Reaction-diffusion models, constraints and control

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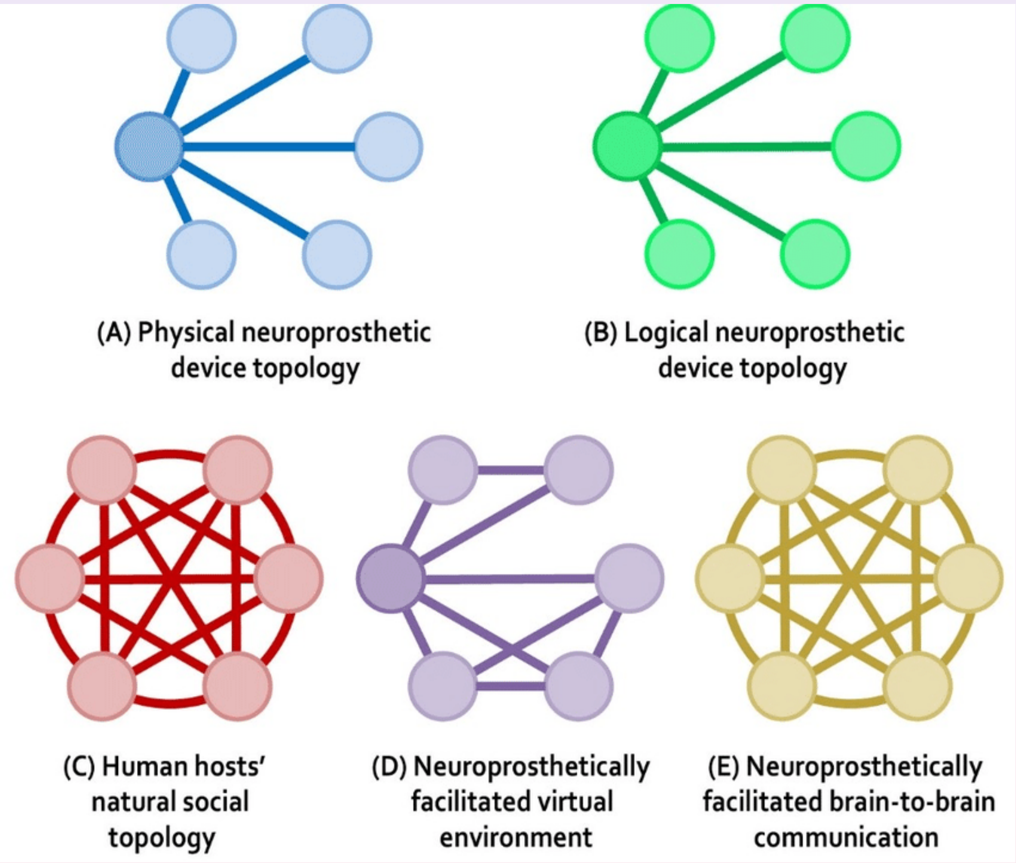
Often the mathematical treatment of real life problems not only involves

1. Modelling
2. Analysis
3. Simulation

but also

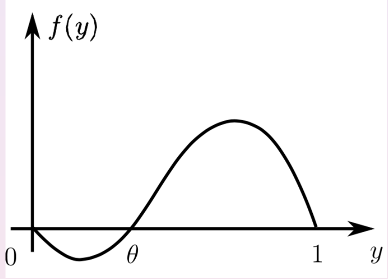
1. Design
2. Optimisation
3. Parameter identification
4. Uncertainty quantification
5. Control



Our original motivation: multilinguism

A Game-Theoretic Analysis of Minority Language Use in Multilingual Societies[1]

José-Ramón Uriarte
University of the Basque Country
October 2015



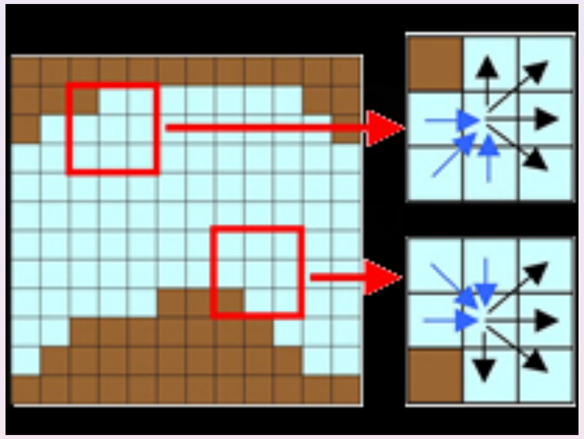
Why, often, minority languages are used less than expected, in view of the percentage of population that masters them?

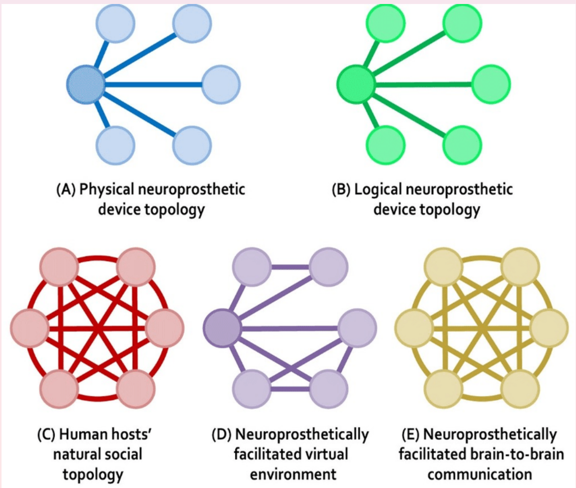
A "politeness equilibrium" emerges as a consequence of a variety of factors, including the fact we try not to annoy the other.


Donde fueres,
haz lo que vieres

This and many other problems in social sciences require a significant mathematical effort and a variety of modelling paradigms can be employed:

1. ODE
2. Stochastic systems of interacting particles
3. Reaction-diffusion equations
4. Mean Field games
5. Kinetic models
6. PDEs on networks





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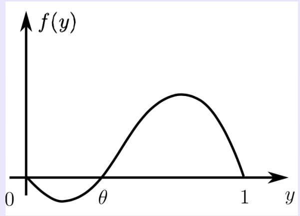
E. Trélat, J. Zhu & E. Zuazua, M3AS, 2018

The **bistable** Cauchy problem: 

$$y_t - y_{xx} = ay(1-y)(y-\theta), \quad y(0,\cdot) = y_0, \quad x \in \mathbf{R}, \quad t \in \mathbf{R}^+,$$

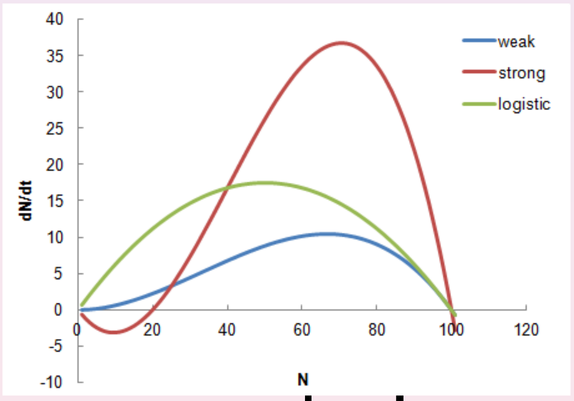
- The state $0 \le y \le 1$ represents the density of individuals.
- Typical application : spread of invading organisms in ecology systems, (cf. e.g. M.A. Lewis and P. Kareiva 1993).
- The role of parameters a, θ:
    - $a > 0$: reproductive rate ($a = 1$ without loss of generality).
    - $\theta \in (0,1)$: local critical density or Allee threshold[2] that determines the sign (positive or negative) the population growth.
- Other applications : population genetics (biology), propagation of nerve tension (neurobiology), propagation waves in chemical reactors (chemistry), etc.

[2] Warder Clyde Allee, 30's

Allee effect

Allee effect : population growth is negative (leading to the extinction) when the density of the population is lower than the Allee threshold θ, otherwise the population will reach carrying capacity.

- increase θ by the sterile male technique, the mating disruption (pest management technique), etc.
- decrease θ by providing protection (e.g. efficient feeding, suppressing natural enemies) to the population



Typical solutions of the system :

- **Steady state** constant solutions : $y \equiv 0$ or θ or 1.
- **Traveling wave** solutions : link two of the three constant solutions, the steady state ones, through a front that propagates in space with a constant velocity.

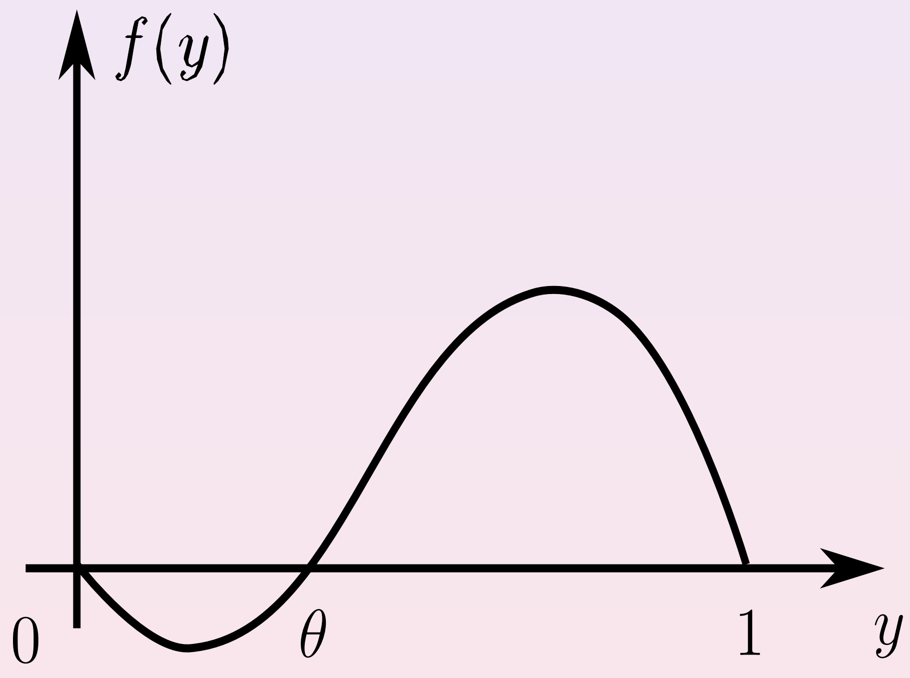
- solution of the form

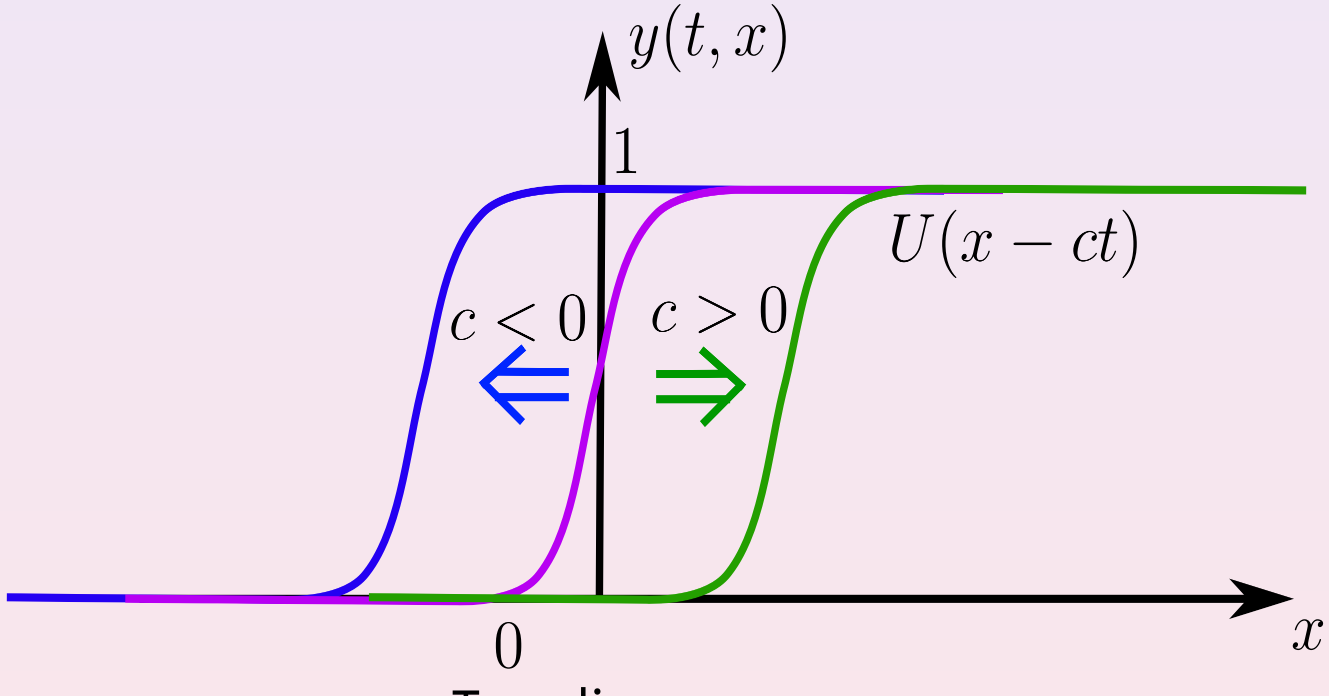
$$y(t,x) = U(x-ct),$$

$$U(\pm\infty) = U_\pm,\ U(\pm\infty)' = 0$$

where $U(x)$ is the wave profile and c is the wave speed.

