Superexponential stabilizability of parabolic equations via bilinear control

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$$\begin{cases} u' = Au + Bu \\ u = \mathbf{p}|_{\partial\Omega} \\ u(0) = u_0 \end{cases}$$

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Multiplicative (or **bilinear**) control problem:

$$\begin{cases} u' = Au + \mathbf{p}Bu \\ u = g|_{\partial\Omega} \\ u(0) = u_0 \end{cases}$$

What are the difficulties?

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Bilinear control:

$$\begin{cases} u' = Au + \mathbf{p}Bu \\ u = g|_{\partial\Omega} \\ u(0) = u_0 \end{cases} \qquad (1)$$

Theorem (Ball, Marsden, Slemrod 1982)

Let X be a Banach space with dim(X)=$+\infty$. Let A generate a C^0-semigroup of bounded linear operators on X and $B : X \to X$ be a bounded linear operator. Let $u_0 \in X$ be fixed, and let $u(t; p, u_0)$ denote the unique solution of (1) for $p \in L^1_{loc}([0, +\infty), \mathbb{R})$. The set of states accessible from u_0 defined by

$$S(u_0) = \{u(t; p, u_0); t \geq 0, p \in L^r_{loc}([0, +\infty), \mathbb{R}), r > 1\}$$

is contained in a countable union of compact subsets of X and, in particular, has a dense complement.

Why, afterall, we want to study these problems?

Multiplicative controls enter the system equations as coefficients. They change (at least some of) the principal parameters of the process at hand.

Examples:

- by embedded *smart* alloys, the natural frequency response of a beam can be changed,
- the rate of a chemical reaction can be altered by various catalysts and/or by the speed at which the reaction ingredients are mechanically mixed

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Nuclear chain reaction

$$u_t = a^2 \Delta u + v(t,x)u$$

- $u(t,x) \geq 0$ neutron density in the reaction,
- $v(t,x) > 0$ neutron amount in the surrounding medium,

$v(t,x)u$ source of netrouns provided by the collision of the particles in the reaction with the surrounding medium

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Schrödinger equation

$$i\psi_t = -\Delta\psi - p(t)\mu(x)\psi$$

- ψ wave function of a particle,
- p amplitude of the electric field,
- μ dipolar moment of the particle

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Definitions

- Fixed a control $\bar{p}$ and an initial condition $\bar{u}_0$, (2) is *locally stabilizable to* $\bar{u}(\cdot; \bar{u}_0, \bar{p})$ if $\exists\, \delta > 0$ such that, $\forall\, u_0 \in B_\delta(\bar{u}_0)$, $\exists\, p$ for which
$$\lim_{t \to +\infty} ||u(t; u_0, p) - \bar{u}(t; \bar{u}_0, \bar{p})|| = 0.$$

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- Given a control $\bar{p}$ and an initial condition $\bar{u}_0$, (2) is *locally exponentially stabilizable to* $\bar{u}(\cdot; \bar{u}_0, \bar{p})$ if $\forall\, \rho > 0$, $\exists\, R(\rho) > 0$ for which, $\forall\, u_0 \in B_{R(\rho)}(\bar{u}_0)$, $\exists\, p$ and $M > 0$ such that
$$||u(t; u_0, p) - \bar{u}(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho t}, \qquad \forall t > 0.$$

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- Given control $\bar{p}$ and an initial condition $\bar{u}_0$, (2) is *locally superexponentially stabilizable to* $\bar{u}(\cdot;\bar{u}_0,\bar{p})$ if $\exists\, M, \omega > 0$ such that, $\forall\, \rho > 0$, $\exists\, R(\rho) > 0$ such that, $\forall\, u_0 \in B_{R(\rho)}(\bar{u}_0)$, $\exists\, p$ for which it holds
$$||u(t;u_0,p) - \bar{u}(t;\bar{u}_0,\bar{p})|| \le Me^{-\rho e^{\omega t}}, \qquad \forall t > 0.$$

