

On the experiment

$L_1 = 27\,\mu\text{m}$
$N_1 = 3.7 \cdot 10^4$

$L_2 = 37\,\mu\text{m}$
$N_2 = 5.4 \cdot 10^4$



Overlap:

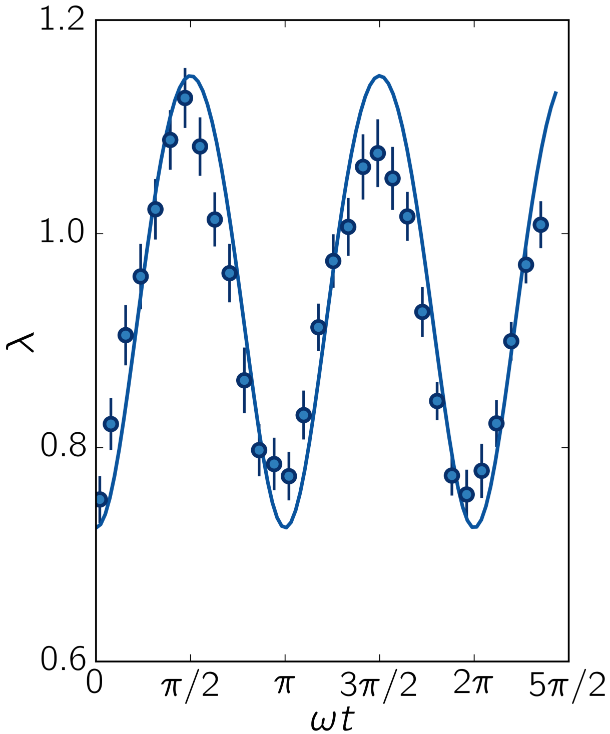
$$\mathcal{O}(n_1, n_2) = \max_{\lambda} \left(\lambda \int d^2 r \frac{n_1(\lambda \boldsymbol{r}) n_2(\boldsymbol{r})}{\|n_1\| \|n_2\|} \right)$$

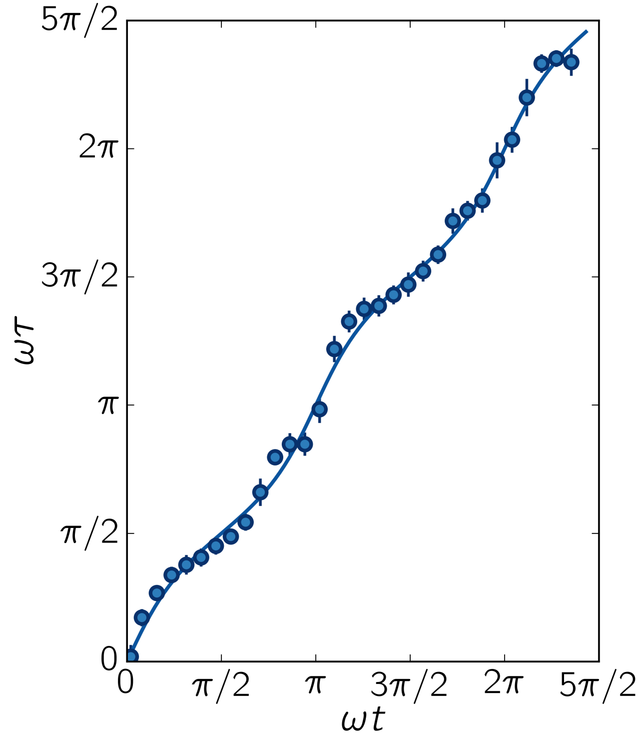
Theory, no free parameter:

$$\lambda(t) = \left[\delta^2 \cos^2(\omega t) + \left(\frac{\mu}{\delta}\right)^2 \sin^2(\omega t) \right]^{-1/2}$$

$$\tan(\omega \tau) = \frac{\mu}{\delta^2} \tan(\omega t)$$

Three experiments to vary
δ, μ, ζ independently:
arXiv:1903.04528





Outline of this talk

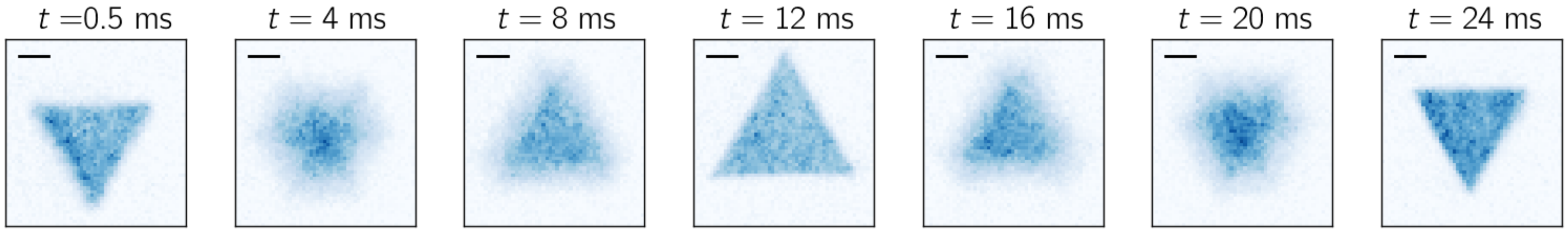
Introduction

1. Symmetry group of the 2D Bose gas
2. Linking evolutions of different clouds
3. Breathers of the Gross–Pitaevskii equation