- sign of the wave speed : $\mathrm{sign} c = -\int_0^1 f(t)dt$





- The profile U is independent of θ: $U(x) = \frac{e^{\sqrt{a/2}x}}{1+e^{\sqrt{a/2}x}}$
- The Allee parameter θ determines the wave speed: $c = \sqrt{a/2}(2\theta - 1)$.

"La ola" / The wave

Control through known dynamical properties

For the system to be controlled or driven to the suitable state configuration, one has to first understand what are its fundamental dynamical properties and how they depend on the parameter θ to be tuned.

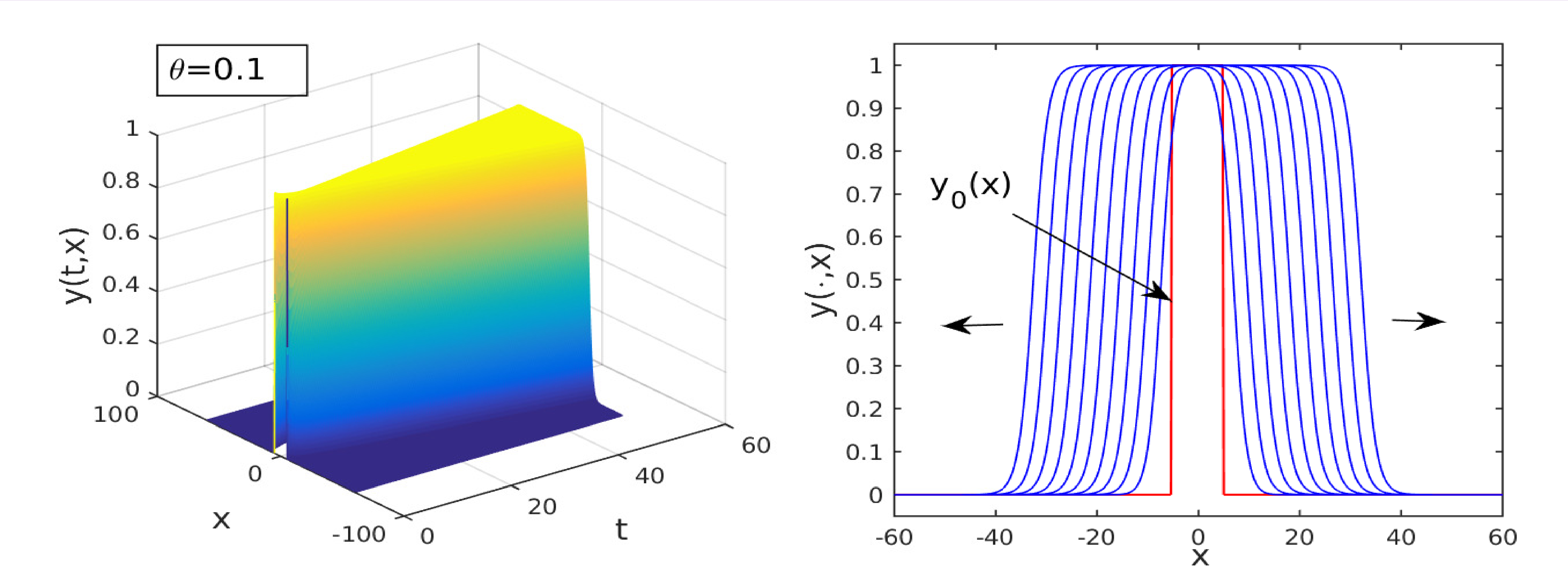
We have to **learn** how the system responds to parameter changes be experiencing on the manipulation of the free parameters. [3][4]

[3] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in PDE and Related Topics, Lecture Notes in Math., vol. 446, Springer, Berlin, 1975, pp. 5-49.

[4] P.C. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions. Arch. Ration. Mech. Anal. **65** (1977) 335-361.

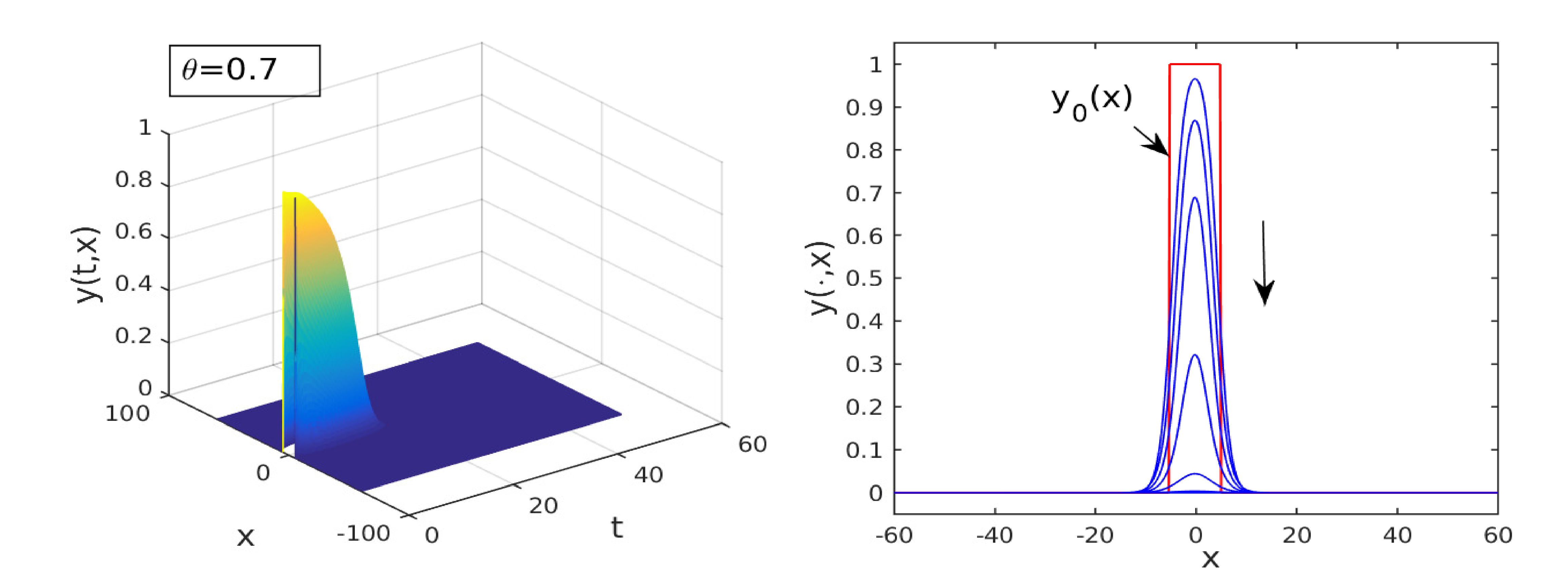
Invasion

- $\theta = 0.1$: invasion of the population



Extintion

- $\theta = 0.7$: extinction of the population



Attractiveness of traveling waves

Lemma

If the initial datum y_0 is such that $y_0(x) \in [0,1]$ for every $x \in \mathbf{R}$, and satisfies

$$\underline{\lim}_{x \to +\infty} y_0(x) > \theta, \qquad \overline{\lim}_{x \to -\infty} y_0(x) < \theta \tag{1}$$

for some $x_1 \in \mathbf{R}$, the solution approaches a the traveling wave $U(x - ct - x_1)$ uniformly in x and exponentially in time for $c = c(\theta)$, i.e., for some positive constants K and γ,

$$\|y(t,x) - U(x - ct - x_1)\|_{L^\infty(\mathbf{R})} < Ke^{-\gamma t}.$$

Formulation of the control problem

[5]

Control problem $\mathcal{P}_c$

Find $\theta(t) \in [0,1]$, $t \in [0, T]$ such that the solution of

$$y_t - y_{xx} = ay(1-y)(y-\theta(t)),$$

$$y(0,\cdot) = y_0(\cdot)$$

develops into an expected wave $U(\cdot)$ at the given time T, minimizing

$$J(\theta) = \Big| y(T,\cdot) - U \Big|^2$$

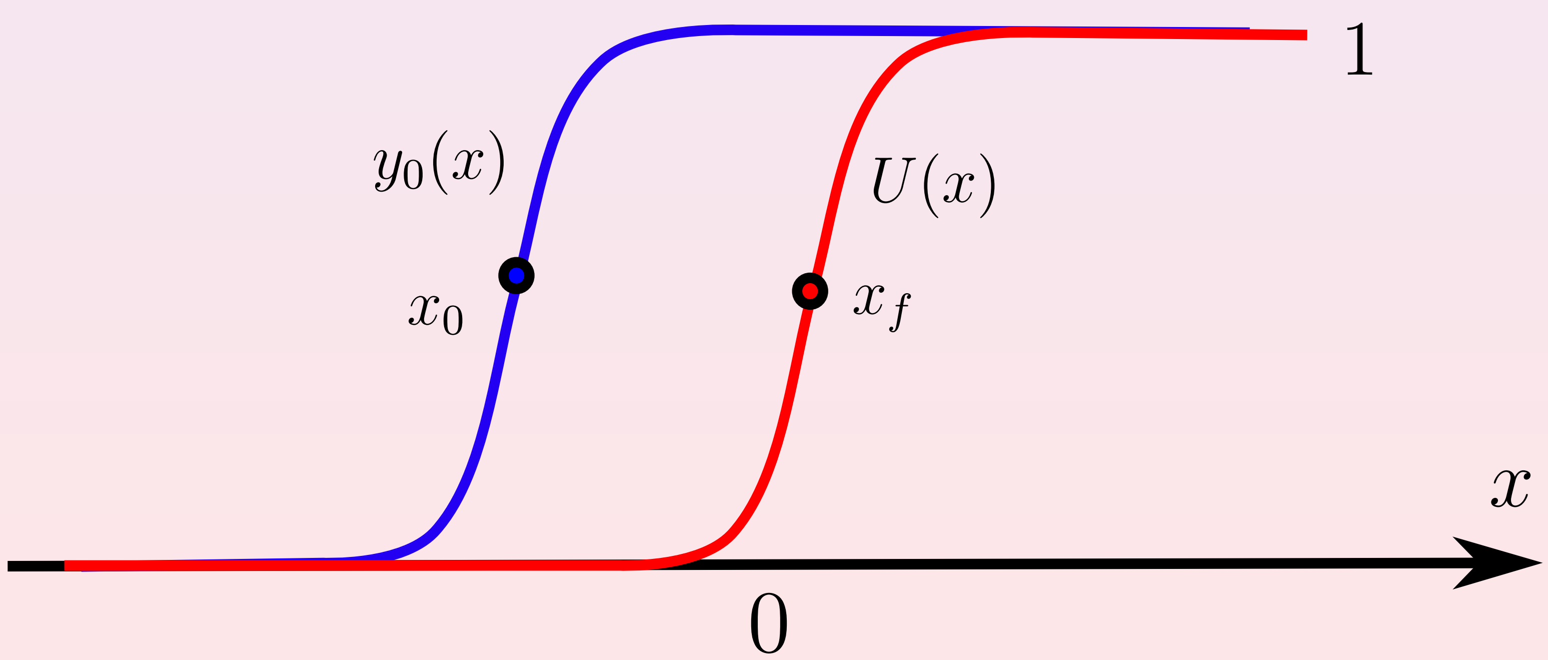
.

[5] P. M. Cannarsa, G. Floridia, A.Y. Khapalov, Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign, JMPA, 2017.

Control in two steps if time T is long enough for initial data satisfying:

$$\lim_{x\to-\infty} y_0(x) < \lim_{x\to+\infty} y_0(x).$$

- Step 1. Choose θ_1 a suitable constant value and keep it long enough $[0, T_1]$ until the solution approximates some traveling wave profile.
- Step 2. Once the solution approaches a traveling wave profile, its location can be tuned by a suitable choice of a second value θ_2 in the time interval $[T_1, T_1 + T_2]$.



