The turnpike property and its use for Model Predictive Control

Chair of Applied Mathematics
Department of Mathematics
University of Bayreuth

20.08.2019

VIII Partial differential equations, optimal design and numerics, Benasque

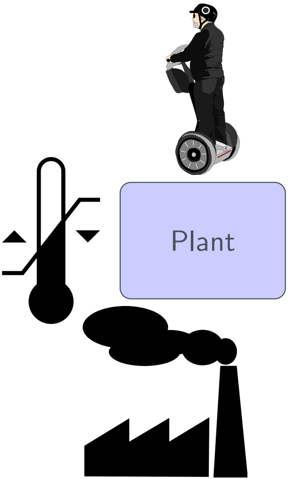
Lars Grüne, Manuel Schaller and Anton Schiela

Outline

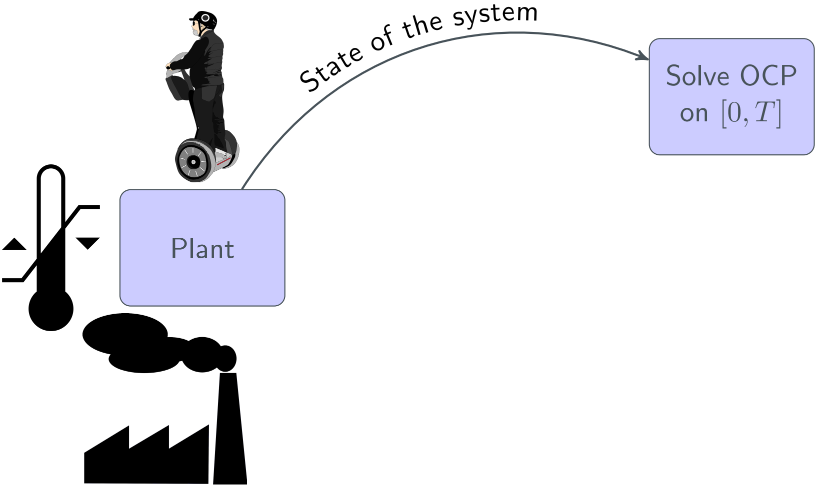
1. Motivation
2. Turnpike analysis for general evolution equations
3. Greedy discretization for MPC

