A semilinear hyperbolic system with space-dependent and nonlinear damping (part I)

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VIII Partial differential equations, optimal design and numerics

Benasque, August 20, 2019

Joint work with Fatima Aqel, Edda Dal Santo (L'Aquila)

The problem

$$\begin{cases} \partial_t \rho + \partial_x J = 0 \\ \partial_t J + \partial_x \rho = -2k(x)g(J) \end{cases} \qquad x \in I = (0,1), \;\; t > 0$$

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Relation with the semilinear wave equation:

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- Fixed string at both ends:

$$u(0,t) = u(1,t) = 0 \qquad \Leftrightarrow \qquad J_b = 0\,, \qquad \int_I \rho_0(x)\, dx = 0\,.$$

Stationary equations

$$\partial_x J = 0\,, \qquad \partial_x \rho = -2k(x)g(J)$$

- Stationary solution:

$$J(x) = J_b\,, \qquad \rho(x) = -2g(J)\int_0^x k(y)\,dy + C\,,$$

C uniquely determined by $\int_I \rho_0$.

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- Without loss of generality:

$$J_b = 0\,, \qquad \int_I \rho_0\,dx = 0\,, \qquad J(x) = \rho(x) = 0 \qquad \forall\, x$$

General target

(A) to study the decay properties of the solutions as $t \to \infty$;

(B) to provide approximations with good accuracy for large t.

What we did... and what we do

- For the Cauchy problem...

Rigorous L^1 error estimates for suitable approximations of

$$\partial_t \rho + \partial_x J = 0\,, \qquad \partial_t J + \partial_x \rho = -2k(x)g(\rho, J)$$

based on Bressan-Liu-Yang functional for hyperbolic systems of conservation laws
[A. - Gosse, Springerbriefs 2015]

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  - No direct approach using the BLY functional, due to the reflection of waves at the two boundaries
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$$\rightsquigarrow \quad \text{Decay in } L^\infty \text{ as } t \to \infty$$

[Haraux (2009): Decay in L^p, $2 \leq p \leq \infty$, sufficiently regular data. Chitour, Marx, Prieur (2019)]

A probabilistic interpretation of telegrapher's equation

Goldstein, Kac (1956)

- Particles moving either to the left or to the right, speed $= \pm 1$.
  Time step τ, space step δ

- $\alpha(x,t)$: probability that a particle at (x,t) arrived from the left,
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- λ: fixed rate of reversal, $1 - \lambda\tau > 0$:

$$\alpha(x,t) = (1-\lambda\tau)\alpha(x-\delta, t-\tau) + \lambda\tau\,\beta(x-\delta, t-\tau)$$
$$\beta(x,t) = (1-\lambda\tau)\beta(x+\delta, t-\tau) + \lambda\tau\,\alpha(x+\delta, t-\tau)$$

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- As $\tau = \delta \to 0$: a linear system

$$\partial_t \alpha + \partial_x \alpha = -\lambda\alpha + \lambda\beta$$
$$\partial_t \beta - \partial_x \beta = \lambda\alpha - \lambda\beta$$

A probabilistic interpretation for the telegrapher's equation

$$\boxed{\begin{aligned} &\partial_t(\alpha+\beta)+\partial_x(\alpha-\beta)=0 \\ &\partial_t(\alpha-\beta)+\partial_x(\alpha+\beta)=-2\lambda(\alpha-\beta) \end{aligned}}$$

- For $v=\alpha+\beta$:

$$\partial_t^2 v-\partial_x^2 v+2\lambda\cdot(\partial_t v)=0\,.$$

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In our case, the rate of reversal λ is not constant but depends on x and on the solution through $g'(J)$.

Question: Is there a counterpart of this derivation for nonlinear damping?

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[Katsoulakis-Tzavaras JSP 1999]

The "Well Balanced" (WB) approximation

Let $N \in 2\mathbb{N}$, $\Delta x = \Delta t = 1/N$, $x_j = j\Delta x$, $j = 0, \dots, N$ and set

$$\mu_N = \sum_{j=1}^{2N-1} \left(\int_{x_{j-1}}^{x_j} k(x)\, dx \right) \delta_{\{x_j\}}$$

Consider

$$\begin{cases} \partial_t \rho + \partial_x J & = 0\,, \\ \partial_t J + \partial_x \rho + 2g(J)\mu_N & = 0\,, \end{cases}$$

with

- initial data $(\rho_0^{\Delta x}, J_0^{\Delta x})$ piecewise constant, being constant on each cell
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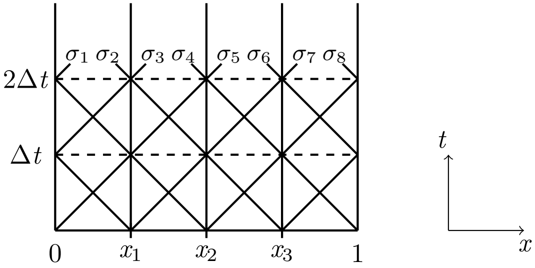
- boundary conditions $J^{\Delta x}(0,t) = J^{\Delta x}(1,t) = 0$.