Setting of the problem

Let $(X, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $A : D(A) \subset X \to X$ a densely defined linear operator with the following properties:

$$
\begin{array}{ll}
(a) & A \text{ is self-adjoint },\\
(b) & \langle Ax, x\rangle \geq 0, \forall x \in D(A),\\
(c) & \exists\, \lambda > 0 \text{ such that } (\lambda I + A)^{-1} : X \to X \text{ is compact }.
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1. there exists an orthonormal basis $\{\varphi_k\}_{k\in\mathbb{N}^*}$ on X of eigenfunctions of A,
2. the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}^*}$ of A are non-negative and $\lambda_k \to +\infty$ as $k \to +\infty$,
3. $-A$ generate a strongly continuous analytic semigroup of contractions e^{-tA}.

Setting of the problem

Given $T > 0$, consider the bilinear control problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, \quad t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{4}$$

where $p \in L^2(0, T)$ is the control function.

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Given $T > 0$, consider the bilinear control problem

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where $p \in L^2(0, T)$ is the control function.
Consider system (4) with $p = 0$:

$$\begin{cases} u'(t) + \mathbf{A}u(t) = 0, & t \in [0, T] \\ u(0) = \boldsymbol{\varphi_1}. \end{cases}$$

The solution $\psi_1(t) = e^{-\boldsymbol{\lambda_1} t}\boldsymbol{\varphi_1}$ is called the ground state solution.

Setting of the problem

Remark

Let $\mathbf{A}$ be strictly accretive. The evolution of the free dynamics with initial condition u_0 can be represented by $u(t) = e^{-t\mathbf{A}}u_0$. Therefore, with $p = 0$, system (4) is locally exponentially stabilizable the trajectory ψ_1. Indeed,

$$||u(t) - \psi_1(t)|| = ||e^{-t\mathbf{A}}u_0 - e^{-t\mathbf{A}}\boldsymbol{\varphi_1}|| \leq e^{-\nu t}||u_0 - \boldsymbol{\varphi_1}||.$$

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Novelty: construction of a control function p that brings $u(t)$ arbitrary close to $\psi_1(t)$ in a very short time. The convergence rate of the controlled solution to the reference trajectory is **doubly-exponential**.

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Weak version of the exact controllability to the ground state solution.

Superexponential stabilizability

Theorem

Let $A : D(A) \to X$ *be a densely defined linear operator satisfying hypothesis* (3) *and suppose that there exists a constant* $\alpha > 0$ *such that the eigenvalues of* A *fulfill the gap condition*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha, \quad \forall k \in \mathbb{N}^*. \tag{5}$$

Let $B : D(B) \subset X \to X$ *be a linear bounded operator with the following properties:*

$$\begin{aligned} &\langle B\varphi_1, \varphi_k \rangle \neq 0, \qquad \forall k \in \mathbb{N}^*, \\ &\exists\, \tau > 0 \text{ such that } \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty. \end{aligned} \tag{6}$$

Then, $\forall\, \rho > 0$, $\exists\, R > 0$ *such that any* $u_0 \in B_R(\varphi_1)$ *admits a control* $p \in L^2_{loc}(0, \infty)$ *such that the corresponding solution* $u(\cdot; u_0, p)$ *of* (4) *satisfies*

$$||u(t) - \psi_1(t)|| \leq M e^{-\rho e^{\omega t} - \lambda_1 t} \qquad \forall t \geq 0, \tag{7}$$

where M *and* ω *are positive constants depending only on* A *and* B*.*

Sketch of the proof, $\lambda_1 = 0$

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0, \end{cases} \qquad \begin{cases} \psi_1'(t) + A\psi_1(t) = 0, & t \in [0, T] \\ \psi_1(0) = \varphi_1. \end{cases}$$