Contents

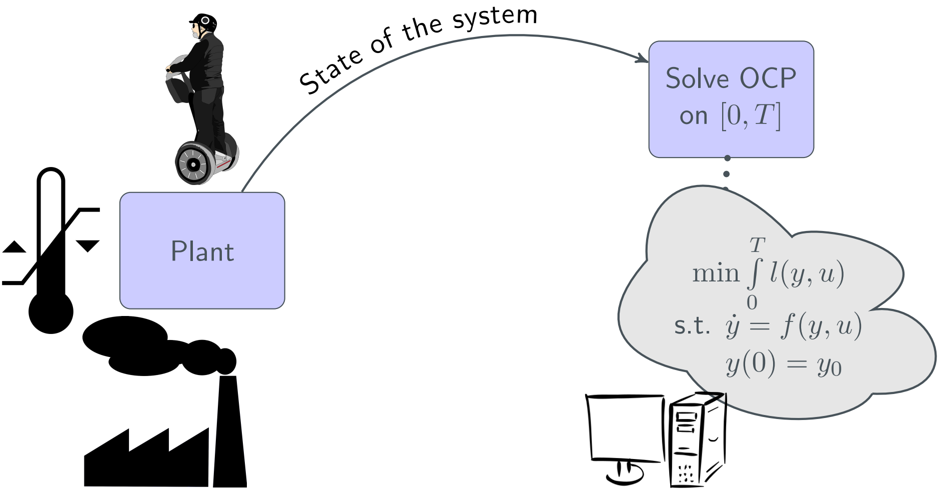
A Basic MPC algorithm



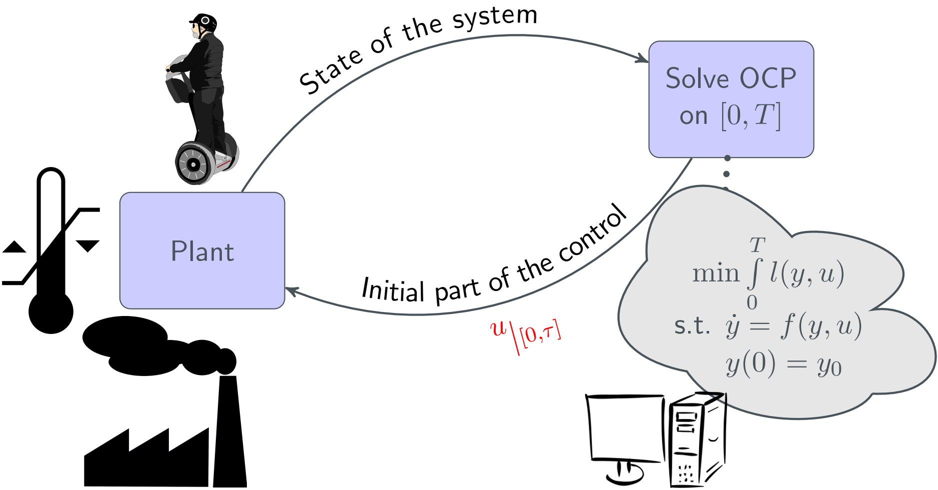
A Basic MPC algorithm



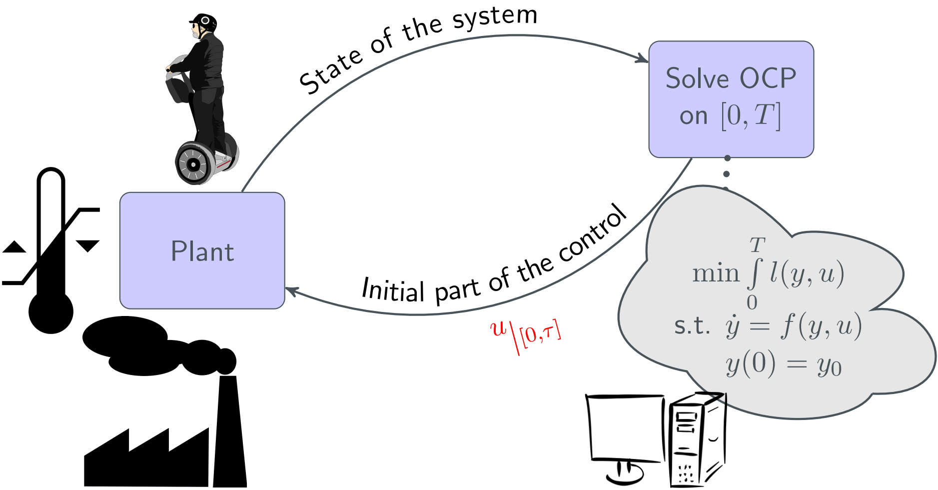
A Basic MPC algorithm



A Basic MPC algorithm

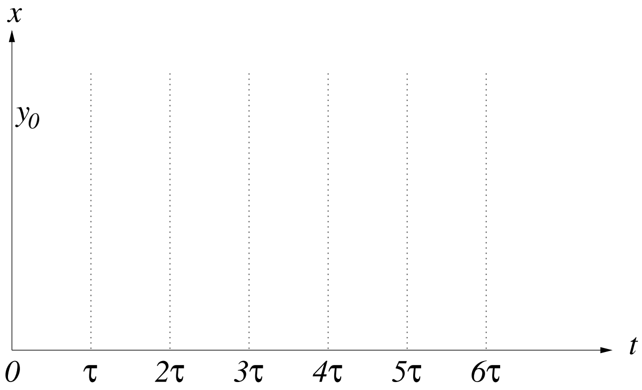


A Basic MPC algorithm

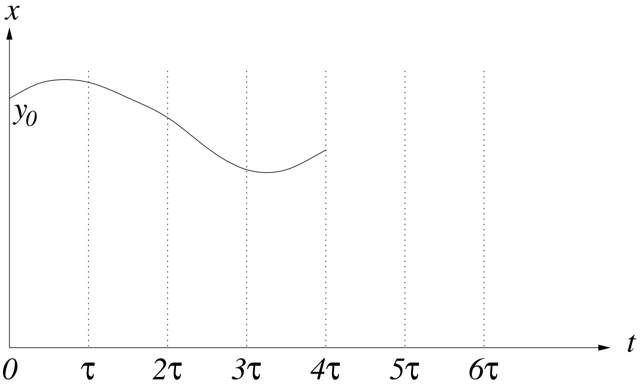


Usually $\tau \ll T < \infty$

Trajectories

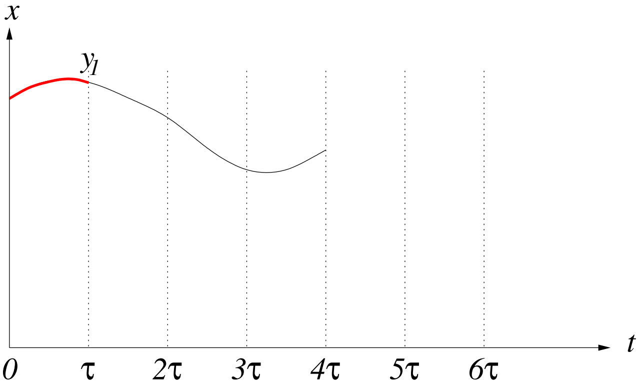


Trajectories



Black = Open loop

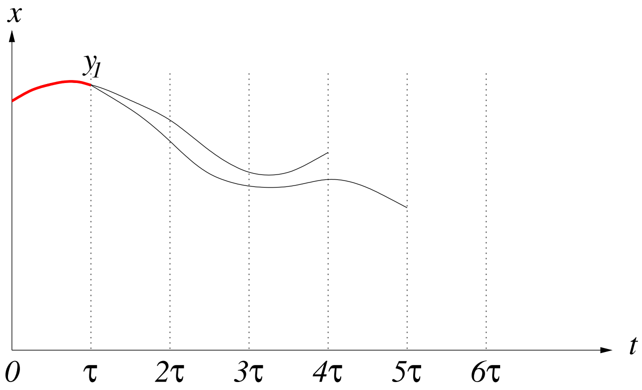
Trajectories



Black = Open loop

Red = MPC closed loop