Computational optimisation: Two-grids

[6]

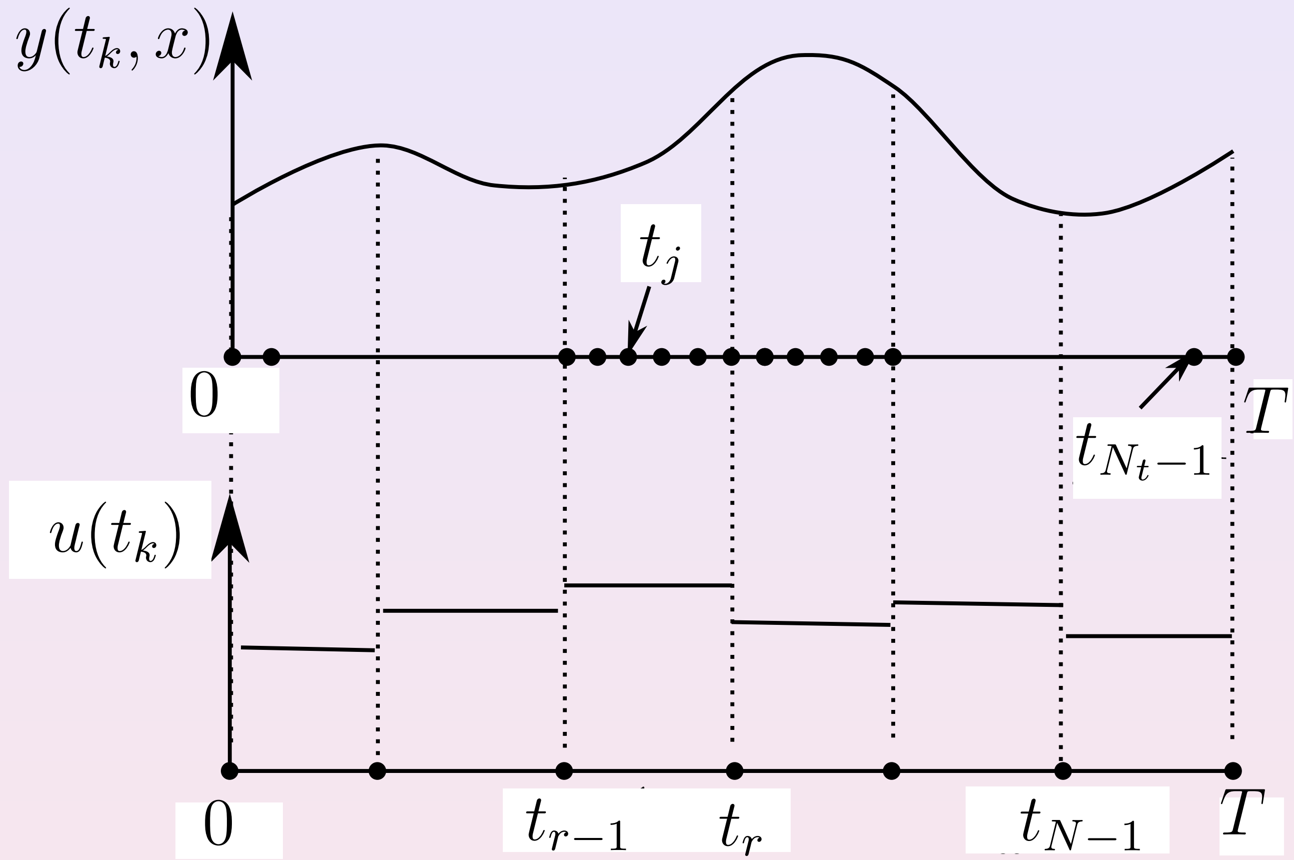
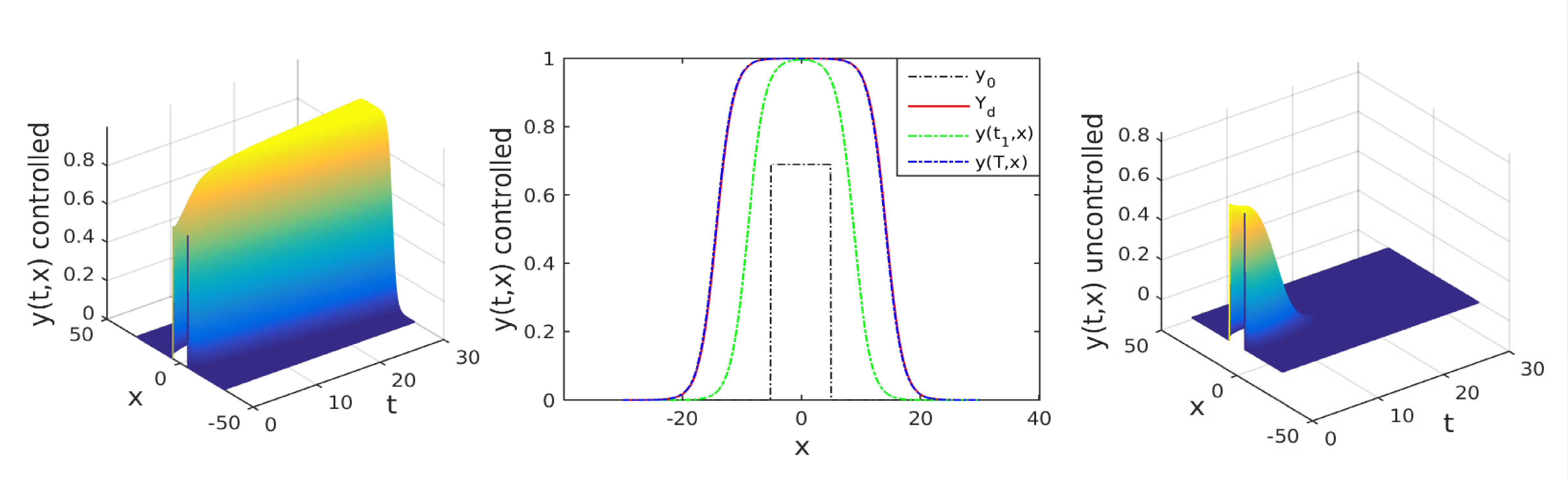


Figure: A fine time-mesh is employed to get a fine approximation of the state y. A coarser one suffices to approximate the control θ.

[6] J. Casado-Diaz, C. Castro, M. Luna-Laynez & E. Z., Numerical approximation of a one-dimensional elliptic optimal design problem, SIAM J. Multiscale Analysis, 2011.

Numerical experiment: Avoiding extintion



Effective tuning of the Allee parameter

Consider the system

$$\frac{\partial}{\partial t}\mathbf{y} - \mathbf{y}_{xx} = F(\mathbf{y}) \tag{2}$$

with $\mathbf{y} = (y_1, y_2)^\top$, y_1 being the density of normal couples and y_2 of the sterile ones and the vector valued reaction term $F = (f_1, f_2)^\top$ with $f_1 = ry_1(1-y)(y-\theta) - uy_1$ and $f_2 = uy_1 - r\theta y_2(1-y)$.

Overall:

$$\frac{\partial}{\partial t}y - y_{xx} = ry(1-y)(y-(\theta + y_2)) \tag{3}$$

with $y = y_1 + y_2$.

Conclusion

The Allee parameter allows controlling the system, but quite mildly (long time, traveling wave profiles) because of the weak effect it produces into the dynamics.

Other types of controls can be implemented as, for instance, the boundary control. In that case the dynamics is confined in a bounded region and the control is applied by regulating the density of population or its flux on the boundary, which mimics the invasion of a population with specific characteristics.

Conclusion

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C. Pouchol, E. Trélat, E. Zuazua. Phase plane control for 1D monostable and bistable reaction-diffusion equations, Nonlinearity, to appear.

The density of individuals $0 \leq y(t,x) \leq 1$ obeys the PDE

$$\begin{cases} y_t - y_{xx} = y(1-y)(y-\theta), \\ y(0) = y_0, \\ y(t,0) = u(t), \; y(t,L) = v(t). \end{cases}$$

with $\theta < 1/2$, on the space interval $x \in (0,L)$. Here u, v stand for the **controls** with constraints $0 \leq u(t), v(t) \leq 1$.

For a $(= 0,$ θ ou $1)$, we say that

- *The system is controllable to a in finite time* if for all $0 \leq y_0 \leq 1$, there exist T, and controls u, v s.t.

$$y(T, \cdot) = a.$$

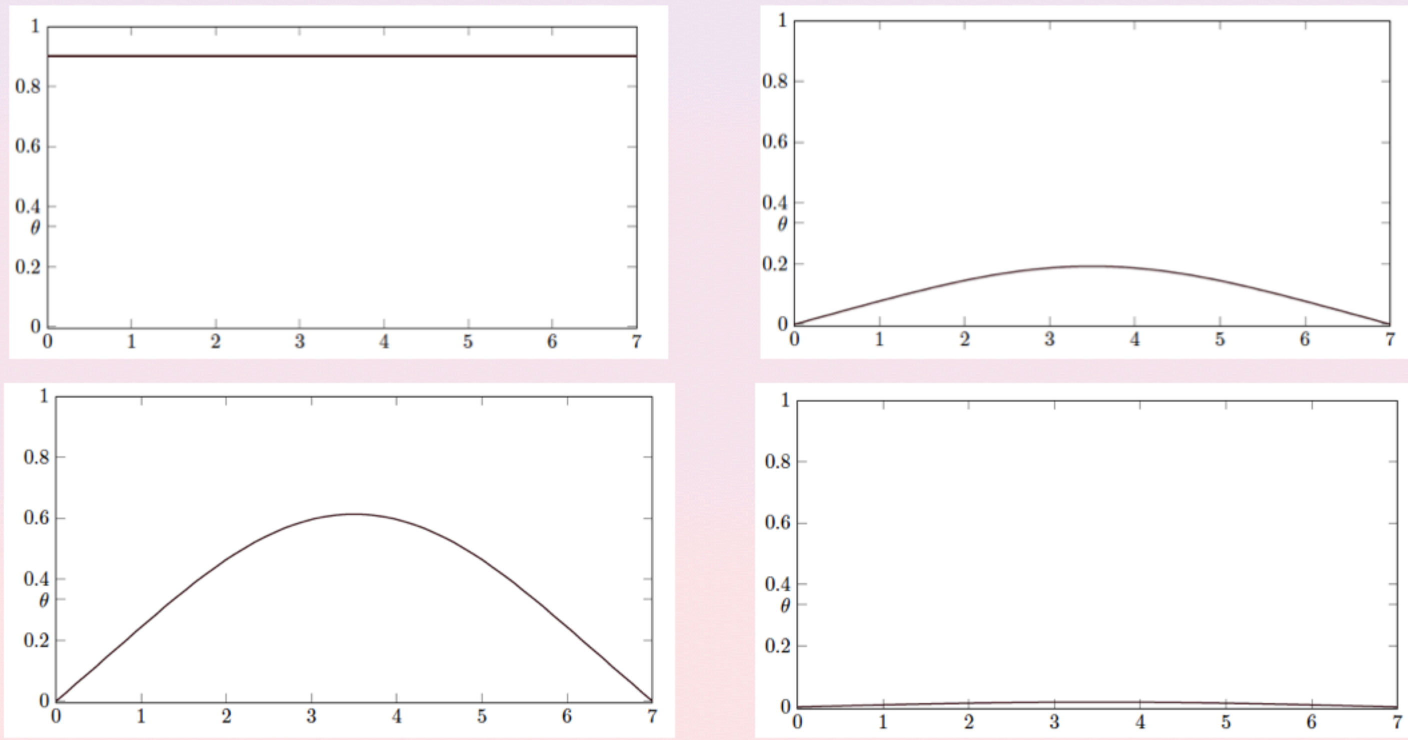
- *In infinite time* when the same occurs asymptotically as $t \to +\infty$.