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$$v := u - \psi_1$$

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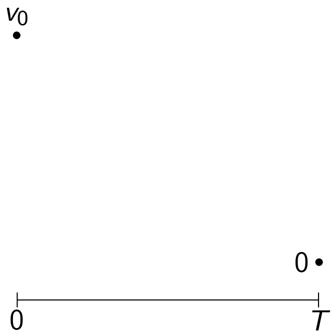

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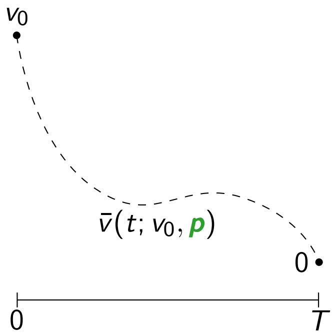

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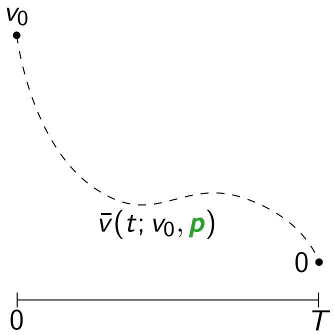

$$||p||_{L^2(0,T)} \leq C(T)\Lambda_T ||v_0||$$

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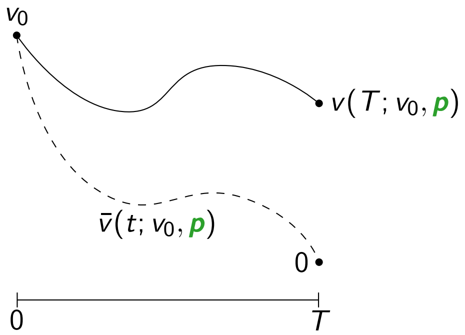

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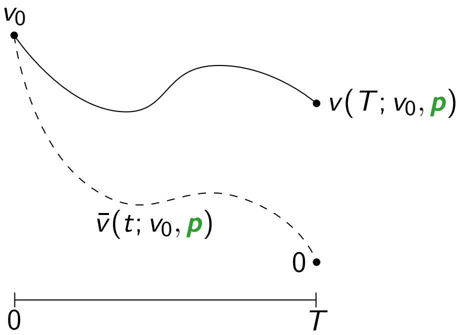

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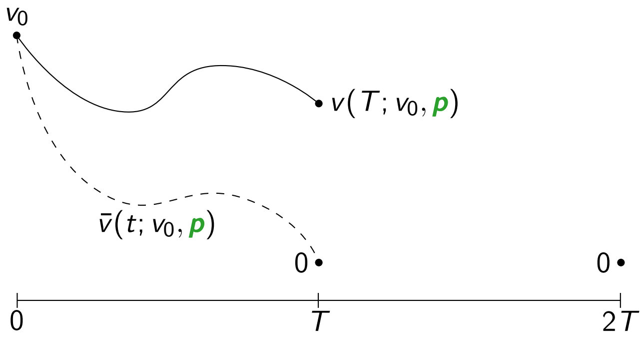

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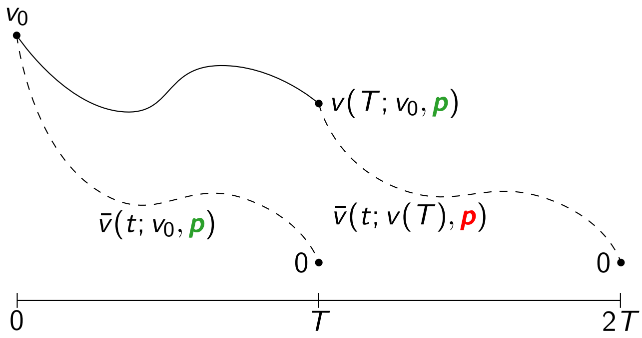