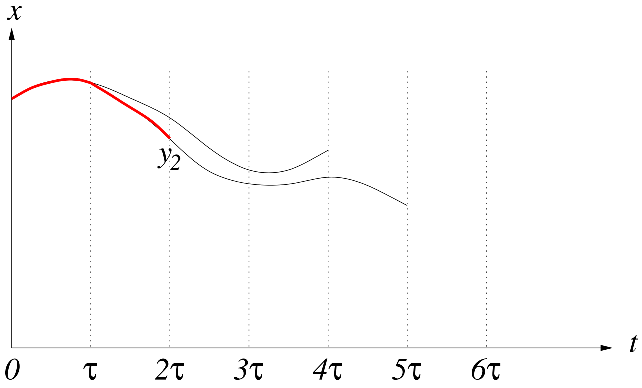
Trajectories



Black = Open loop

Red = MPC closed loop

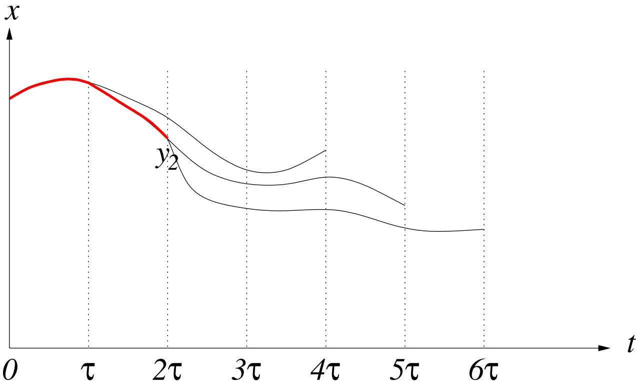
Trajectories



Black = Open loop

Red = MPC closed loop

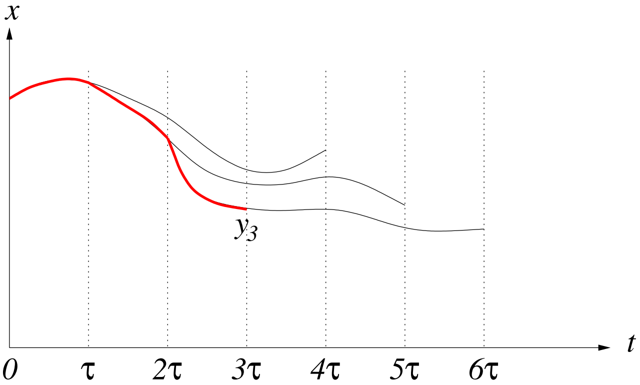
Trajectories



Black = Open loop

Red = MPC closed loop

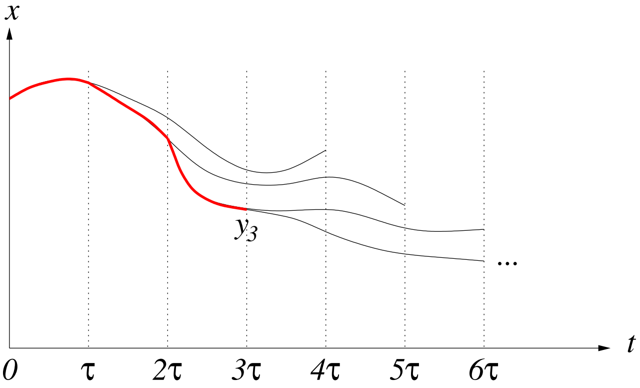
Trajectories



Black = Open loop

Red = MPC closed loop

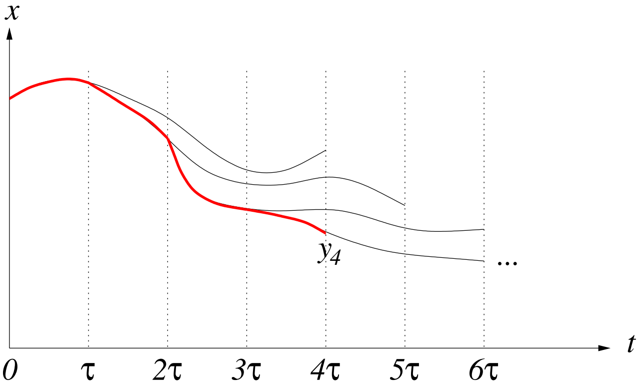
Trajectories



Black = Open loop

Red = MPC closed loop

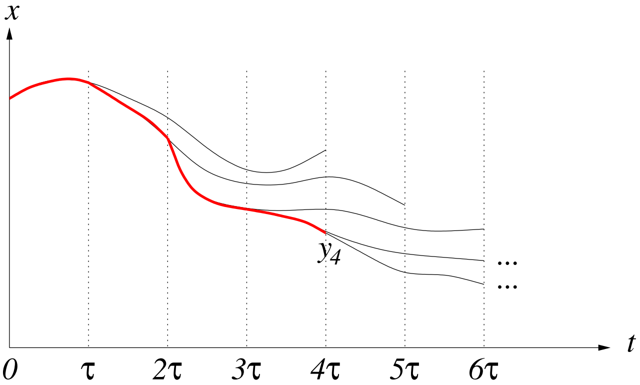
Trajectories