Lack of obstructions: L small

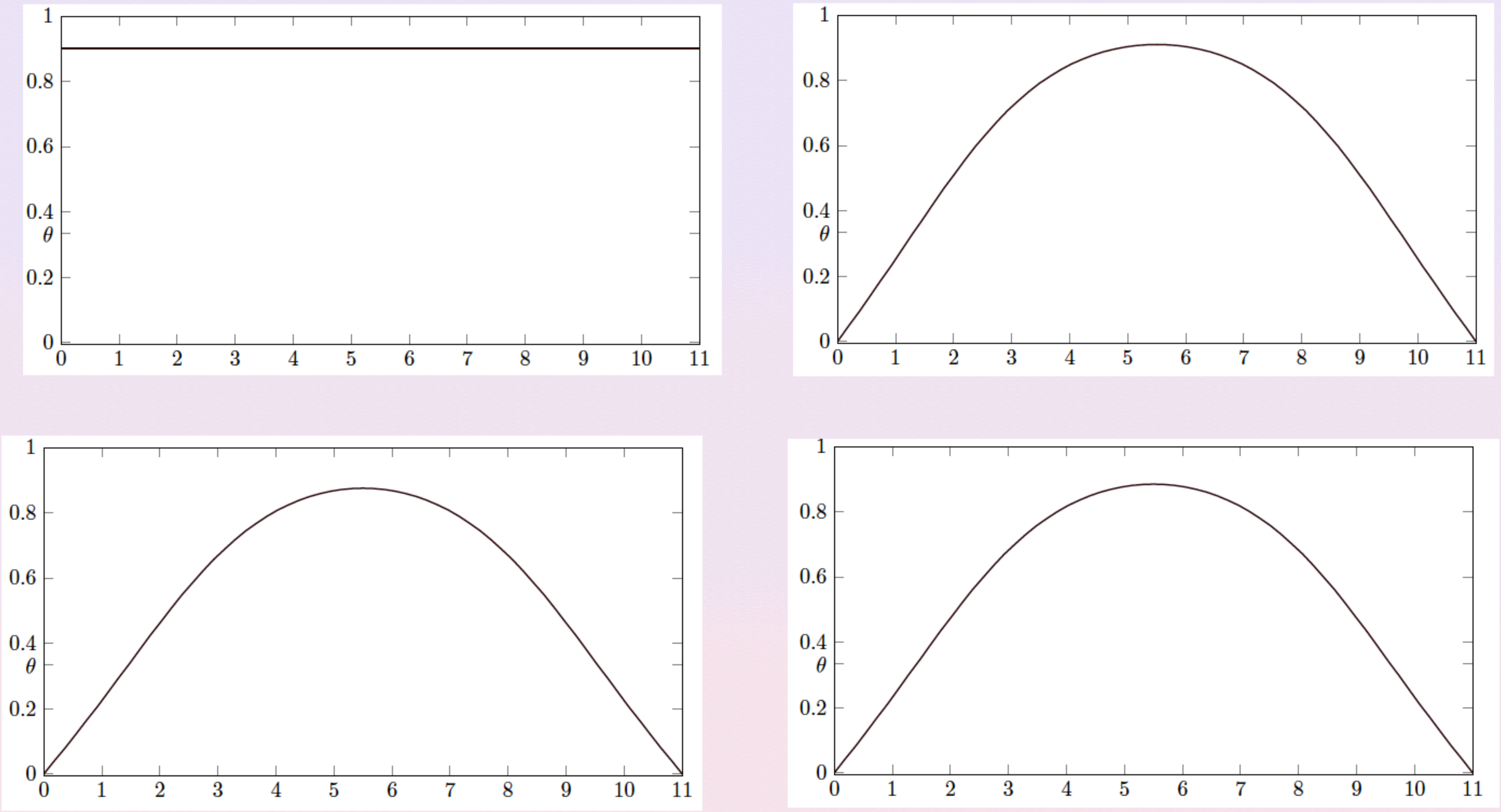
The *static strategy* to control to 0, L small:

$$\begin{cases} y_t - y_{xx} = f(y), \\ y(0) = y_0, \\ y(t,0) = 0, \ y(t,L) = 0. \end{cases}$$

with $f(y) = y(1-y)(y-\theta)$ with $\theta = \frac{1}{3}$, $y_0 = 0.9$, $L = 7$.



Obstructions: L large ($L = 11$)



Known facts (1)

The static strategy to reach $a = 0$, θ, or 1 consists on keeping the time-independent control a:

$$\begin{cases} y_t - y_{xx} = f(y), \\ y(0) = y_0, \\ y(t,0) = a, \; y(t,L) = a. \end{cases}$$

Matano's Theorem (1978) :
$y(t,\cdot)$ converges towards a steady state solution $0 \le w \le 1$:

$$\begin{cases} -w_{xx} = f(w), \\ w(0) = a, \; w(L) = a. \end{cases} \tag{4}$$

$w \equiv a$ is a steady stae solution for $a = 0, \theta$ and 1. But is it the only one?

P. L. Lions's Theorem (1982) : There exists a threshold L_a such that

- If $L < L_a$, $w = a$ is the unique steady-state solution.
- If $L > L_a$, there is another non-trivial steady state solution.

Conclusion

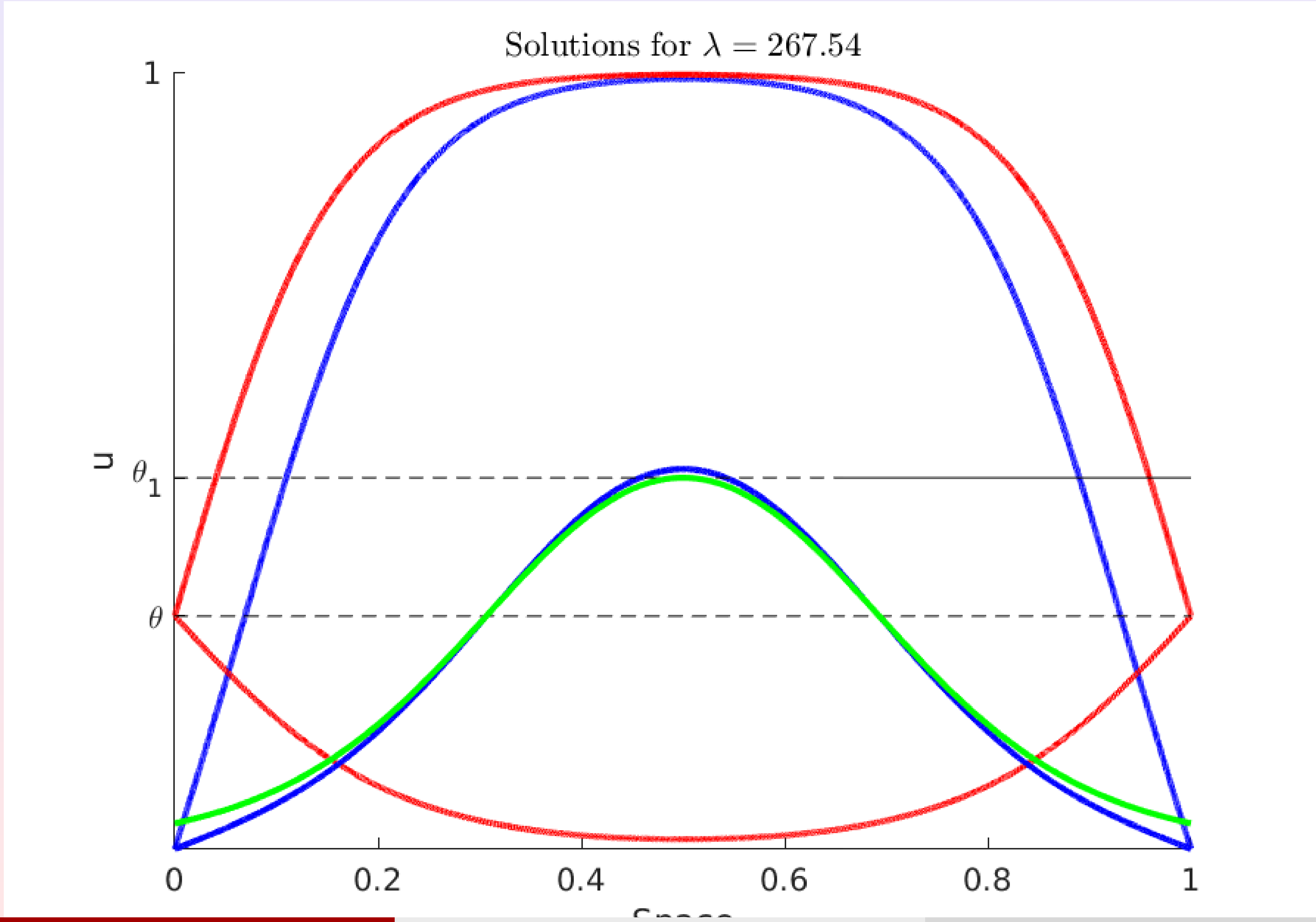
- If $L < L_a$, the system is asymptotically controllable towards a.
- If $L > L_a$, there is a barrier function making this impossible.

This issue is closely related to the question of whether the minimiser of the functional

$$J(y) = \frac{1}{2} \int_0^L |y_x|^2 dx - \int_0^L F(y) dx$$

in $H_0^1(0, L)$ is the trivial one $y \equiv 0$ or not.
Obviously, large L implies the first Dirichlet eigenvalue to be small, this weakens the coercivity of the H_0^1-norm, and facilitates the existence of non-trivial solutions.

Some non-trivial steady-states



Application

SCIENTIFIC REPORTS

OPEN

The minimum area requirements (MAR) for giant panda: an empirical study

Received: 28 July 2015
Accepted: 01 November 2016
Published: 08 December 2016

Jing Qing[1,2,*], Zhisong Yang[1,*], Ke He[1], Zejun Zhang[3], Xiaodong Gu[4], Xuyu Yang[4], Wen Zhang[5], Biao Yang[6], Dunwu Qi[7] & Qiang Dai[2]

Habitat fragmentation can reduce population viability, especially for area-sensitive species. The Minimum Area Requirements (MAR) of a population is the area required for the population's long-term persistence. In this study, the response of occupancy probability of giant pandas against habitat patch size was studied in five of the six mountain ranges inhabited by giant panda, which cover over 78% of the global distribution of giant panda habitat. The probability of giant panda occurrence was positively associated with habitat patch area, and the observed increase in occupancy probability with patch size was higher than that due to passive sampling alone. These results suggest that the giant panda is an area-sensitive species. The MAR for giant panda was estimated to be 114.7 km^2 based on analysis of its occupancy probability. Giant panda habitats appear more fragmented in the three southern mountain ranges, while they are large and more continuous in the other two. Establishing corridors among habitat patches can mitigate habitat fragmentation, but expanding habitat patch sizes is necessary in mountain ranges where fragmentation is most intensive.

Application

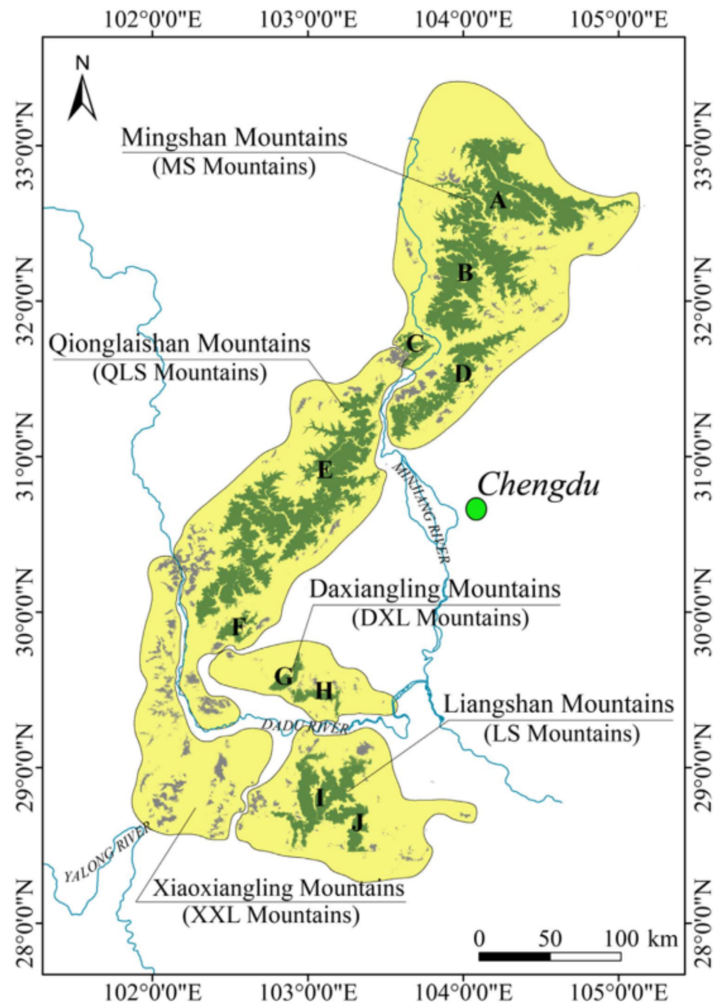


Figure 3. Distribution of habitat patches in the five mountain ranges. Yellow regions show the schematic of the mountain ranges. Habitat patches greater than MAR are green and patches smaller than MAR are grey. The map is made by ArcGIS 9.1 software, http://www.arcgis.com/features.