An approximate solution $(\rho^{\Delta x}, J^{\Delta x})(x,t)$ is an **exact** solution to the initial-boundary value problem above.

Waves in the WB approximation



WB scheme, $N = 4$. The segments with speed 0, ± 1 correspond to the location of the discontinuities.

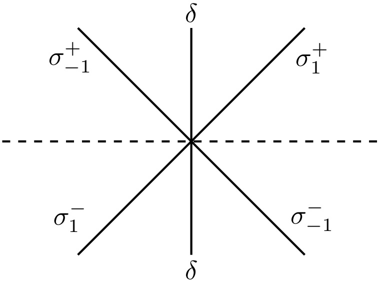
The size of waves:

± 1-waves: $\boxed{\sigma_{\pm 1} = \Delta J = \pm \Delta \rho}$, $\quad$ 0-waves: $\boxed{\delta = \int_{x_{j-1}}^{x_j} k(y)\, dy}$

The vector size:

$$\boldsymbol{\sigma}(t) = (\sigma_1, \ldots, \sigma_{2N})\,(t)$$

Interactions



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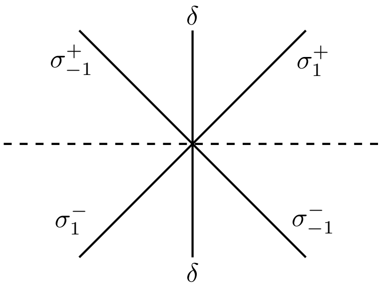

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Wave sizes change according to:

$$\boxed{\begin{pmatrix}\sigma_{-1}\\ \sigma_1\end{pmatrix}^{+} = \begin{pmatrix}1-c & c\\ c & 1-c\end{pmatrix}\begin{pmatrix}\sigma_{-1}\\ \sigma_1\end{pmatrix}^{-}} \qquad \boxed{c \sim g'(J)\delta}.$$

$\rightsquigarrow$ A 2×2 doubly stochastic matrix if $g' \geq 0$

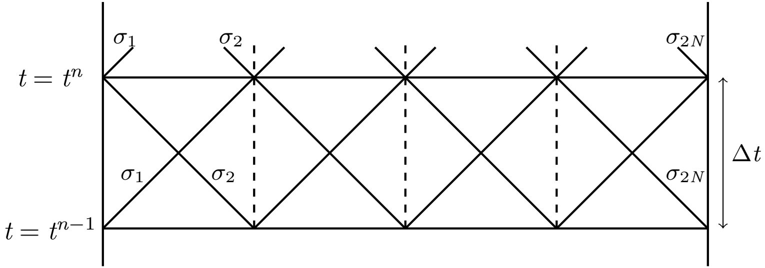
The transition matrix

Let

$$\boldsymbol{\sigma}(t) = (\sigma_1, \ldots, \sigma_{2N}) \in \mathbb{R}^{2N}, \qquad N = (\Delta x)^{-1}$$

be the vector of sizes of waves at time t, in increasing space order. Then for $t^n = n\Delta t$, $n \geq 1$:

$$\boldsymbol{\sigma}(t^n+) = B(\boldsymbol{c}^n)\,\boldsymbol{\sigma}(t^{n-1}+)$$



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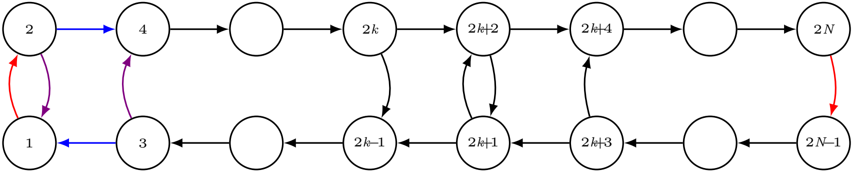
GOAL:

Determine spectral properties of the matrices $B(\boldsymbol{c})$

The transition matrix

$$B(\boldsymbol{c}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 1-c_1 & \cdots & 0 & 0 & 0 & 0 \\ 1-c_1 & 0 & 0 & c_1 & & \vdots & \vdots & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & c_{N-1} & 0 & 0 & 1-c_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 1-c_{N-1} & 0 & 0 & c_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}$$



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$$B(\boldsymbol{c}) = (1 - c)B(\mathbf{0}) + cB_1$$

is a Birkhoff decomposition for $B(\boldsymbol{c})$, where $B(\mathbf{0})$, B_1 are permutation matrices.