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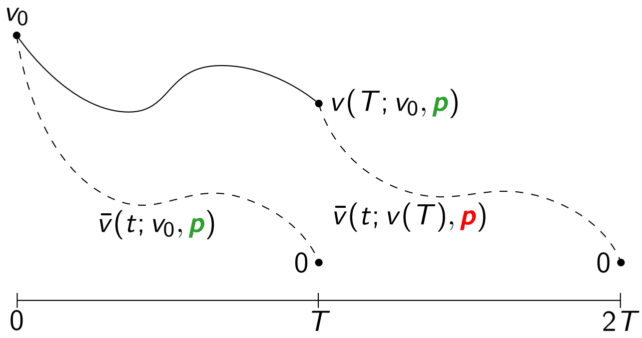

$$||p||_{L^2(T,2T)} \leq C(T)\Lambda_T||v(T)||$$

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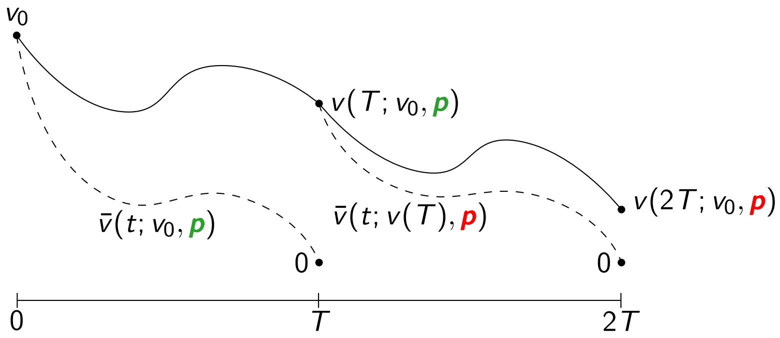

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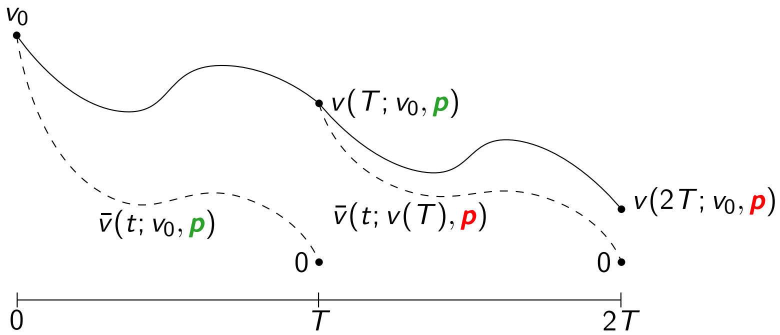

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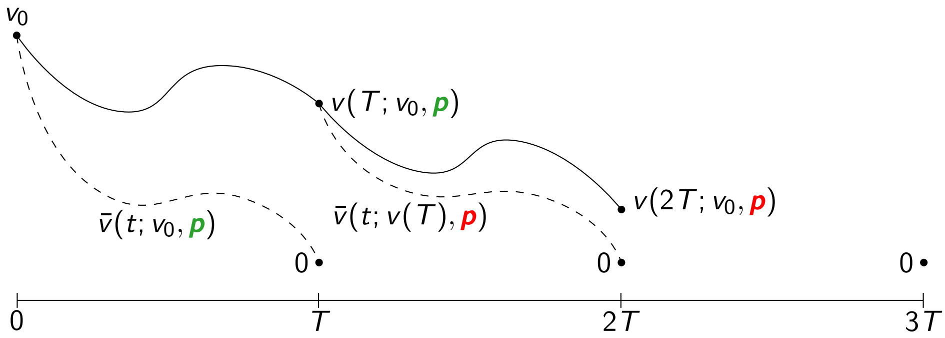