Known facts (2)

For $a = 0$ and $a = 1$, because of the comparison principle this is the best we can do, i.e. the control properties cannot be improved taking controls other than the trivial one since every solution of

$$\begin{cases} y_t - y_{xx} = f(y), \\ y(0) = y_0, \\ y(t,0) = u(t),\ y(t,L) = v(t) \end{cases}$$

is such that

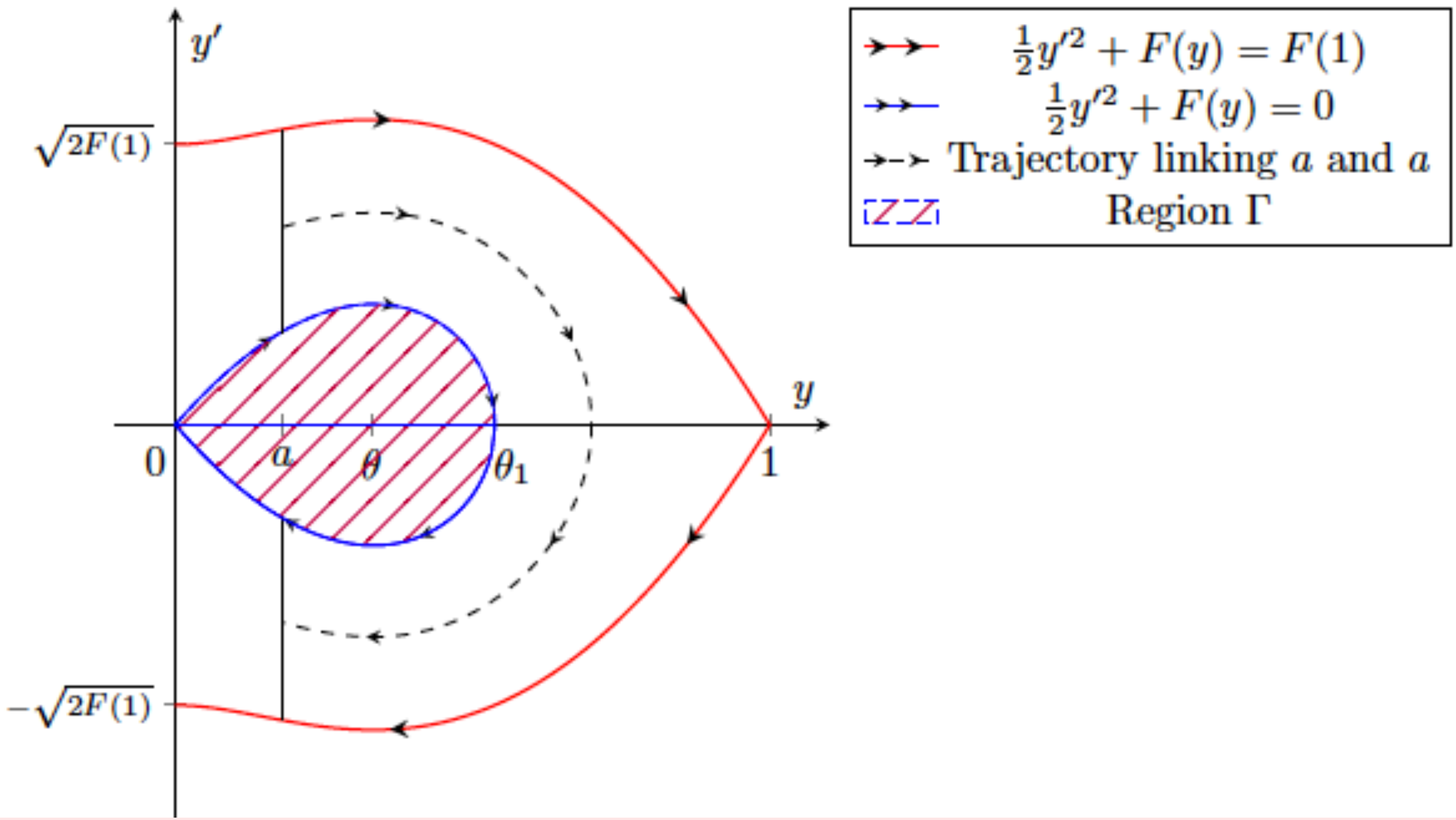
$$y(t,x) \geq z(t,x) > 0 \text{ over } (0,L).$$

where

$$\begin{cases} z_t - z_{xx} = f(z), \\ z(0) = y_0, \\ z(t,0) = 0,\ z(t,L) = 0 \end{cases}$$

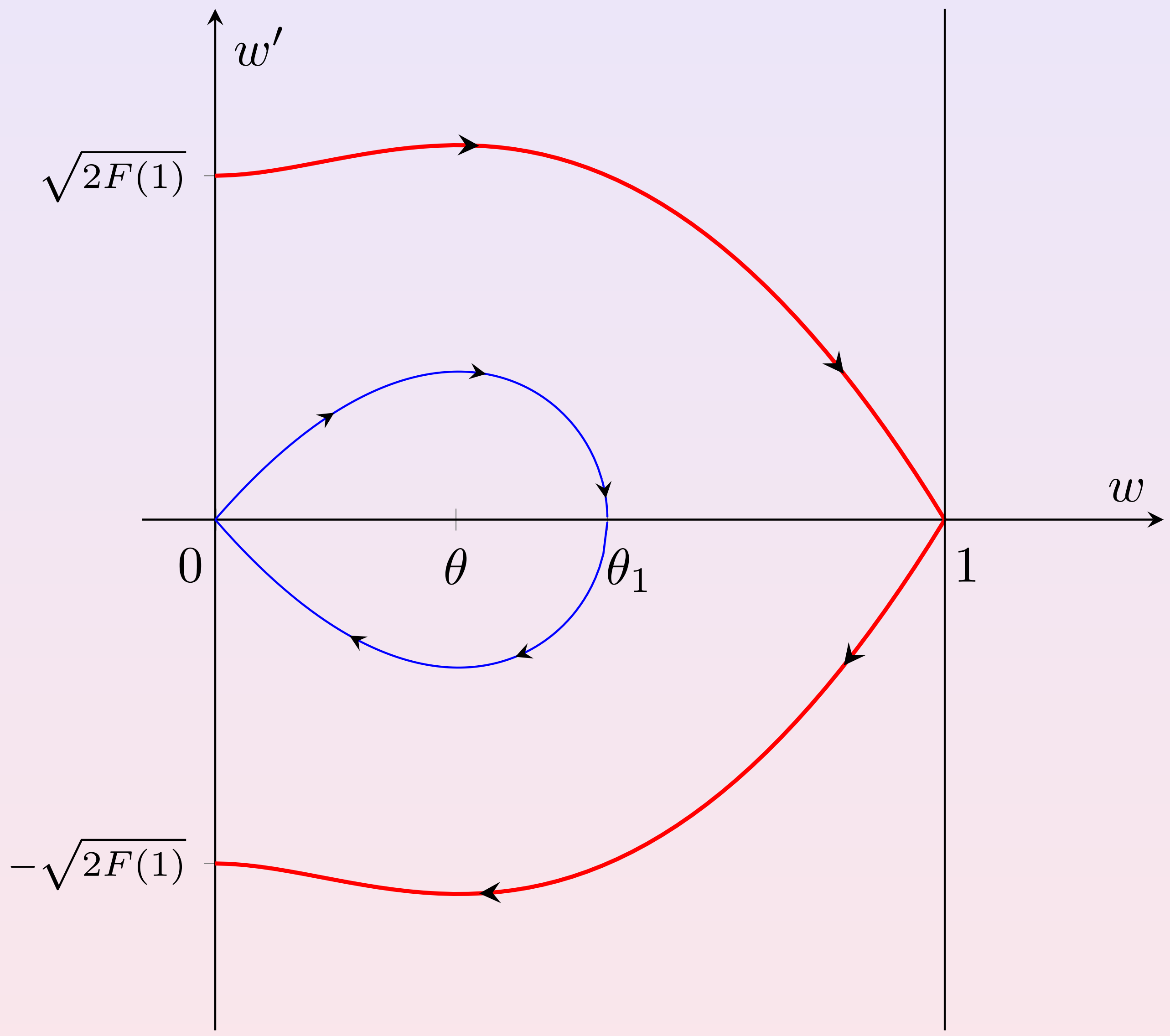
Phase portrait for $-w'' = f(w)$

We set $F(y) := \int_0^y f(z)\,dz$, and suppose that $F(1) \geq 0$ ($\Leftrightarrow \theta \leq \frac{1}{2}$ with $f(y) = y(1-y)(y-\theta)$).



Case $a = 1$: $L_1 = \infty$

$w = 1$ is the only steady-state solution

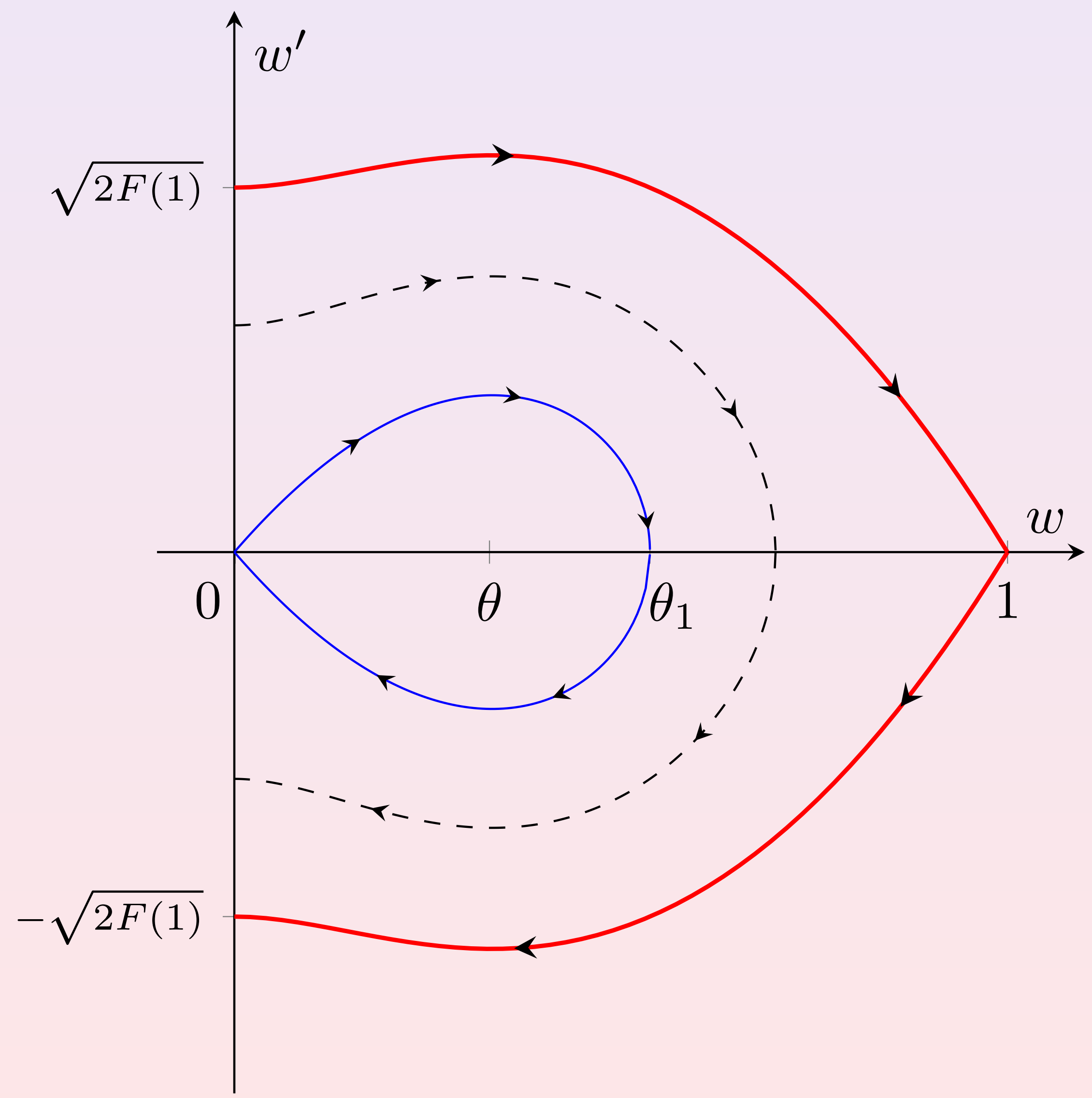


Consequently, the constant control $= 1$ assures that the system reaches asymptotically the equilibrium $y \equiv 1$ in infinite time, and this for all L.