Black = Open loop

Red = MPC closed loop

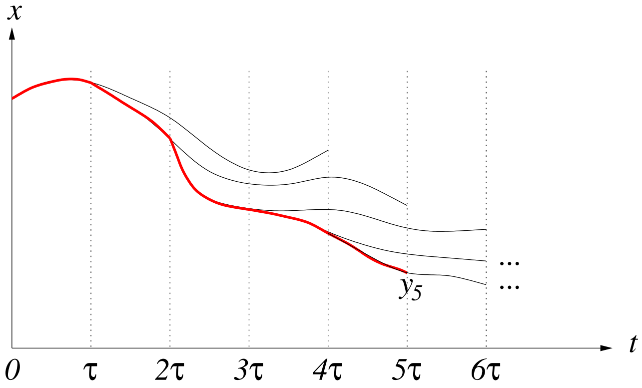
Trajectories



Black = Open loop

Red = MPC closed loop

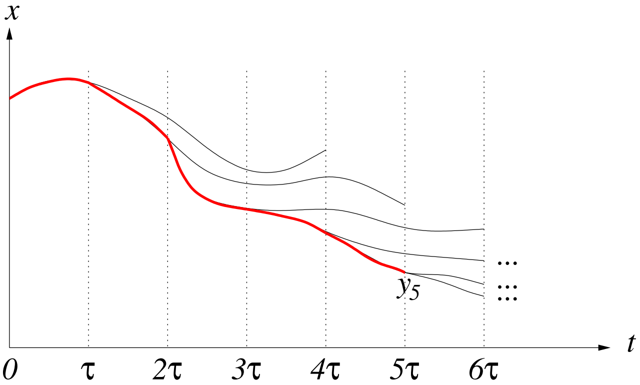
Trajectories



Black = Open loop

Red = MPC closed loop

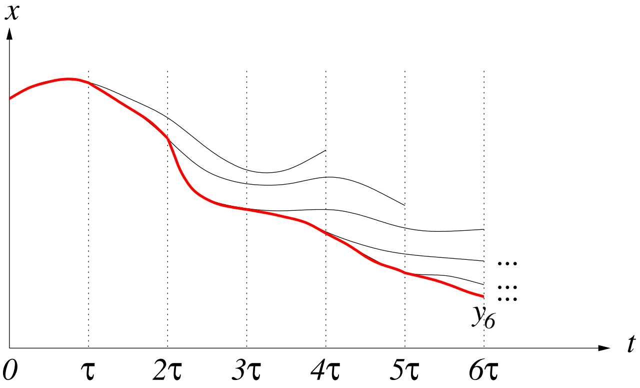
Trajectories



Black = Open loop

Red = MPC closed loop

Trajectories

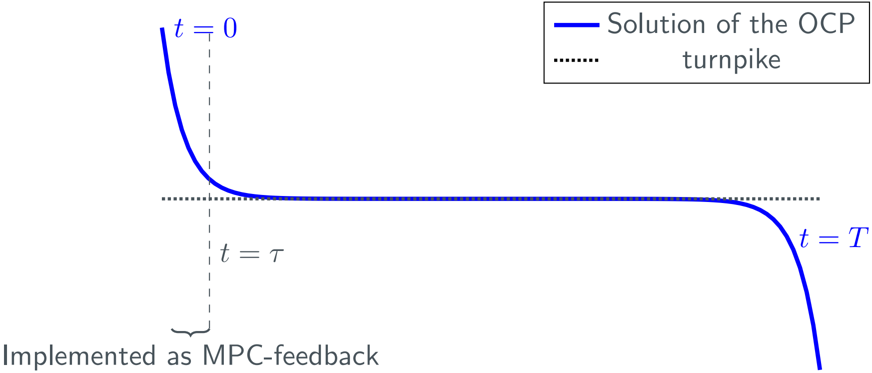


Black = Open loop

Red = MPC closed loop

Turnpike Property

- Optimal solutions stay close to an optimal equilibrium for the majority of the time interval