Conclusion and outlook

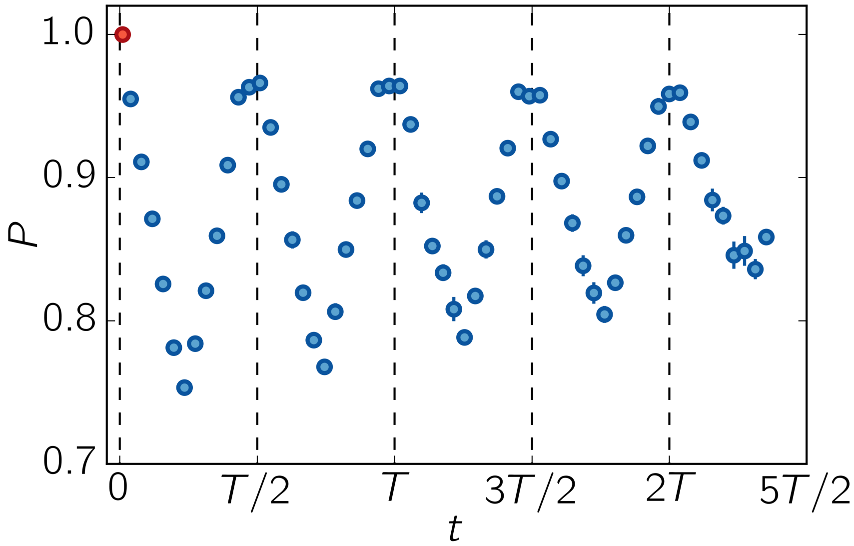
Periodic evolution of a wave function

- Wave function ψ in a harmonic trap: not necessarily periodic.
- A triangle seems to be periodic, period $T/2 \to$ breather of the GPE



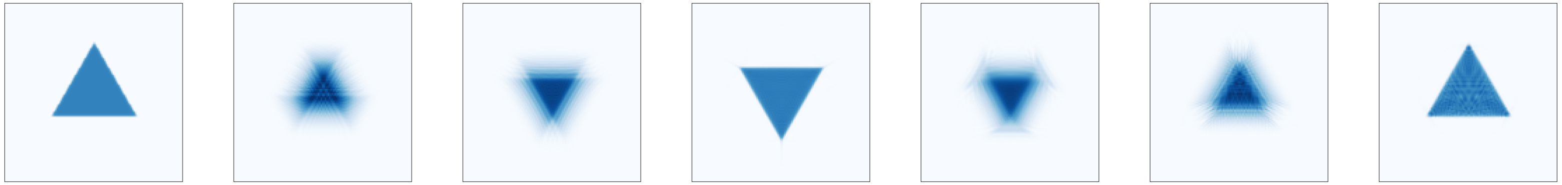
T=48 ms

- Overlap of the images: $P = \frac{\int \mathrm{d}^2 r\, n(\boldsymbol{r},0)\, n(\boldsymbol{r},t)}{\|n(\boldsymbol{r},0)\|\|n(\boldsymbol{r},t)\|}$

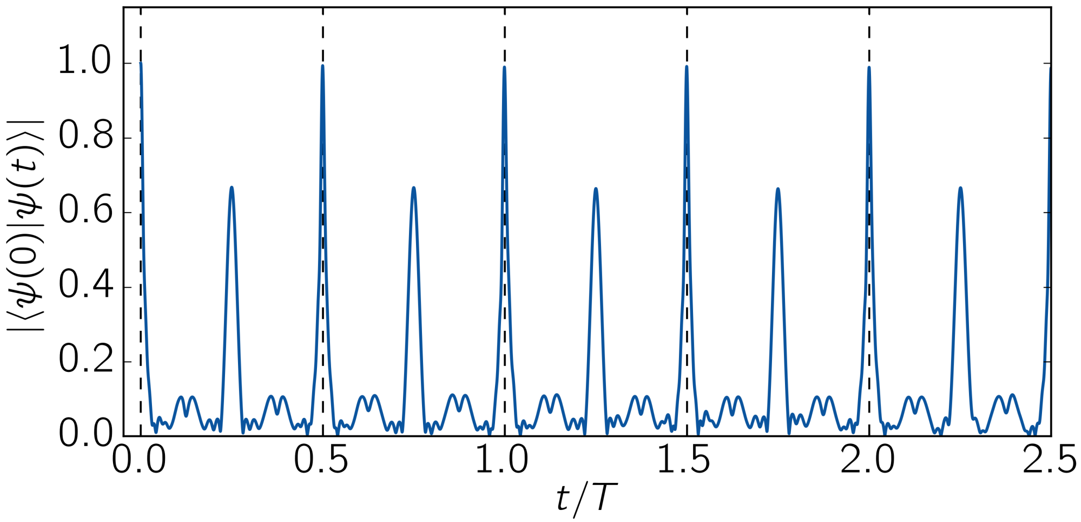


Numerical simulations

- Grid size $N_s \times N_s$ with $N_s = 128$, healing length $\xi = 0.5\ell$



- Overlap between the wave functions: $|\langle\psi(0)|\psi(t)\rangle|$

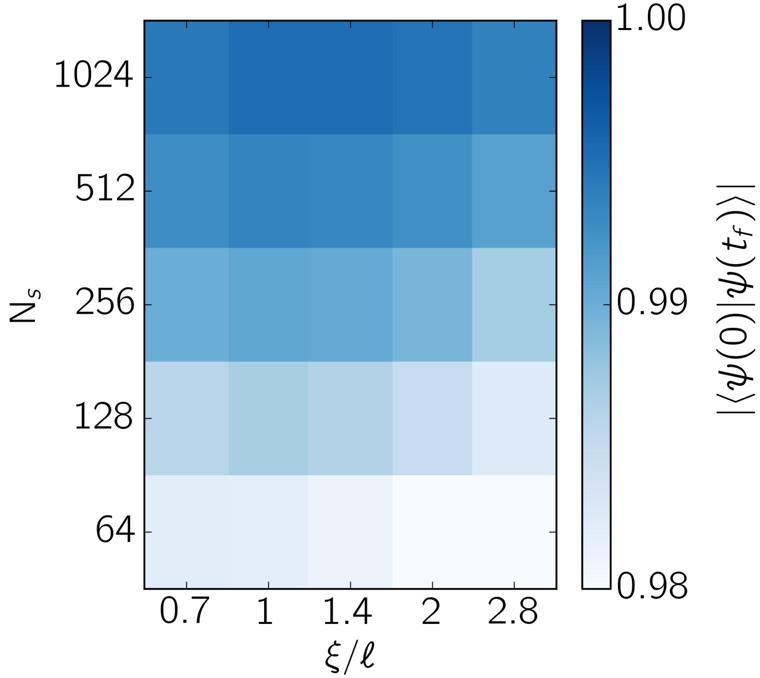


- No analytical proof of periodicity
- Finite-size scaling on the numerical simulations