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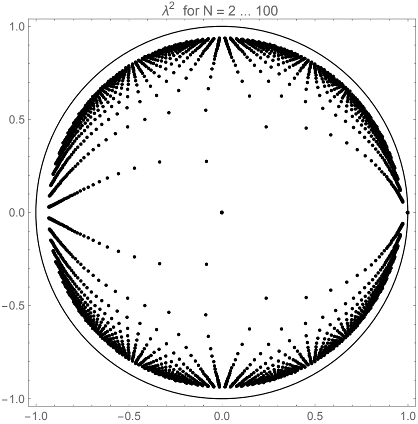
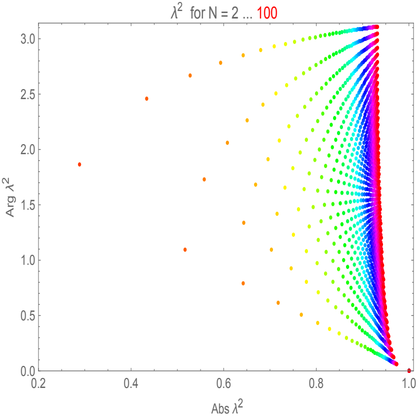
- References on doubly stochastic matrices (books): Horn–Johnson, Bapat–Raghavan, D.Serre

Some plots



λ^2 for N = 2 ... 100
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0



λ^2 for N = 2 ... 100
Arg λ^2
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.2
0.4
0.6
0.8
1.0
Abs λ^2

Iterations of the transition matrix

- By iteration,

$$\boldsymbol{\sigma}(t^n+) = \mathcal{B}_n \boldsymbol{\sigma}(0+)\,, \qquad \mathcal{B}_n \doteq \left[B^{(n)} B^{(n-1)} \cdots B^{(2)} B^{(1)}\right] \in M_{2N}\,.$$

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Good news:

Because of the boundary conditions $J(0,t) = J(1,t)$, the projection of $\boldsymbol{\sigma}(t)$ onto the eigenspace for $\lambda = 1$ is zero. Also the one for $\lambda = -1$ is harmless.

Problem:

Estimate the second maximal modulus of the eigenvalues of $B^{(n)}$ and of $\mathcal{B}_n$.

Two parameters, both $\to \infty$: n and $N = \Delta x^{-1}$

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Possible tools:

Nonhomogeneous Markov chains... Joint Spectral Radius... Matrix theory!

An estimate for the eigenvalues

Write $B(\boldsymbol{c})$ as a perturbation:

$$B(\boldsymbol{c}) = B(\mathbf{0}) + E(\boldsymbol{c})$$

with

$$|||E(\boldsymbol{c})||| = \max_{\|v\|=1} \|E(\boldsymbol{c})v\| = 2 \max_{j=1,\dots,N-1} c_j \to 0 \qquad N \to \infty$$

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A Rayleigh quotient formula

$$\lambda_\ell = \mu_\ell + \frac{\boldsymbol{y}_\ell^* E(\boldsymbol{c}) \boldsymbol{x}_\ell}{\boldsymbol{y}_\ell^* \boldsymbol{x}_\ell} + \text{"}\mathcal{O}\left(|||E(\boldsymbol{c})|||^2\right)\text{"}, \qquad \mu_\ell = \mathrm{e}^{\frac{i\pi\ell}{N}}$$

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Lemma:

If $k(x) \geq \bar{k} \cdot \chi_{(\alpha,\beta)}$ with $\bar{k} > 0$ and if $\inf g' > 0$, then there exists $C = C(\alpha, \beta) > 0$ such that for every $\mu_\ell \neq \pm 1$ and N large enough one has

$$\left| \mu_\ell + \frac{\boldsymbol{y}_\ell^* E(\boldsymbol{c}) \boldsymbol{x}_\ell}{\boldsymbol{y}_\ell^* \boldsymbol{x}_\ell} \right| \leq 1 - \frac{\bar{k}(\inf g')}{N} C.$$

END OF PART I

... To be continued ...