MPC and turnpike

OCP has turnpike property $\Rightarrow$ MPC-closed loop approximately optimal on infinite horizon (Grüne '13, Grüne/Stieler '14, Grüne/Pirkelmann '18)

MPC and turnpike

OCP has turnpike property $\Rightarrow$ MPC-closed loop approximately optimal on infinite horizon (Grüne '13, Grüne/Stieler '14, Grüne/Pirkelmann '18)

When does the turnpike property hold?

- strict dissipativity $\Longrightarrow$ Turnpike (Carlson et al. '91, Grüne '13, Grüne/Stieler/Pirkelmann '18)

MPC and turnpike

OCP has turnpike property $\Rightarrow$ MPC-closed loop approximately optimal on infinite horizon (Grüne '13, Grüne/Stieler '14, Grüne/Pirkelmann '18)

When does the turnpike property hold?

- strict dissipativity $\Longrightarrow$ Turnpike (Carlson et al. '91, Grüne '13, Grüne/Stieler/Pirkelmann '18)
- Turnpike + Controllability $\Longrightarrow$ str. dissipativity (Grüne/Müller '16)

MPC and turnpike

OCP has turnpike property $\Rightarrow$ MPC-closed loop approximately optimal on infinite horizon (Grüne '13, Grüne/Stieler '14, Grüne/Pirkelmann '18)

When does the turnpike property hold?

- strict dissipativity $\Longrightarrow$ Turnpike (Carlson et al. '91, Grüne '13, Grüne/Stieler/Pirkelmann '18)
- Turnpike + Controllability $\Longrightarrow$ str. dissipativity (Grüne/Müller '16)
- System stabilizable: Turnpike $\Longleftrightarrow$ Detectability (Grüne/Guglielmi '18)

MPC and turnpike

OCP has turnpike property $\Rightarrow$ MPC-closed loop approximately optimal on infinite horizon (Grüne '13, Grüne/Stieler '14, Grüne/Pirkelmann '18)

When does the turnpike property hold?

- strict dissipativity $\Longrightarrow$ Turnpike (Carlson et al. '91, Grüne '13, Grüne/Stieler/Pirkelmann '18)
- Turnpike + Controllability $\Longrightarrow$ str. dissipativity (Grüne/Müller '16)
- System stabilizable: Turnpike $\Longleftrightarrow$ Detectability (Grüne/Guglielmi '18)
- Stabilizability and Detectability $\Rightarrow$ Turnpike for OC of PDEs (Trélat, Zhang, Zuazua, Poretta, Gugat, Zamorano, Breiten et al. ... '13–'19)

Contents

Setting

Spaces.

- X Hilbert space with norm $\|\cdot\|$
- $L_p(0,T;X)$, $1 \leq p < \infty$ with norm $\|y\|_{L_p(X)} := (\int_0^T \|y(t)\|^p\,dt)^{1/p}$
- $C(0,T;X)$ with norm $\max_{t\in[0,T]} \|x(t)\|$

Setting

Spaces.

- X Hilbert space with norm $\|\cdot\|$
- $L_p(0,T;X)$, $1 \leq p < \infty$ with norm $\|y\|_{L_p(X)} := (\int_0^T \|y(t)\|^p \, dt)^{1/p}$
- $C(0,T;X)$ with norm $\max_{t\in[0,T]} \|x(t)\|$

Dynamics.

For $y_0 \in X$, $f \in L_1(0,T;X)$ we consider the dynamics

$$\dot{y}(t) = Ay(t) + Bu(t) + f \quad \text{for } t \geq 0, \qquad y(0) = y_0$$

with

- $A : D(A) \subset X \to X$ generator of C_0-semigroup $(T(t))_{t\geq 0}$
- B admissible control operator for $(T(t))_{t\geq 0}$

Example: Boundary control of wave equation

$\Omega \subset \mathbb{R}^n$ bounded C^2-domain.