Finite-size scaling

- Two choices to improve a simulation:
  (a) Increase N_s (reduce pixel size ℓ) and keep ξ the same
  (b) Reduce ξ: more in the Thomas-Fermi regime

Overlap between $\psi(0)$ and $\psi(T/2)$:

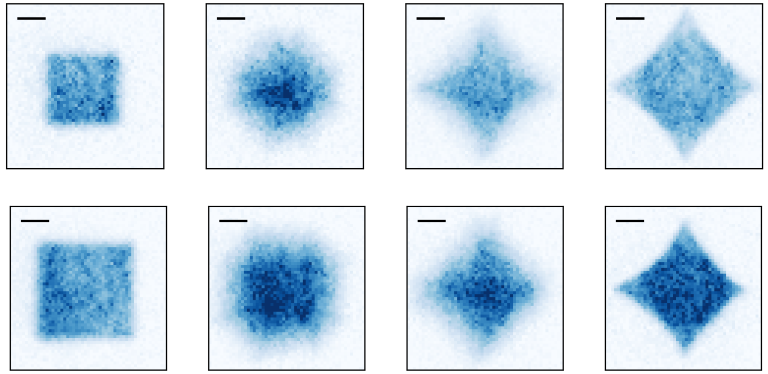


- Best overlap here: > 0.995

- Other shapes? Square, pentagon, hexagon, star, etc. not periodic

  But a disk-shaped cloud also seems to evolve periodically.

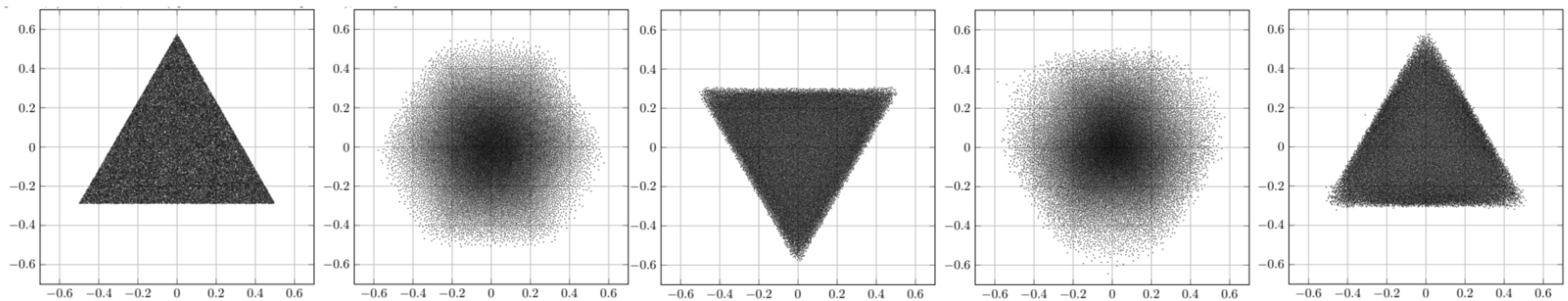
Conclusion



- Dynamical symmetries of the 2D Bose gas:
  Vary L, N, $\tilde{g}$, $\omega \rightarrow$ same universal dynamics
- Breathers of the GPE: this dynamics is periodic
- **Open questions:**
  $\rightarrow$ Periodic only with contact interactions?
  $\rightarrow$ Resilient to quantum anomaly (increase $\tilde{g}$)?
  Olshanii et al. PRL, **105**, 095302 (2010)
  Holten et al. PRL, **121**, 120401 (2018)
  $\rightarrow$ Effect of temperature?
  $\rightarrow$ Breathers for unitary Fermi gas in 3D?
  $\rightarrow$ ...

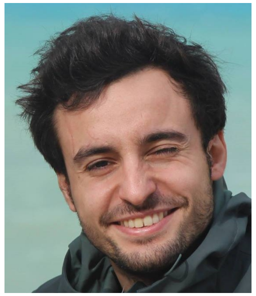
Conclusion

- Classical particles with interaction $\propto 1/\boldsymbol{r}^2$
- Numerical simulation with 20 000 particles
- Initial condition: uniform filling of a triangle, $\boldsymbol{v}_i = 0$



$t = 0$ $t = T/8$ $t \approx T/4$ $t \approx 3T/8$ $t \approx T/2$

The team



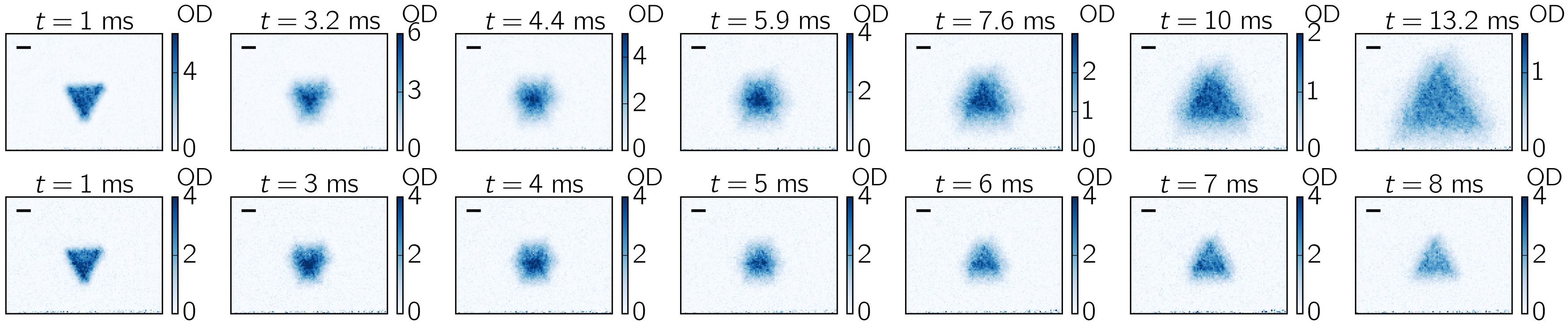
Brice

Thank you for your attention!