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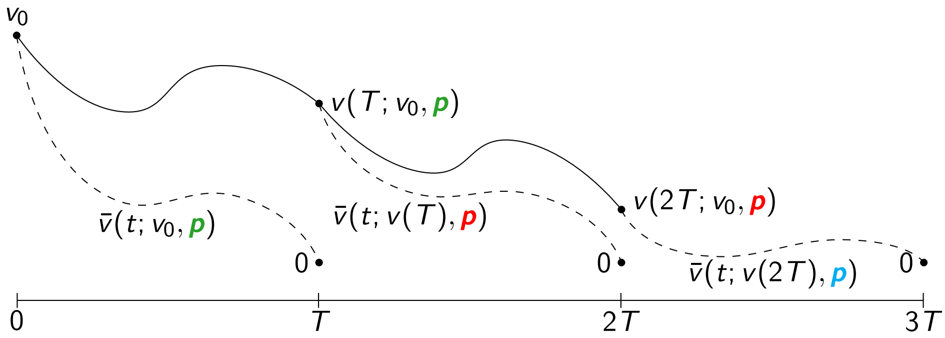

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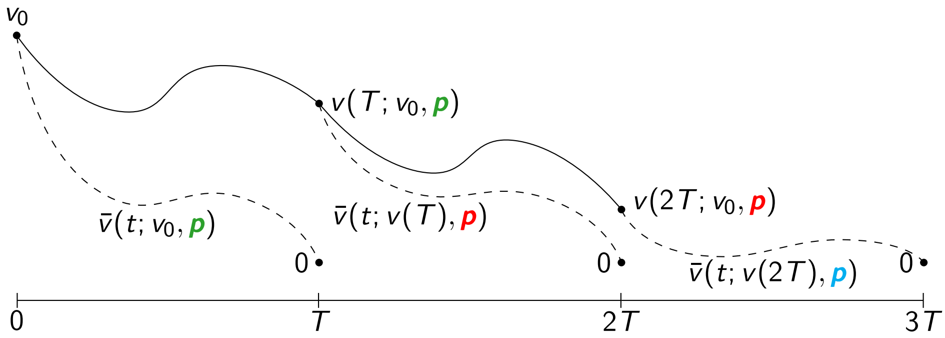

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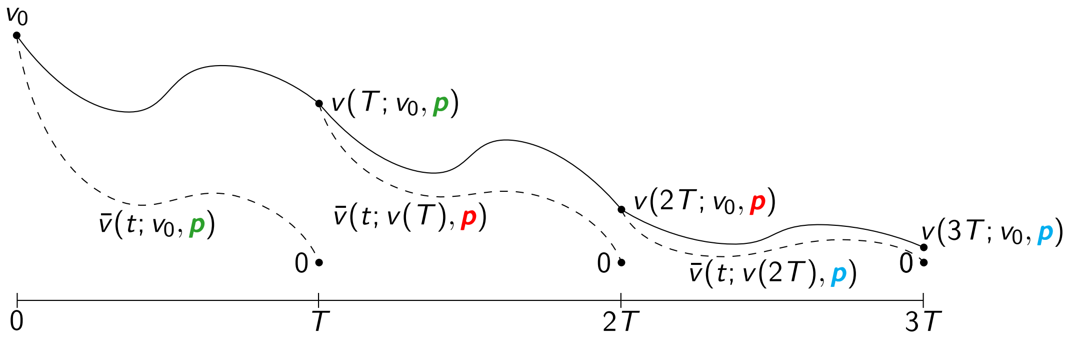