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \Delta w && \text{on } \Omega \times (0,T) \\ w &= 0 && \text{on } \partial\Omega \setminus \Gamma \times (0,T) \\ w &= u && \text{on } \Gamma \times (0,T) \end{aligned}$$

$$w(x,0) = f(x), \;\; \frac{\partial w}{\partial t}(x,0) = g(x) \qquad x \in \Omega,$$

where $f \in L_2(\Omega), g \in H^{-1}(\Omega)$ and $u \in L_2(0,T;L_2(\Gamma))$.

Example: Boundary control of wave equation

$\Omega \subset \mathbb{R}^n$ bounded C^2-domain.

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \Delta w && \text{on } \Omega \times (0,T) \\ w &= 0 && \text{on } \partial\Omega \setminus \Gamma \times (0,T) \\ w &= u && \text{on } \Gamma \times (0,T) \end{aligned}$$

$$w(x,0) = f(x), \ \ \frac{\partial w}{\partial t}(x,0) = g(x) \qquad x \in \Omega,$$

where $f \in L_2(\Omega), g \in H^{-1}(\Omega)$ and $u \in L_2(0,T;L_2(\Gamma))$.

- Can derive well-posed boundary control system on $X = L_2(\Omega) \times H^{-1}(\Omega)$, see Tucsnak&Weiss 2009 with generator $A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}$, $D(\Lambda) = H_0^1(\Omega) \times L_2(\Omega)$, where A_0 is the Dirichlet Laplacian

Example: Boundary control of wave equation

$\Omega \subset \mathbb{R}^n$ bounded C^2-domain.

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \Delta w && \text{on } \Omega \times (0,T) \\ w &= 0 && \text{on } \partial\Omega \setminus \Gamma \times (0,T) \\ w &= u && \text{on } \Gamma \times (0,T) \end{aligned}$$

$$w(x,0) = f(x), \quad \frac{\partial w}{\partial t}(x,0) = g(x) \qquad x \in \Omega,$$

where $f \in L_2(\Omega), g \in H^{-1}(\Omega)$ and $u \in L_2(0,T;L_2(\Gamma))$.

- Can derive well-posed boundary control system on $X = L_2(\Omega) \times H^{-1}(\Omega)$, see Tucsnak&Weiss 2009 with generator $A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}$, $D(\Lambda) = H_0^1(\Omega) \times L_2(\Omega)$, where A_0 is the Dirichlet Laplacian
- (Ω, Γ, T) Geometric Control Condition (GCC) $\Longrightarrow$ controllability, see Rauch et. al 1974, Bardos et. al 1992.

Linear-quadratic OCP

Optimal Control problem.

$$\min_{y,u} \frac{1}{2}\int_0^T \|C(y(t)-y_d)\|_Y^2 + \|R(u(t)-u_d)\|_U^2 \, dt$$

$$\text{s.t.} \quad \dot{y} = Ay + Bu + f$$

$$y(0) = y_0$$

with

- Y, U Hilbert Spaces
- (A, B, C) form well posed system (Staffans 2005, Tucsnak&Weiss 2014)
- $R \in L(U, U)$, $\|Ru\|_U^2 \geq \alpha \|u\|_U^2$ for $\alpha > 0$

Optimality conditions

(y,u) optimal, Lagrange multiplier $\lambda \in C(0,T;X)$ s.t.

$$C^*Cy - \lambda' - A^*\lambda = C^*Cy_d, \qquad \lambda(T) = 0$$
$$R^*Ru - B^*\lambda = R^*Ru_d,$$
$$y' - Ay - Bu = f, \qquad y(0) = y_0$$

Optimality conditions

(y,u) optimal, Lagrange multiplier $\lambda \in C(0,T;X)$ s.t.

$$\begin{aligned} C^*Cy - \lambda' - A^*\lambda &= C^*Cy_d, \qquad \lambda(T) = 0 \\ R^*Ru - B^*\lambda &= R^*Ru_d, \\ y' - Ay - Bu &= f, \qquad y(0) = y_0 \end{aligned}$$

With $u = \underbrace{(R^*R)}_{=:Q}{}^{-1}B^*\lambda + u_d$, $E_t y = y(t)$, equivalent to

$$\underbrace{\begin{pmatrix} C^*C & -\frac{d}{dt} - A^* \\ 0 & E_T \\ \frac{d}{dt} - A & -BQ^{-1}B^* \\ E_0 & 0 \end{pmatrix}}_{=:M} \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix}$$

Corresponding steady state problem

Assumption

Data is time independent, i.e. $(y_d(t), u_d(t), f(t)) \equiv (y_d, u_d, f)$

Corresponding steady state problem

Assumption

Data is time independent, i.e. $(y_d(t), u_d(t), f(t)) \equiv (y_d, u_d, f)$

Corresponding steady-state problem:

$$\min_{\bar{y},\bar{u}} \ \frac{1}{2}\|C(\bar{y} - y_d)\|_Y^2 + \frac{1}{2}\|R(\bar{u} - u_d)\|_U^2$$
$$\text{s.t.} \qquad A\bar{y} + B\bar{u} + f = 0.$$