Evolution with different ω's

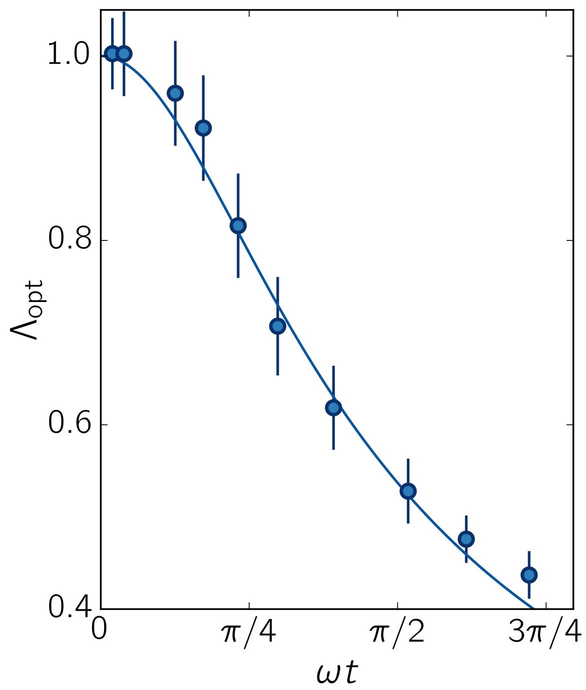
$\omega_1 = 0$

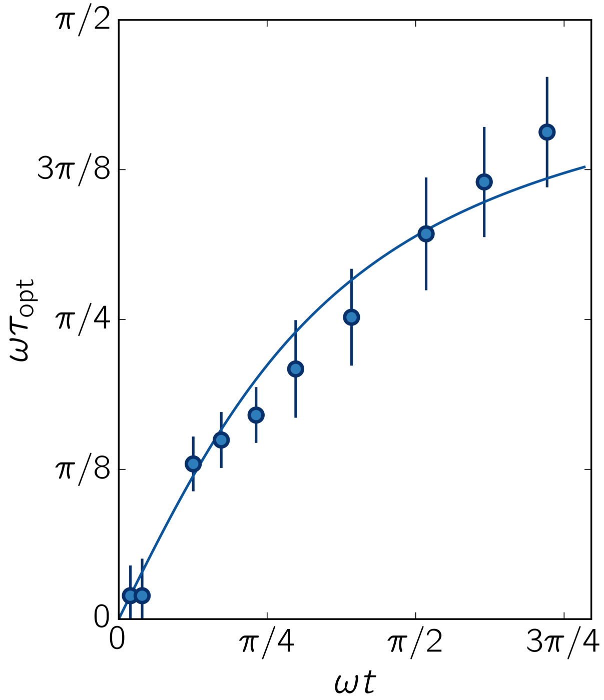
$\frac{\omega_2}{2\pi} \approx 20\,\mathrm{Hz}$



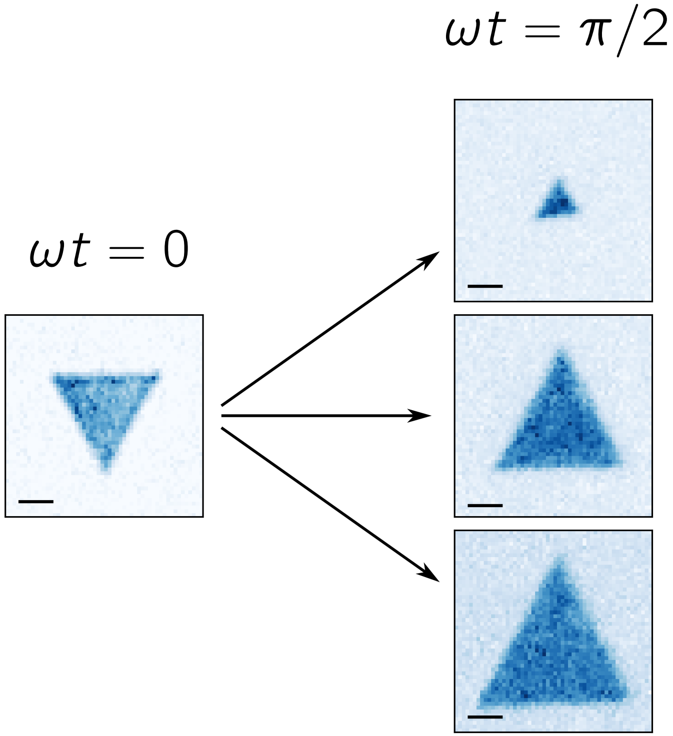
$$\Lambda(t) = \left(1 + \omega_2^2 t^2\right)^{-1/2}$$