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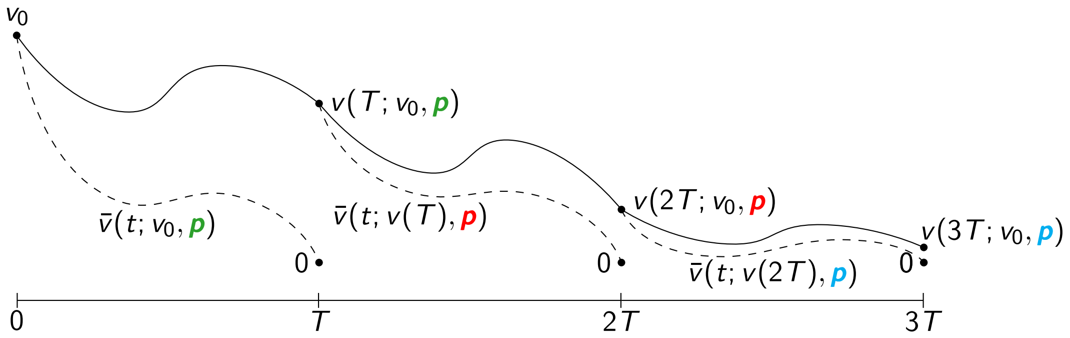

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- $$||v(nT)|| \leq \frac{1}{K_T}(K_T||v_0||)^{2^n} \tag{8}$$
  in every $[nT, (n+1)T]$.
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  $$||u(t) - \psi_1(t)|| \leq \frac{C_T}{K_T}\theta^{2^{t/T-1}} \qquad \forall t \geq 0,$$

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- Let $\theta \in (0,1)$ and let $\rho > 0$ be the value for which $\theta = e^{-2\rho}$, then $\exists\ R_\rho > 0$ such that if $||u_0 - \boldsymbol{\varphi_1}|| \leq R_\rho$, then
  $$||u(t) - \boldsymbol{\varphi_1}|| \leq M_T e^{-\rho e^{\omega_T t}}, \forall t \geq 0.$$
  with $M_T, \omega_T > 0$ suitable constants.

Sketch of the proof, $\lambda_1 > 0$

We introduce the operator

$$A_1 := \boldsymbol{A} - \boldsymbol{\lambda_1} I.$$

Observe that $A_1 : D(A_1) \subset X \to X$ is self-adjoint, accretive and $-A_1$ generates a strongly continuous analytic semigroup of contraction. Its eigenvalues are given by

$$\mu_k = \boldsymbol{\lambda_k} - \boldsymbol{\lambda_1}, \qquad \forall k \in \mathbb{N}^*$$

(in particular, $\mu_1 = 0$) and it has the same eigenfunctions as $\boldsymbol{A}$, $\{\boldsymbol{\varphi_k}\}_{k\in\mathbb{N}^*}$. Moreover, the family $\{\mu_k\}_{k\in\mathbb{N}^*}$ satisfies the same gap condition that is satisfied by the eigenvalues of $\boldsymbol{A}$.

Sketch of the proof, $\lambda_1 > 0$

We introduce $z(t) = e^{\lambda_1 t}u(t)$, then z solves

$$\begin{cases} z'(t) + A_1 z(t) + p(t)Bz(t) = 0, & t > 0, \\ z(0) = u_0. \end{cases}$$

So, we can apply the previous analysis to this problem:

$$||z(t) - \varphi_1|| \leq M_T e^{-\rho e^{\omega_T t}}, \qquad \forall t \geq 0.$$

and therefore

$$||u(t) - \psi_1(t)|| = ||e^{-\lambda_1 t}z(t) - e^{-\lambda_1 t}\varphi_1|| = e^{-\lambda_1 t}||z(t) - \varphi_1|| \leq M_T e^{-(\rho e^{\omega_T t} + \lambda_1 t)}, \quad \forall t \geq 0.$$

Superexponential stabilizability → Exact controllability

Theorem

Let $A : D(A) \to X$ *be a densely defined linear operator satisfying hypothesis* (3) *and suppose that there exists a constant* $\alpha > 0$ *such that the eigenvalues of* A *fulfill the gap condition*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha, \quad \forall k \in \mathbb{N}^*.$$

Let $B : D(B) \subset X \to X$ *be a linear bounded operator with the following properties:*

$$\langle B\varphi_1, \varphi_k \rangle \neq 0, \qquad \forall k \in \mathbb{N}^*,$$

$$\exists\, \tau > 0 \text{ such that } \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty.$$