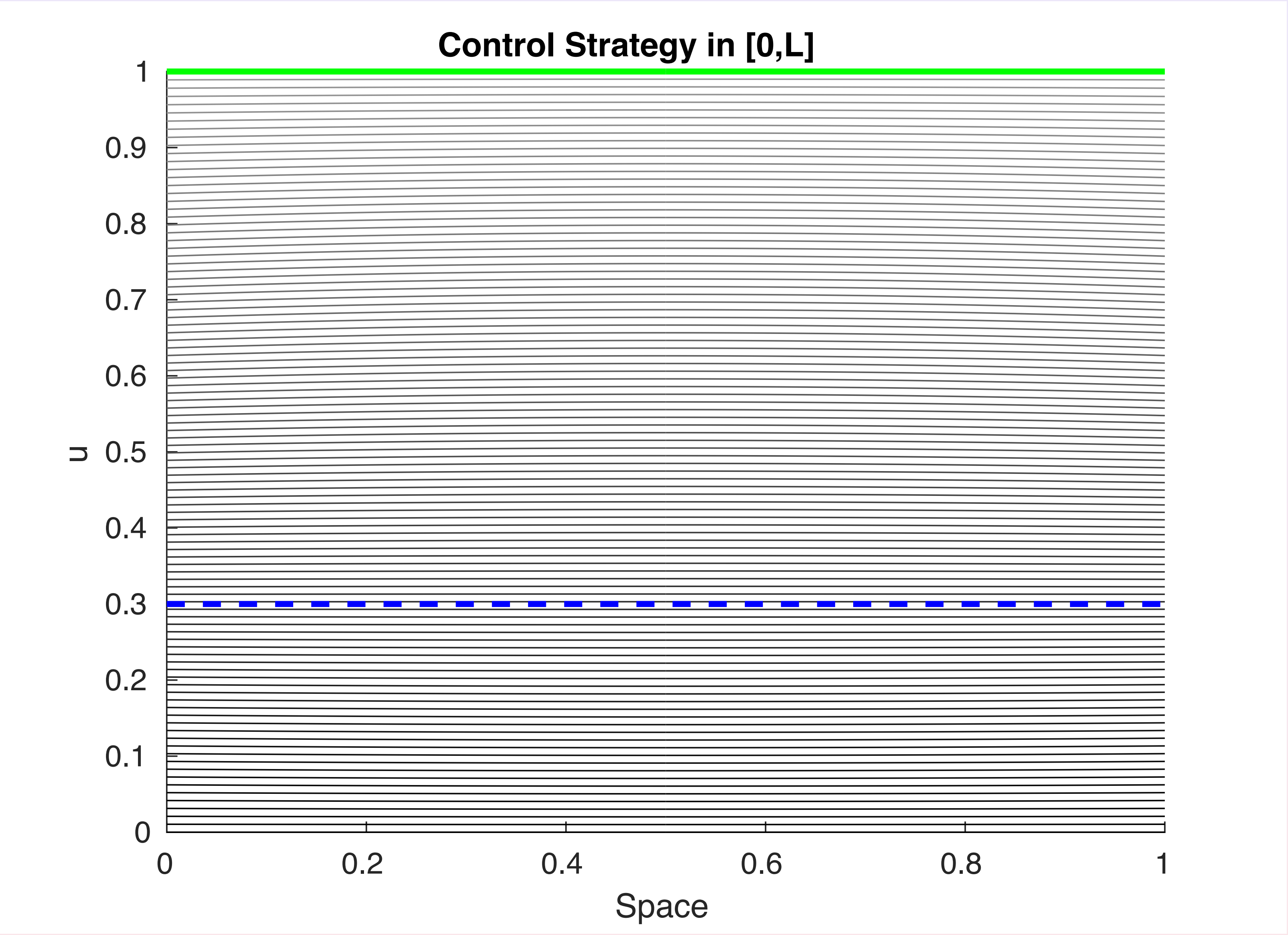
Case $a = 0$: $L_0 < \infty$

$w = 0$ is the unique steady state solution if $L < L_0 = L^\star$,

$$L^\star = \inf_{\beta \in (\theta_1, 1)} \sqrt{2} \int_0^\beta \frac{dy}{\sqrt{F(\beta) - F(y)}}.$$



Lack of obstructions: L small



Control to the politeness equilibrium θ

Local controllability.

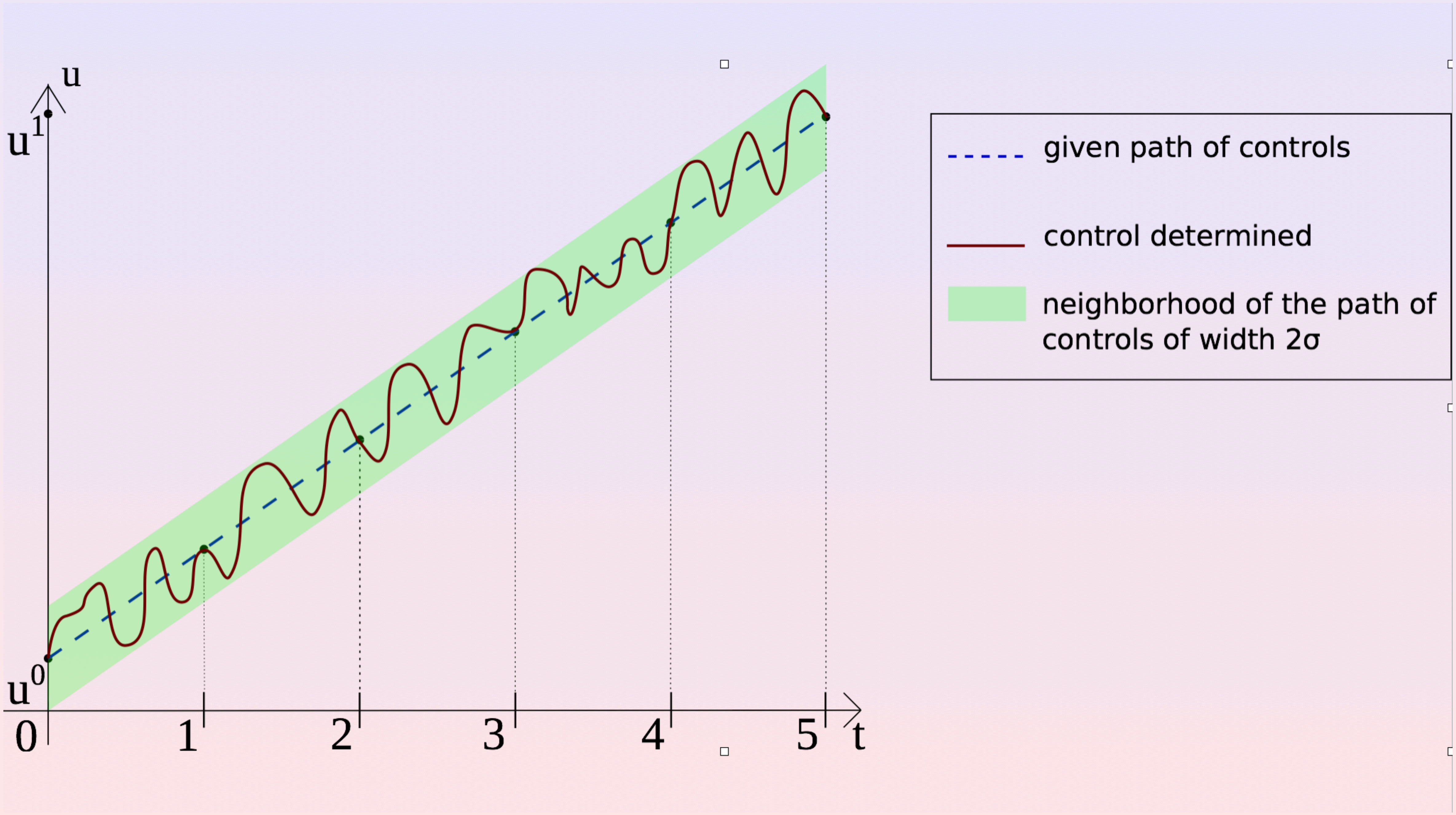
Classical but technically complex results on parabolic control allow showing that initial data y_0 close enough to $w = \theta$ can be driven to $w = \theta$ in any time $T > 0$ with solutions that oscillate very little around $w = \theta$, and in particular $0 \leq y \leq 1$.

Question

The question is whether we can get close to $w = \theta$ for $L > L_\theta$ with a more complex strategy than simply taking constant controls $u = v = \theta$ and this for all initial data, not necessarily close to θ.

The quasi-static os staircase method

Build a path of steady states and follow it slowly, so to guarantee that solutions oscillate very little. This imposes the time of control to be long.



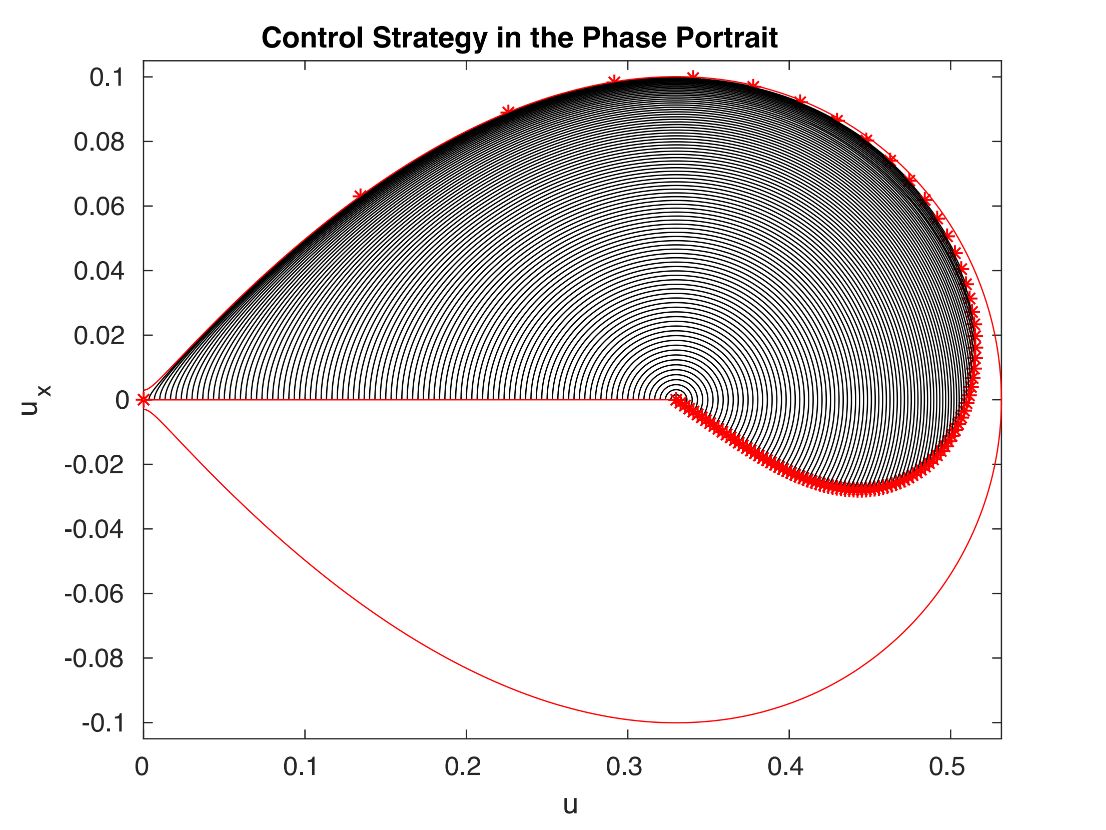
beginframeConstruction of admissible paths

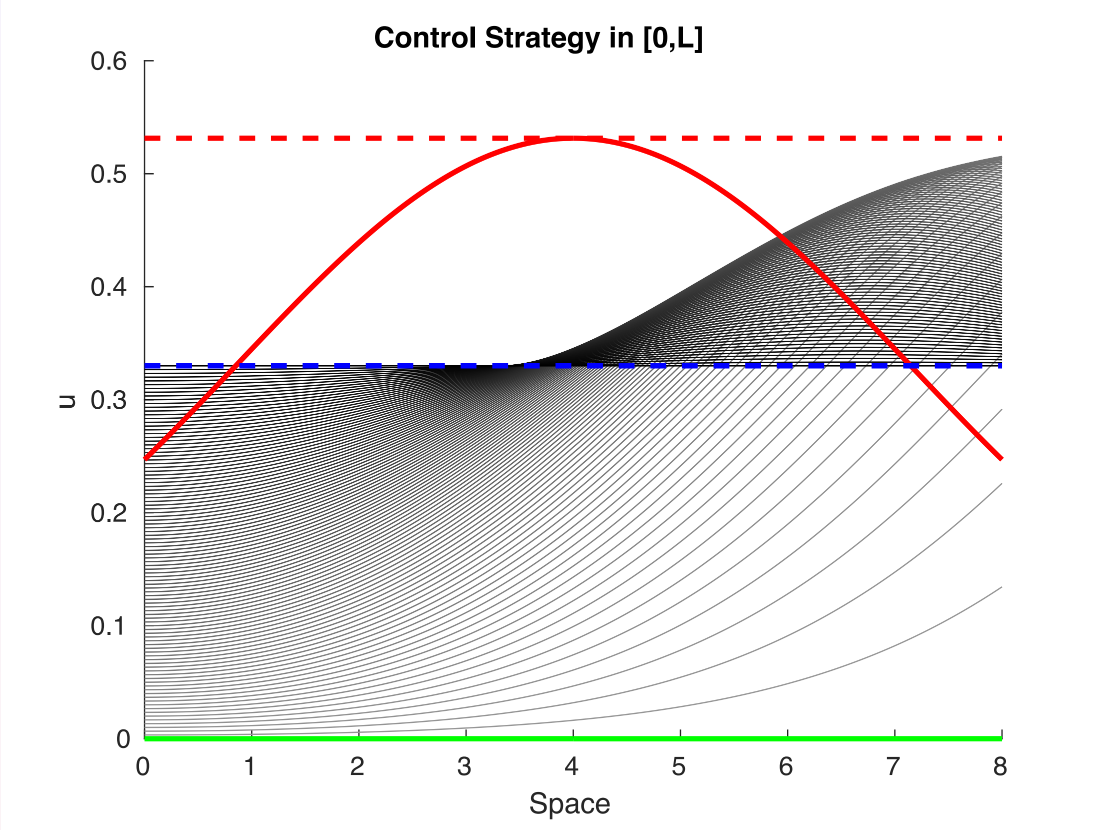
- Γ is an invariant region in the phase plane and it is admissible.
- $(\theta, 0) \in \Gamma$
- We choose a parameterized initial data $\gamma(s) : [0,1] \to \Gamma \subset \mathbb{R}^2$ to fulfill $\gamma(0) = (0,0)$ and $\gamma(1) = (\theta, 0)$. Then we solve the following problem:

$$\begin{pmatrix} y \\ y_x \end{pmatrix}_x = \begin{pmatrix} y_x \\ -f(y) \end{pmatrix} \tag{5}$$

from $[0, L]$ for every initial datum $\gamma(s)$

Construction of the paths in the phase portrait





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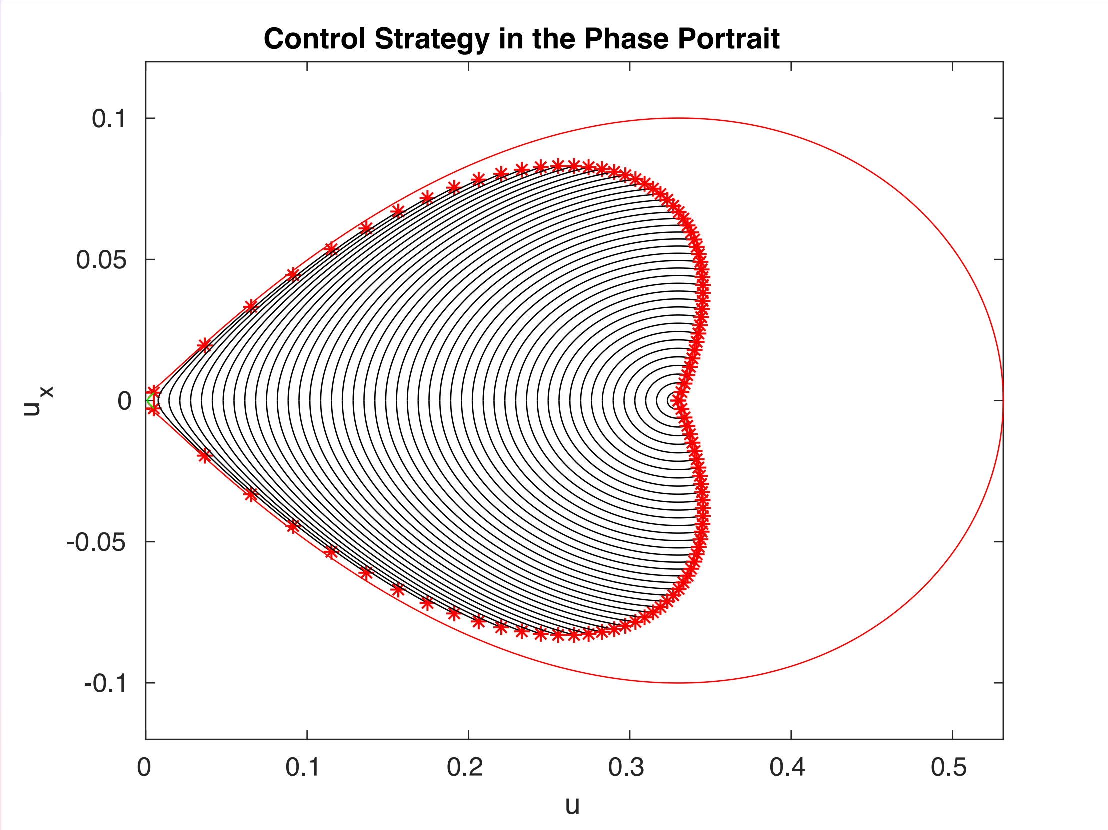
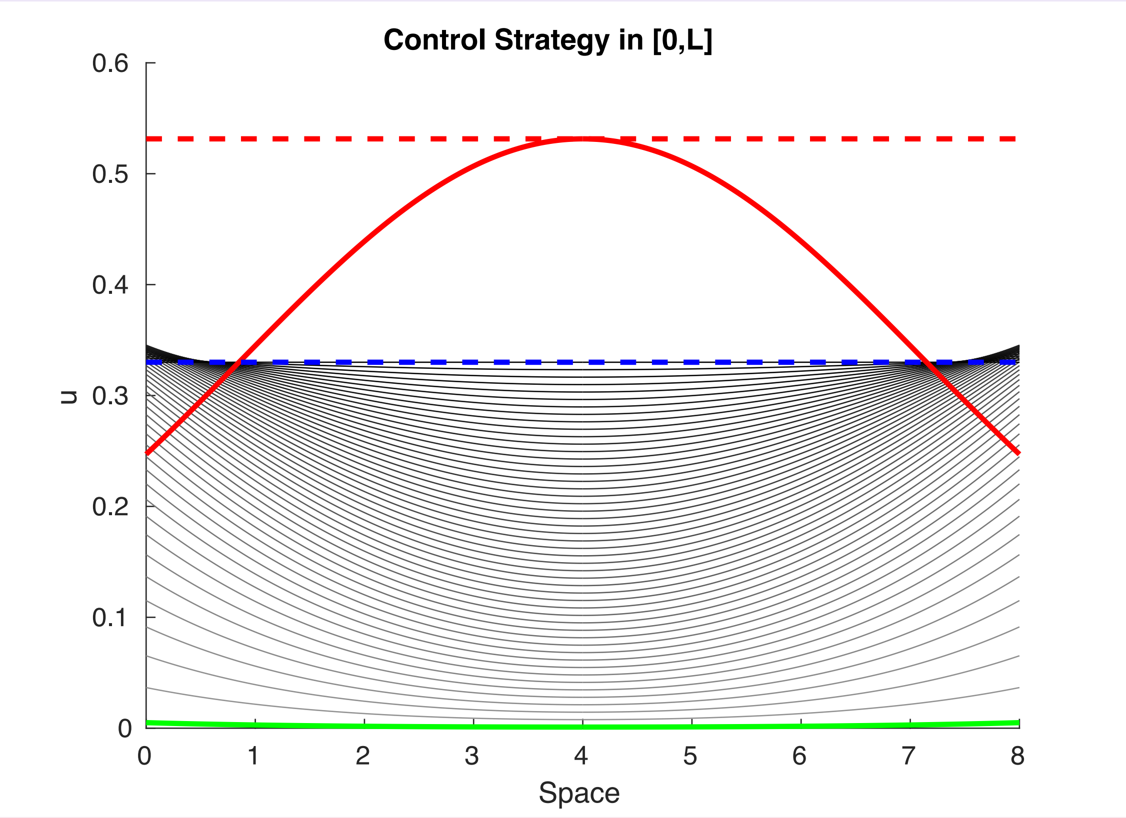
Construction of symmetric paths

In order to create a symmetric path with respect to $L/2$ we can take the initial condition in $L/2$ and to solve the ODE forward $[L/2, L]$. By symmetry the result will be the same from $[0, L/2]$.

Construction of symmetric paths





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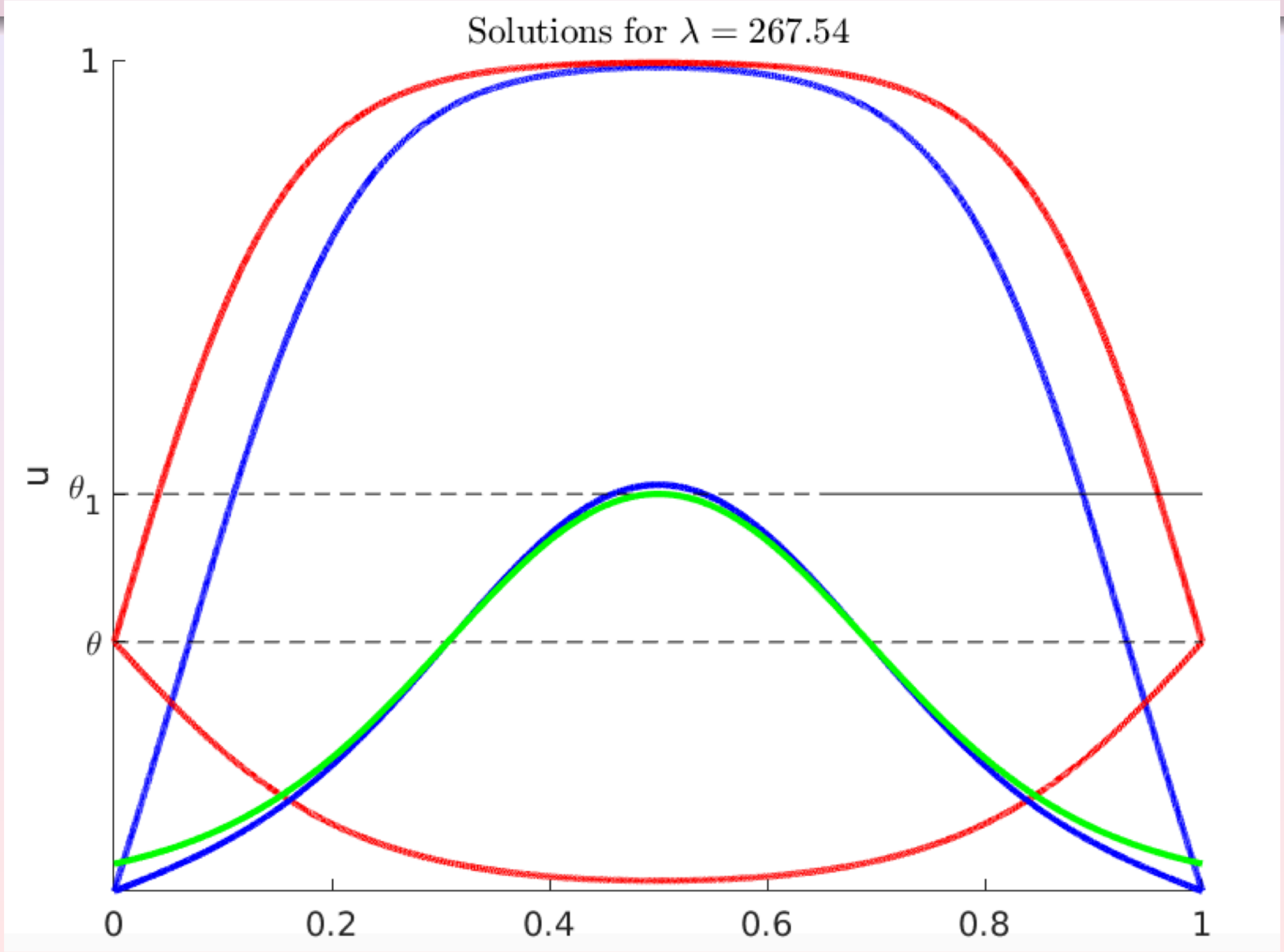
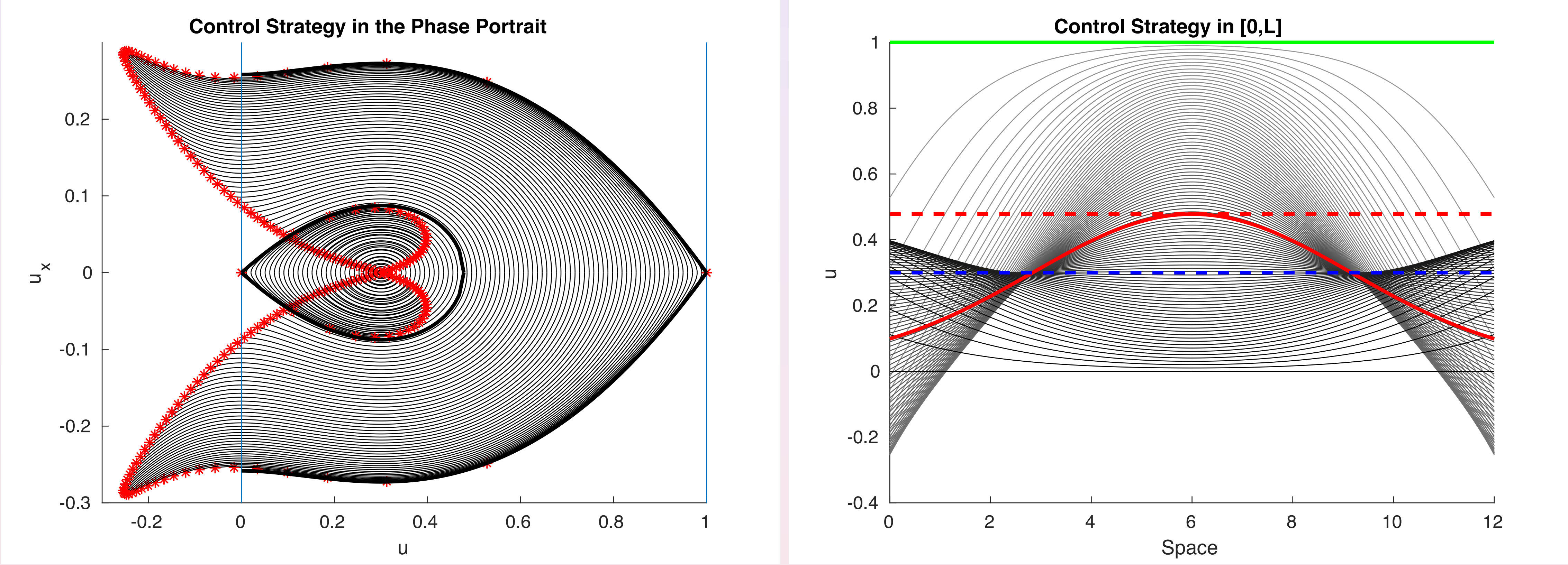
Note that beyond L^* the non-trivial steady state solution with null boundary data introduces a threshold in the space of solutions that makes it impossible the control of all data. The result is optimal!