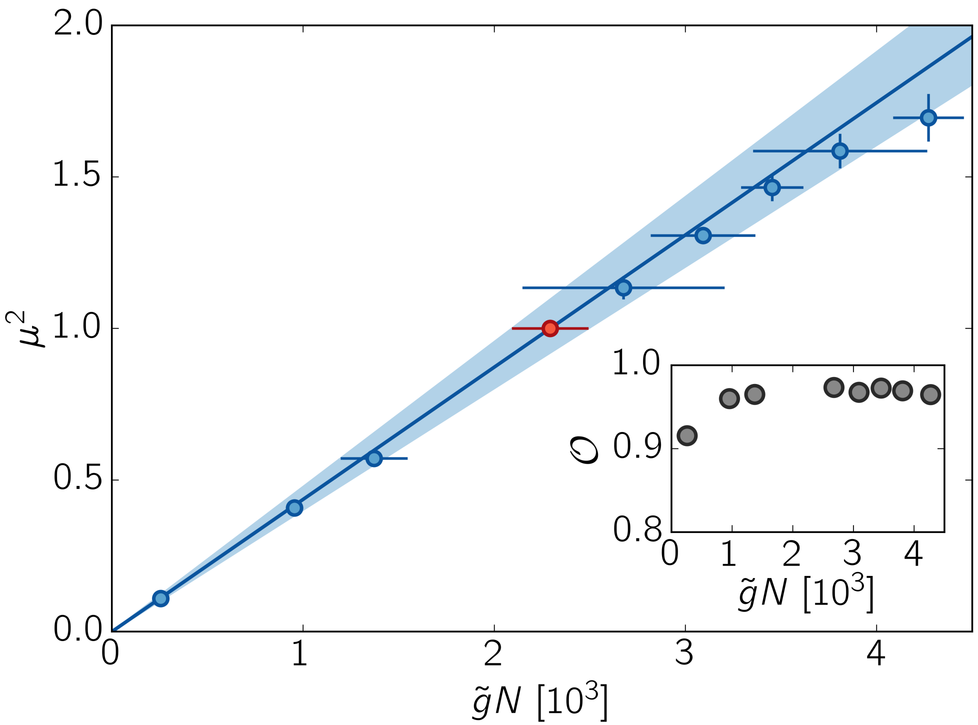
$$\omega_2 \tau(t) = \arctan(\omega_2 t)$$





Evolution with different $\tilde{g}N$'s

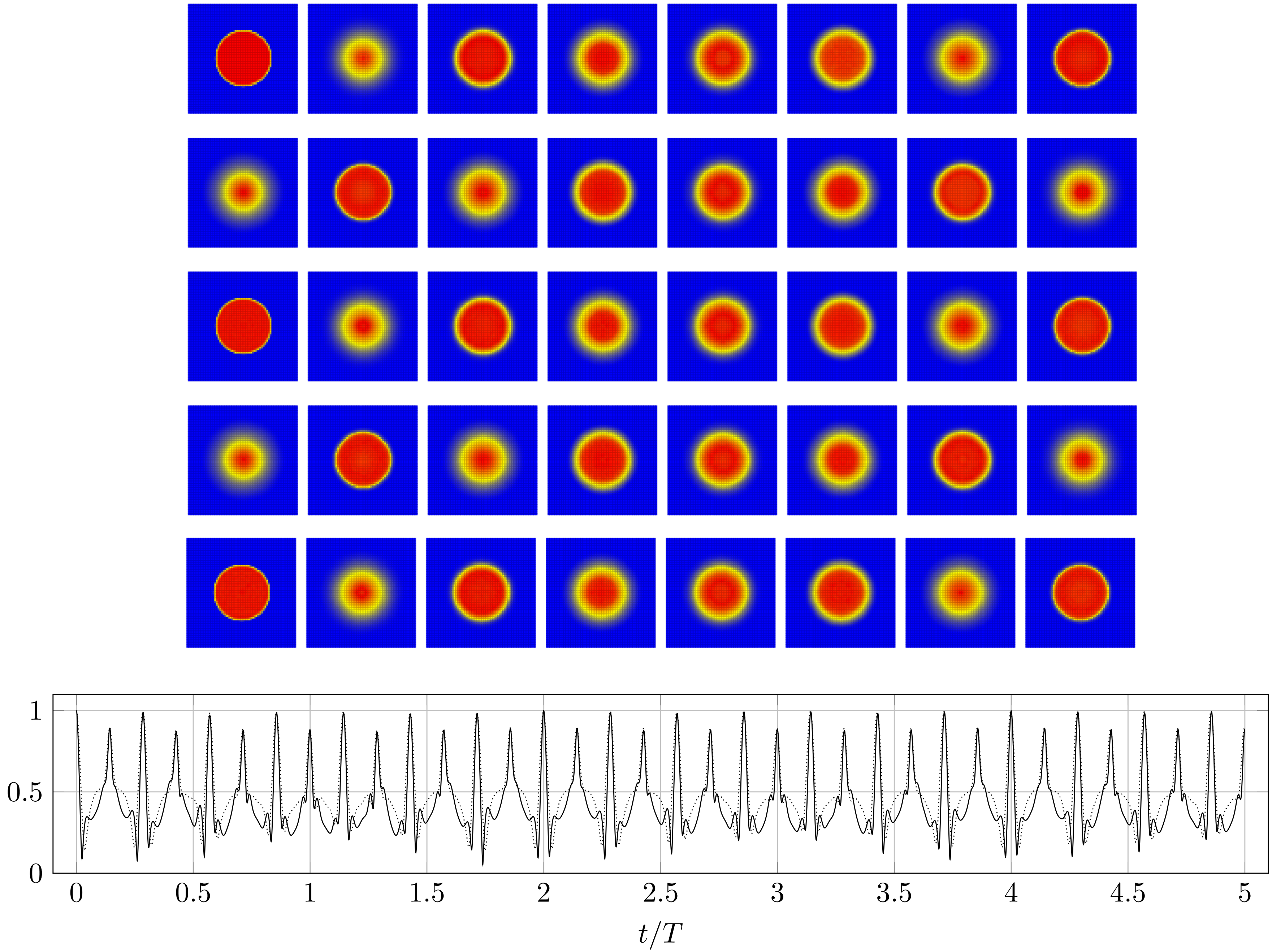




$$\lambda(t) = \left[\delta^2 \cos^2(\omega_2 t) + \left(\frac{\mu}{\delta\zeta}\right)^2 \sin^2(\omega_2 t)\right]^{-1/2}$$