Corresponding steady state problem

Assumption

Data is time independent, i.e. $(y_d(t), u_d(t), f(t)) \equiv (y_d, u_d, f)$

Corresponding steady-state problem:

$$\min_{\bar{y},\bar{u}} \ \frac{1}{2}\|C(\bar{y} - y_d)\|_Y^2 + \frac{1}{2}\|R(\bar{u} - u_d)\|_U^2$$

$$\text{s.t.} \qquad A\bar{y} + B\bar{u} + f = 0.$$

$$\Longrightarrow \begin{pmatrix} C^*C & A^* \\ A & -BQ^{-1}B^* \end{pmatrix} \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ Bu_d \end{pmatrix}$$

Corresponding steady state problem

Assumption

Data is time independent, i.e. $(y_d(t), u_d(t), f(t)) \equiv (y_d, u_d, f)$

Corresponding steady-state problem:

$$\min_{\bar{y},\bar{u}} \ \frac{1}{2}\|C(\bar{y}-y_d)\|_Y^2 + \frac{1}{2}\|R(\bar{u}-u_d)\|_U^2$$

$$\text{s.t.} \qquad A\bar{y} + B\bar{u} + f = 0.$$

$$\implies \begin{pmatrix} C^*C & A^* \\ A & -BQ^{-1}B^* \end{pmatrix} \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ Bu_d \end{pmatrix}$$

$$\implies \underbrace{\begin{pmatrix} C^*C & -\frac{d}{dt} - A^* \\ 0 & E_T \\ \frac{d}{dt} - A & -BQ^{-1}B^* \\ E_0 & 0 \end{pmatrix}}_{=M} \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ \bar{\lambda} \\ Bu_d \\ \bar{y} \end{pmatrix}$$

A linear system for the distance to the turnpike

Dynamic problem

$$M\begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix}$$

A linear system for the distance to the turnpike

Dynamic problem

$$M\begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix}$$

Steady state problem

$$M\begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ \bar{\lambda} \\ Bu_d + f \\ \bar{y} \end{pmatrix}$$

A linear system for the distance to the turnpike

Dynamic Problem

$$M\begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} \xcancel{C^*Cy_d} \\ 0 \\ \xcancel{Bu_d + f} \\ y_0 \end{pmatrix}$$

Steady state problem

$$M\begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \xcancel{C^*Cy_d} \\ \bar{\lambda} \\ \xcancel{Bu_d + f} \\ \bar{y} \end{pmatrix}$$

$$M\begin{pmatrix} y - \bar{y} \\ \lambda - \bar{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\bar{\lambda} \\ 0 \\ y_0 - \bar{y} \end{pmatrix}$$

A linear system for the distance to the turnpike

Dynamic Problem

$$M\begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} \xcancel{C^*Cy_d} \\ 0 \\ \xcancel{Bu_d + f} \\ y_0 \end{pmatrix}$$

Steady state problem

$$M\begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \xcancel{C^*Cy_d} \\ \bar{\lambda} \\ \xcancel{Bu_d + f} \\ \bar{y} \end{pmatrix}$$

$$M\begin{pmatrix} y - \bar{y} \\ \lambda - \bar{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\bar{\lambda} \\ 0 \\ y_0 - \bar{y} \end{pmatrix}$$

How does $(-\bar{\lambda}, y_0 - \bar{y})$ influence $(y(t) - \bar{y}(t), \lambda(t) - \bar{\lambda}(t))$ for $t \in [0, T]$?

Notation

- $\|M^{-1}\|_{L_2,C} := \|M^{-1}\|_{(L_2(0,T;X)\times X)^2\to C(0,T;X)^2}$
- $\|M^{-1}\|_{L_2,L_2} := \|M^{-1}\|_{(L_2(0,T;X)\times X)^2\to L_2(0,T;X)^2}$

Towards a turnpike property

Theorem (Grüne, S., Schiela, 2018)

Assume $(\bar{y}, \bar{u}, \bar{\lambda})$ *solves the steady state problem,* (y, u, λ) *the dynamic problem and set* $0 \leq \mu < \frac{1}{\|M^{-1}\|_{L_2,L_2}}$

- $(\delta y, \delta u, \delta\lambda) := (y, u, \lambda) - (\bar{y}, \bar{u}, \bar{\lambda})$

Then for all $t \in [0, T]$

$$\|\delta y(t)\| + \|\delta\lambda(t)\| \leq C(e^{-\mu t} + e^{-\mu(T-t)})\|M^{-1}\|_{L_2,C}(\|\bar{\lambda}\|_{L_2(\Omega)} + \|y_0 - \bar{y}\|_{L_2(}$$