Then, $\forall\, \rho > 0$, $\exists\, R > 0$ *such that any* $u_0 \in B_R(\varphi_1)$ *admits a control* $p \in L^2_{loc}(0, \infty)$ *such that the corresponding solution* $u(\cdot; u_0, p)$ *of* (4) *satisfies*

$$||u(t) - \psi_1(t)|| \leq M e^{-\rho e^{\omega t} - \lambda_1 t} \qquad \forall t \geq 0,$$

where M *and* ω *are positive constants depending only on* A *and* B.

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$$u(T; u_0, p) = \psi_1(T).$$

Examples

Let $T > 0$, $\Omega = [0,1]$ and consider the bilinear control system

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\mu(x)u(t,x) = 0, & (t,x) \in [0,T] \times \Omega \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Examples

Let $T > 0$, $X = L^2(\Omega)$ and consider the bilinear control system

$$\begin{cases} u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0(x). \end{cases} \tag{10}$$

where A and B are defined by

$$\begin{array}{ll} D(A) = H^2 \cap H_0^1(\Omega), & A\varphi = -\frac{d^2\varphi}{dx^2} \\ D(B) = X, & B\varphi = \mu\varphi. \end{array} \tag{11}$$

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We want to study the superexponential stabilizability of (10)-(11) to the ground state solution $\psi_1 = e^{-\lambda_1 t}\varphi_1$.

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$$\begin{aligned}\langle B\varphi_1, \varphi_k\rangle &= \int_0^1 2\mu(x)\sin(\pi x)\sin(k\pi x)dx = \\ &= \frac{4}{k^3}\Big((-1)^{k+1}\mu'(1) - \mu'(0)\Big) + \\ &\quad - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu(x)\varphi_1(x))''' \cos(k\pi x)dx\end{aligned}$$

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If $\langle B\varphi_1, \varphi_k\rangle \neq 0\ \forall k \in \mathbb{N}^*$ and $\mu'(1) \pm \mu'(0) \neq 0$, then we have

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EXAMPLE: $B\varphi(x) = x^2\varphi(x)$

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- the series

$$\sum_{k\in\mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k\rangle|^2}$$

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Examples

Let $T > 0$, $\Omega = [0,1]$, $X = L^2(\Omega)$ and consider the degenerate control system

$$\begin{cases} u_t - (x^\alpha u_x)_x + p(t)x^{2-\alpha}u = 0, & (t,x) \in [0,T] \times \Omega \\ u(t,1) = 0, \quad \begin{cases} u(t,0) = 0, & \text{if } \alpha \in [0,1), \\ (x^\alpha u_x)(t,0) = 0, & \text{if } \alpha \in [1,2), \end{cases} \\ u(0,x) = u_0(x). \end{cases} \tag{19}$$

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Define the quantities

$$\nu_\alpha := \frac{|1-\alpha|}{2-\alpha}, \qquad k_\alpha := \frac{2-\alpha}{2}.$$

Then, $A : D(A) \subset X \to X$, that is a **self-adjoint accretive** operator with **compact resolvent**, have the following eigenvalues and eigenfunctions

$$\boldsymbol{\lambda_{\alpha,k}} = k_\alpha^2 j_{\alpha,k}^2,$$

$$\boldsymbol{\varphi_{\alpha,k}}(x) = \frac{\sqrt{2k_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha}\left(j_{\nu_\alpha,k} x^{k_\alpha}\right).$$

Examples

- gap condition:

$$\alpha \in [0,1) \Rightarrow \sqrt{\lambda_{\alpha,k+1}} - \sqrt{\lambda_{\alpha,k}} \geq \frac{7}{16}\pi, \quad \forall k \in \mathbb{N}^*,$$

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Gracias! Thank you! Grazie! Merci! Dank!