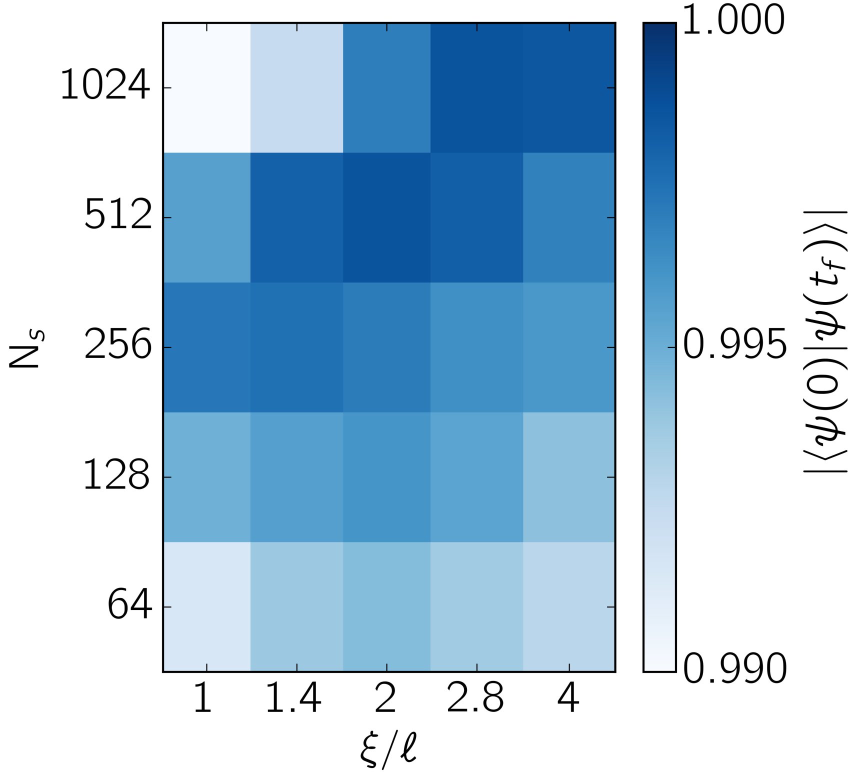
$$\lambda(t = \pi/2\omega) = \frac{\zeta\delta}{\mu}$$