Beyond constraints (I)

When the nontrivial solution with 0 Dirichlet boundary appears, there cannot be an admissible path of steady states connecting 1 and 0.



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BUT we are always able to control to any steady state close to 1 by employing a dynamic strategy.

Beyond constraints (I)

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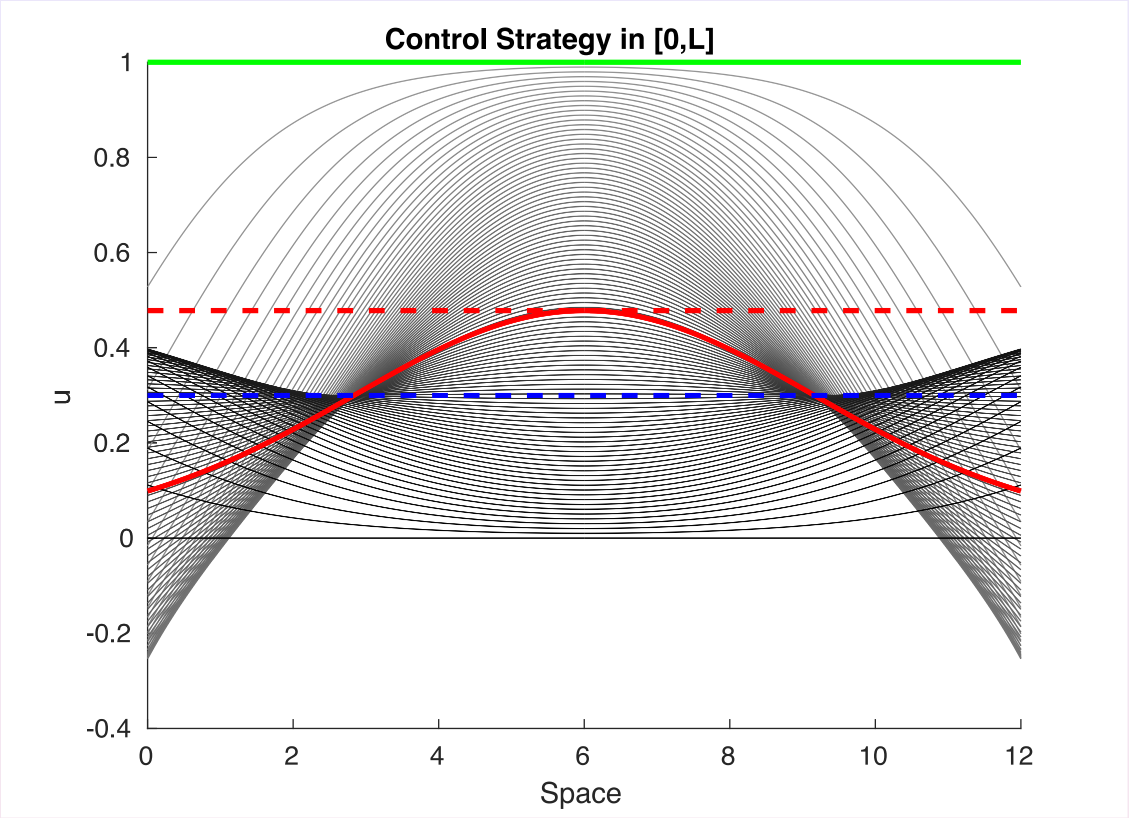
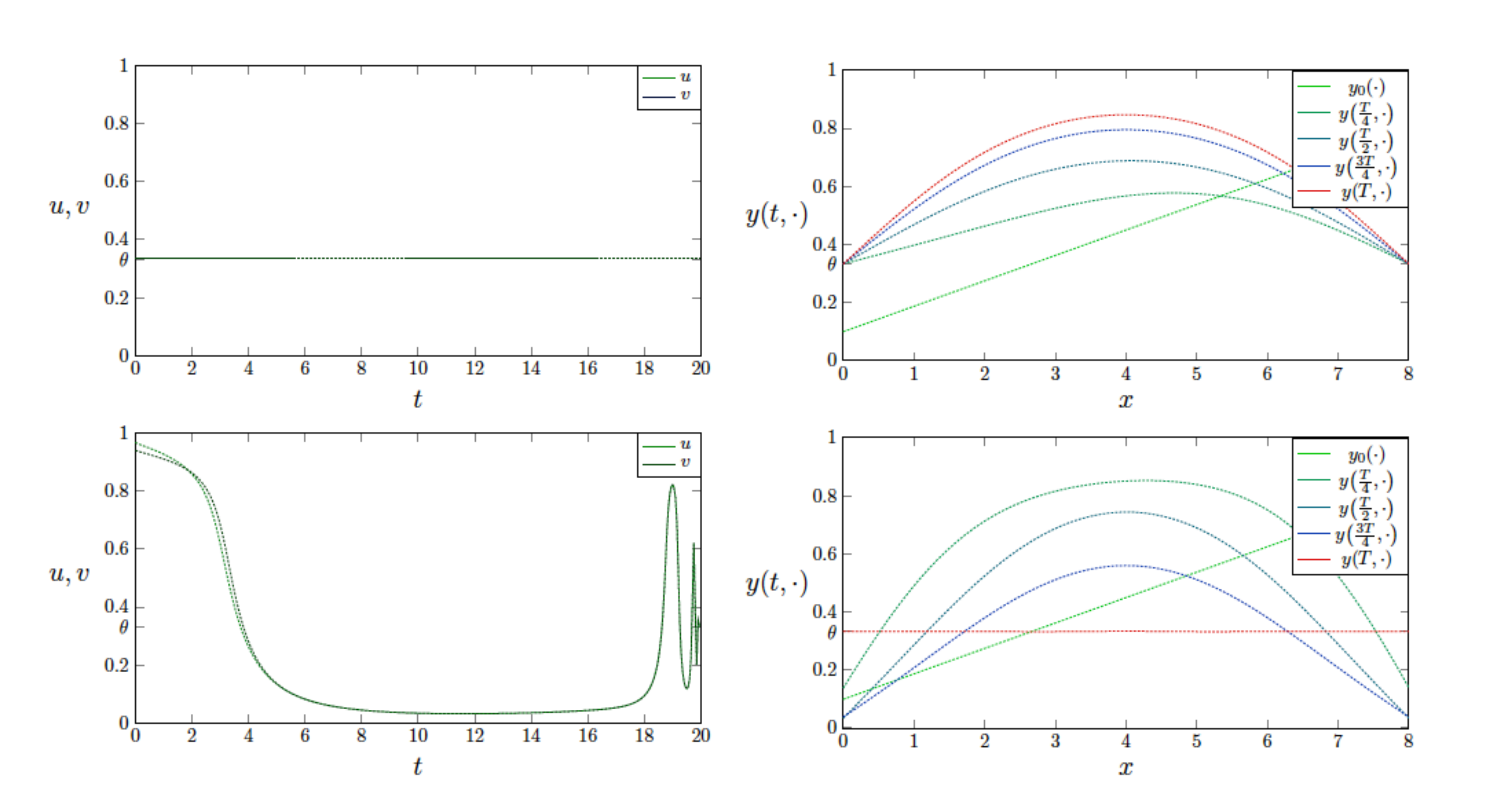


Remark

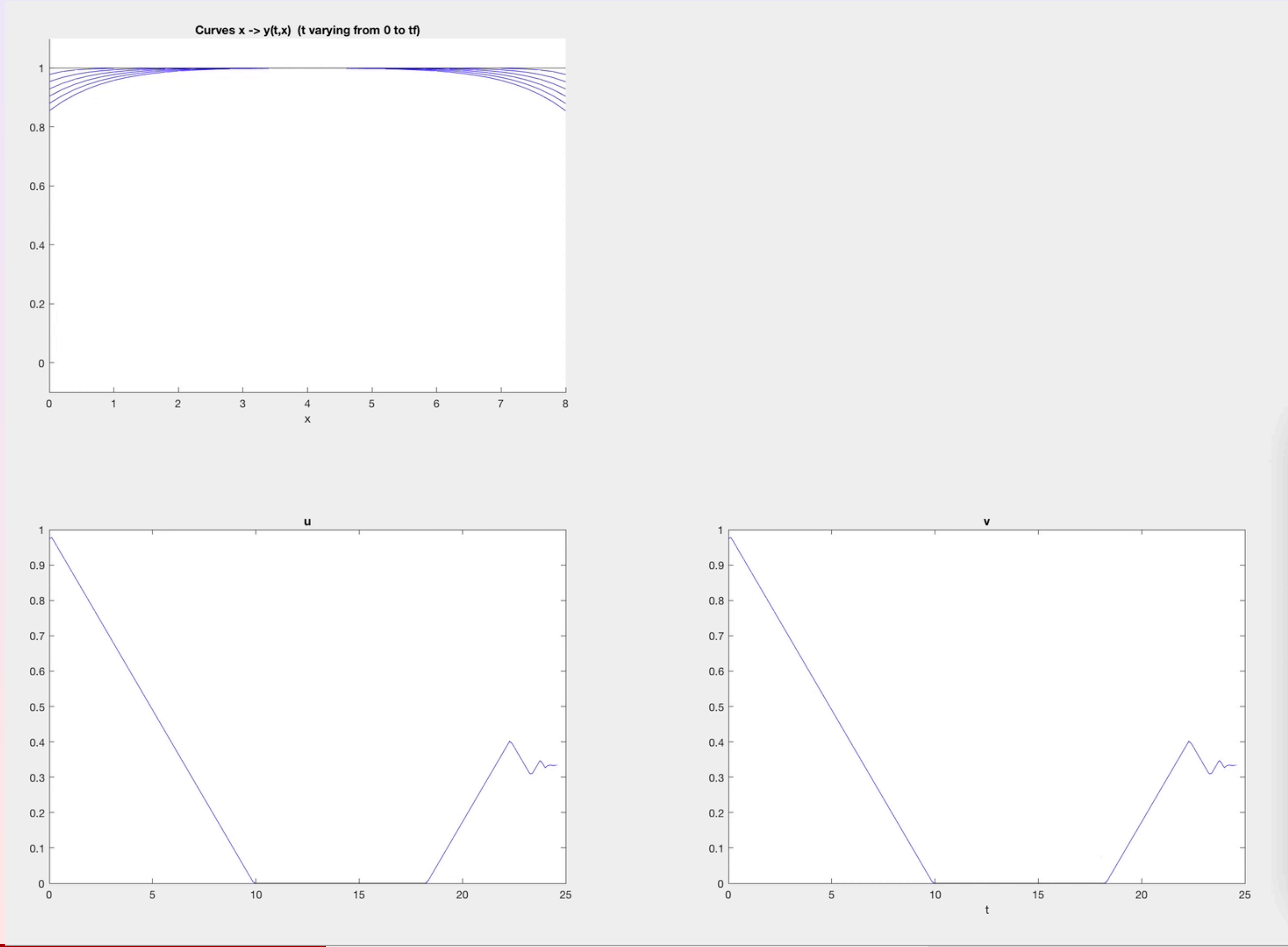
If we admit that we can go below until -0.3 we would be able to control. The path is not blowing up, but in general we should proof that our path is not blowing up in finite time.

Experiment

When $L_\theta < L = 8 < L^*$ the static control strategy does not work, but there are other more complex paths to follow and that turn out to be efficient.



Minimal time control in action



Conclusion

The optimal control strategies are not necessarily simple or intuitive. The lanscape of the set of steady states can be complex and there might be unexpected bridges indicating the path to follow for control.

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Perspectives

- Significant work to be done from a modeling perspective to get closer to real social or biological issues.
- Plenty still to be done to gain understanding of these models from a control perspective.
- Extensions to multi-d and to systems raises interesting new questions about the nature of set of steady state solutions, their stability, etc.
- **The great challenge** of making our analysis ti be not only qualitatively sound but quantitatively efficient. Important modelling efforts are needed.