Breathers: disk



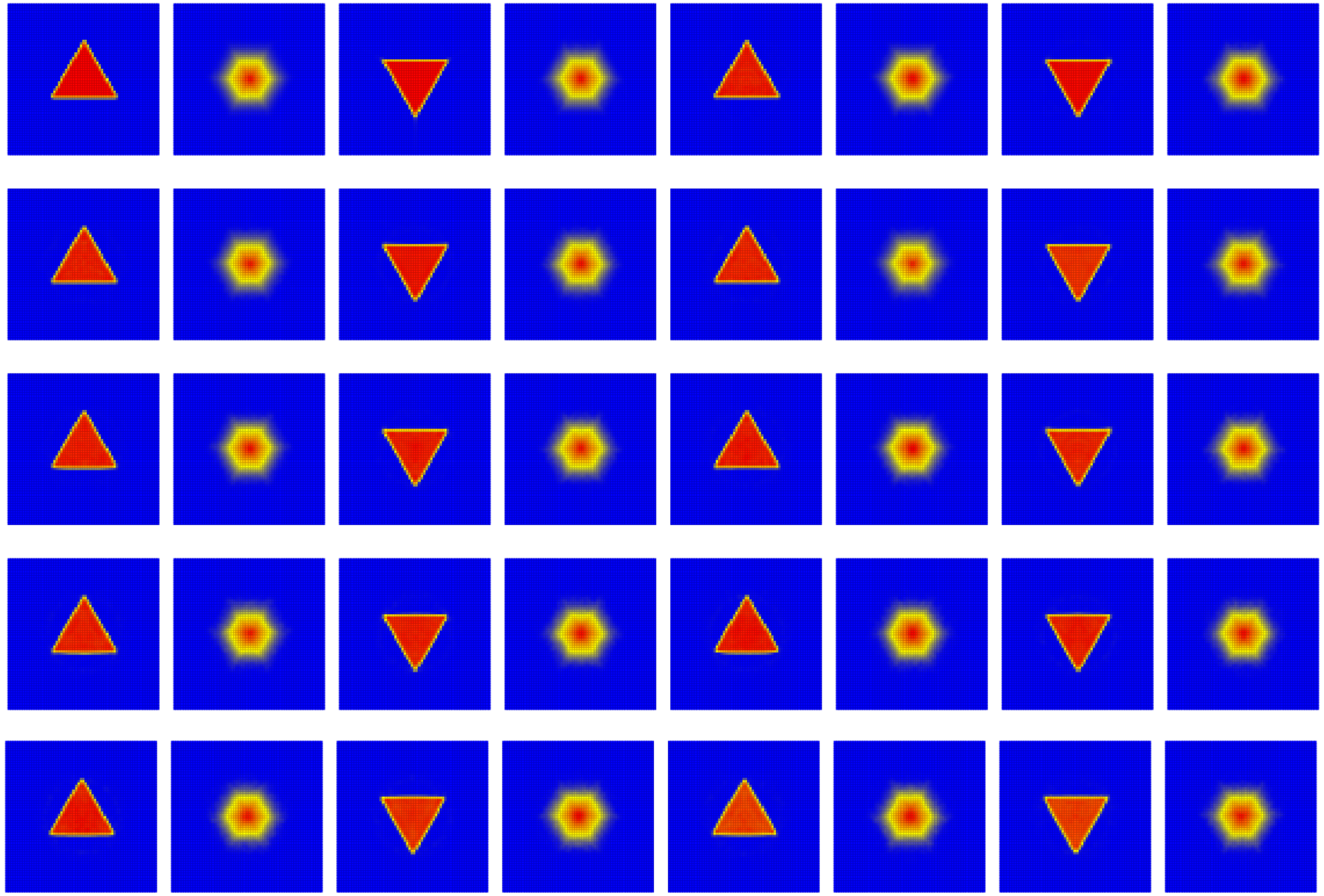
Finite-size scaling: disk

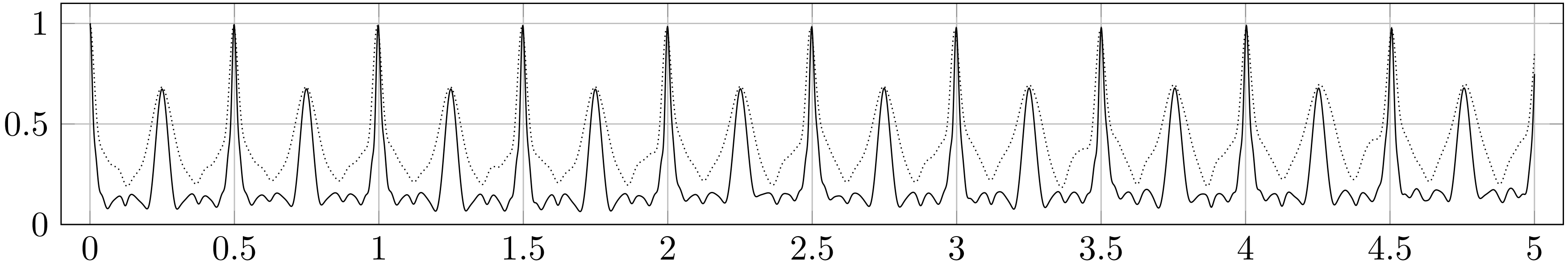
Overlap between $\psi(t=0)$ and $\psi(t=4\pi/\omega)$.



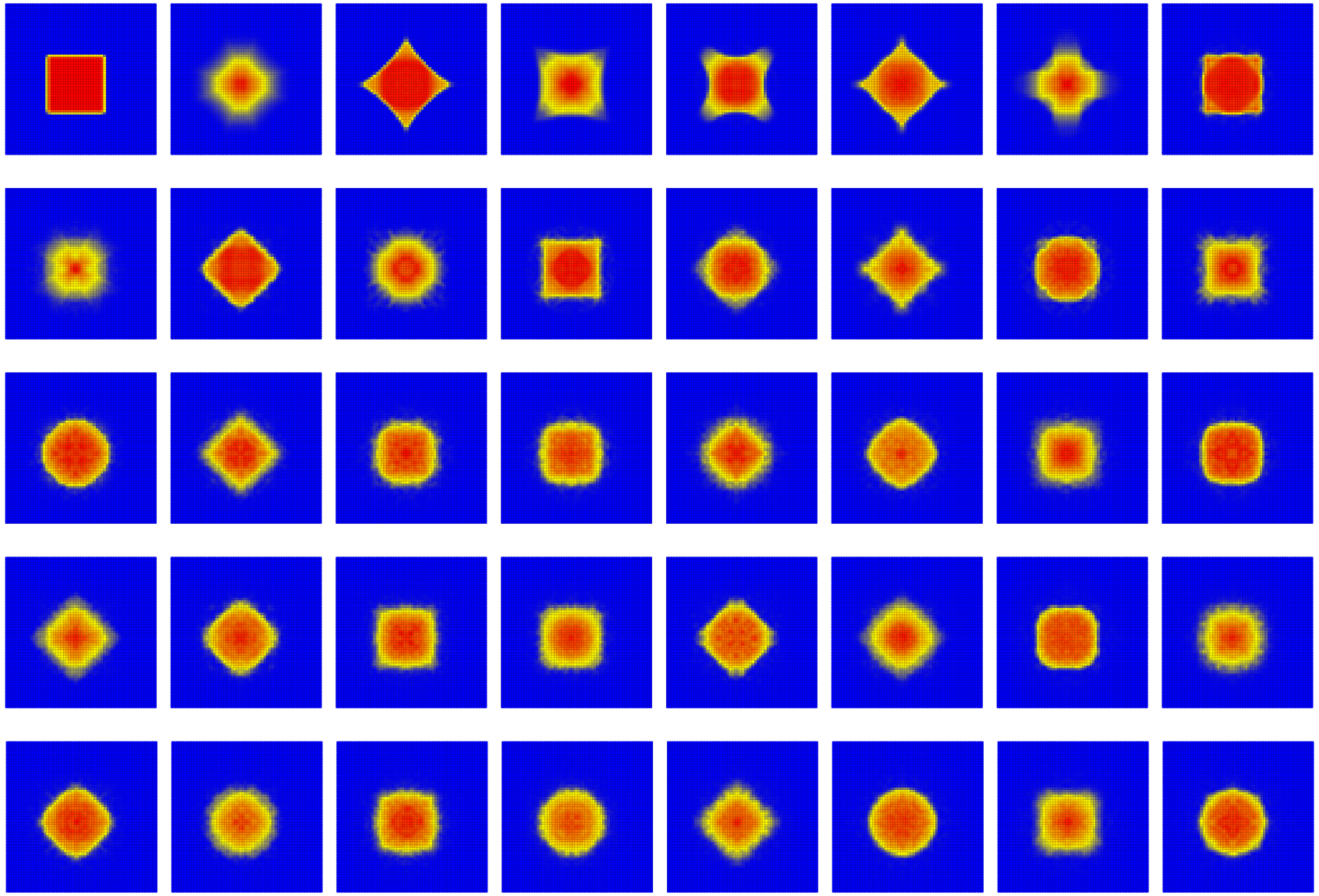
Best overlap > 0.998.

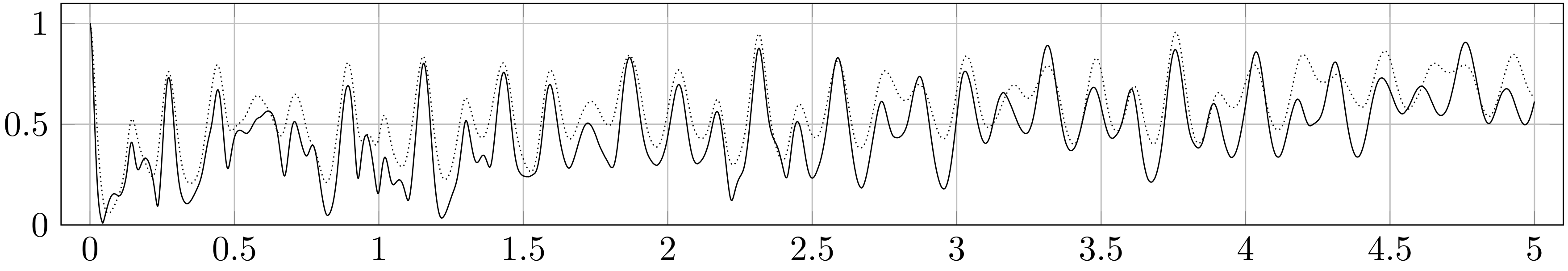
Numerical simulations: Triangle



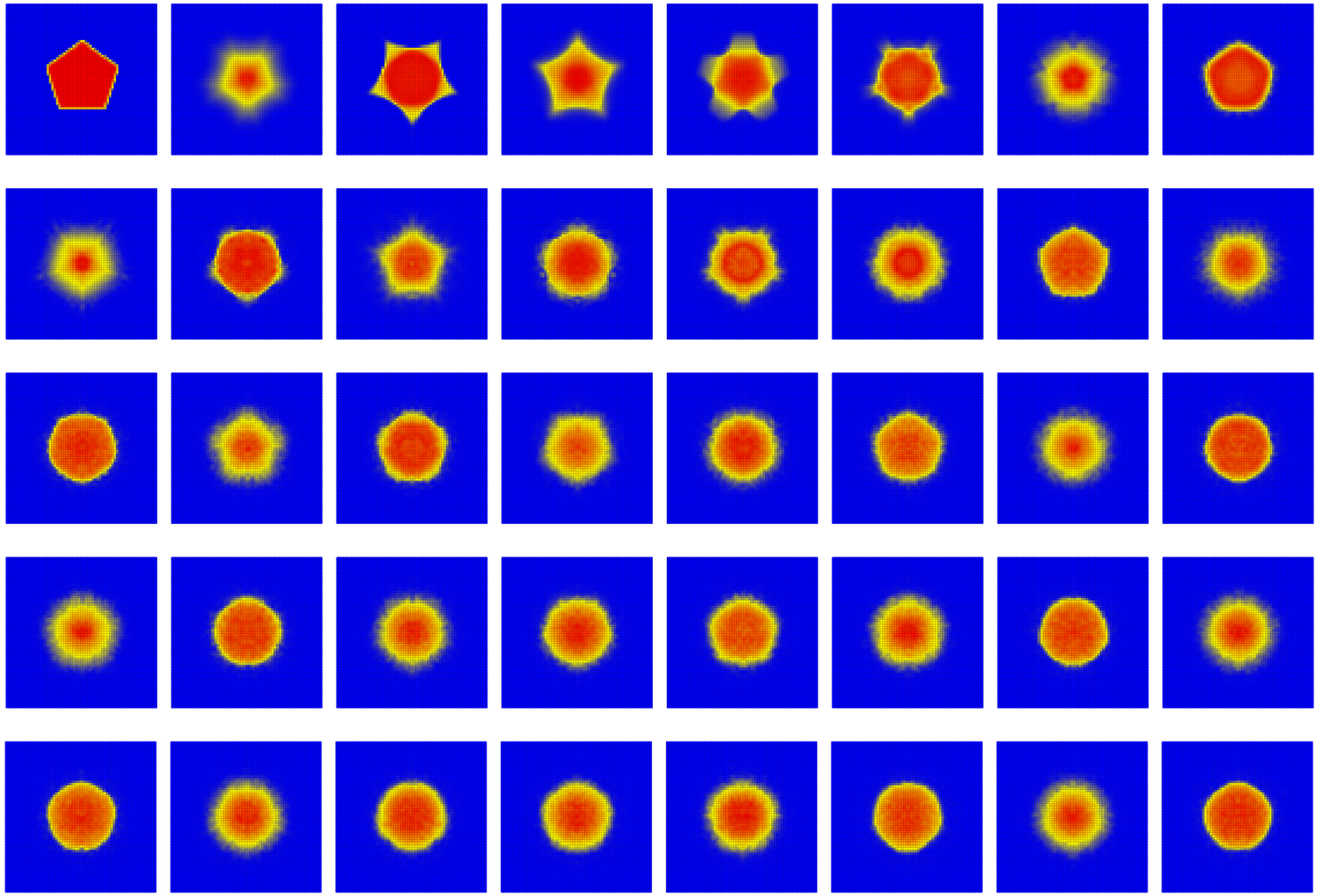


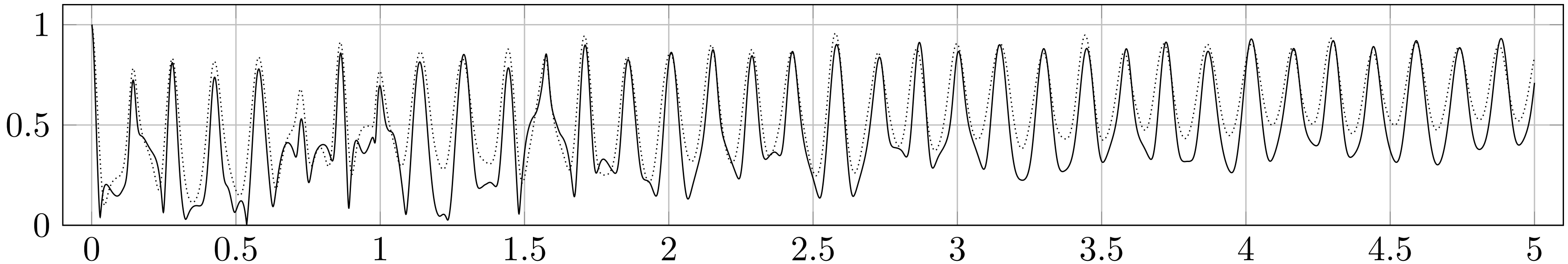
Numerical simulations: Square





Numerical simulations: Pentagon





Numerical simulations: Hexagon