Towards a turnpike property

Theorem (Grüne, S., Schiela, 2018)

Assume $(\bar{y}, \bar{u}, \bar{\lambda})$ *solves the steady state problem,* (y, u, λ) *the dynamic problem and set* $0 \leq \mu < \frac{1}{\|M^{-1}\|_{L_2,L_2}}$

- $(\delta y, \delta u, \delta \lambda) := (y, u, \lambda) - (\bar{y}, \bar{u}, \bar{\lambda})$

Then for all $t \in [0, T]$

$$\|\delta y(t)\| + \|\delta \lambda(t)\| \leq C(e^{-\mu t} + e^{-\mu(T-t)})\|M^{-1}\|_{L_2,C}(\|\bar{\lambda}\|_{L_2(\Omega)} + \|y_0 - \bar{y}\|_{L_2(}$$

$$\|\frac{1}{e^{-\mu t} + e^{-\mu(T-t)}}\delta u\|_{L_2(U)} \leq C\|M^{-1}\|_{L_2,C}(\|\bar{\lambda}\|_{L_2(\Omega)} + \|y_0 - \bar{y}\|_{L_2(\Omega)})$$

Towards a turnpike property

Theorem (Grüne, S., Schiela, 2018)

Assume $(\bar{y}, \bar{u}, \bar{\lambda})$ *solves the steady state problem,* (y, u, λ) *the dynamic problem and set* $0 \leq \mu < \frac{1}{\|M^{-1}\|_{L_2, L_2}}$

- $(\delta y, \delta u, \delta \lambda) := (y, u, \lambda) - (\bar{y}, \bar{u}, \bar{\lambda})$

Then for all $t \in [0, T]$

$$\|\delta y(t)\| + \|\delta \lambda(t)\| \leq C(e^{-\mu t} + e^{-\mu(T-t)})\|M^{-1}\|_{L_2,C}(\|\bar{\lambda}\|_{L_2(\Omega)} + \|y_0 - \bar{y}\|_{L_2(}$$

$$\|\frac{1}{e^{-\mu t} + e^{-\mu(T-t)}}\delta u\|_{L_2(U)} \leq C\|M^{-1}\|_{L_2,C}(\|\bar{\lambda}\|_{L_2(\Omega)} + \|y_0 - \bar{y}\|_{L_2(\Omega)})$$

If $B \in L(U, X)$*, then for a.e.* $t \in [0, T]$

$$\|\delta u(t)\| + \leq C(e^{-\mu t} + e^{-\mu(T-t)})\|M^{-1}\|_{L_2,C}(\|\bar{\lambda}\|_{L_2(\Omega)} + \|y_0 - \bar{y}\|_{L_2(\Omega)})$$

A stability result for the solution operator

$$\min_{y,u} \frac{1}{2}\int_0^T \|C(y-y_d)\|_Y^2 + \|R(u-u_d)\|_U^2 \, dt$$

$$\text{s.t.} \qquad y' = Ay + Bu + f, \qquad y(0) = y_0$$

Recall: M operator for optimality conditions.

Theorem (Grüne, S., Schiela, 2018)

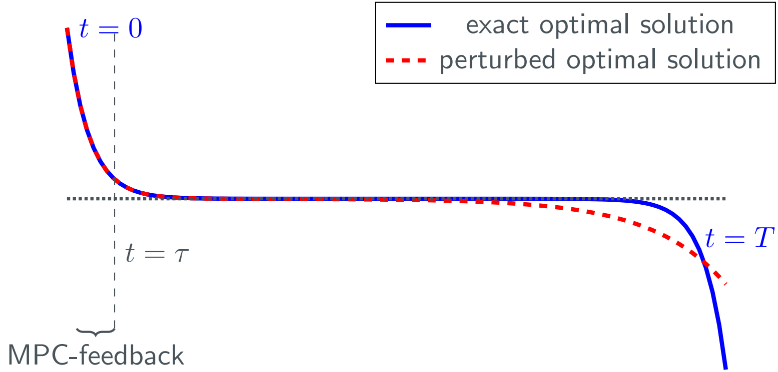
If (A,B) is exp. stabilizable, (A,C) exp. detectable, then

- $\|M^{-1}\|_{(L_2(0,T;X)\times X)^2 \to C(0,T;X)^2}$
- $\|M^{-1}\|_{(L_2(0,T;X)\times X)^2 \to L_2(0,T;X)^2}$
- $\|M^{-1}\|_{(L_1(0,T;X)\times X)^2 \to C(0,T;X)^2}$
- $\|M^{-1}\|_{(L_1(0,T;X)\times X)^2 \to L_2(0,T;X)^2}$

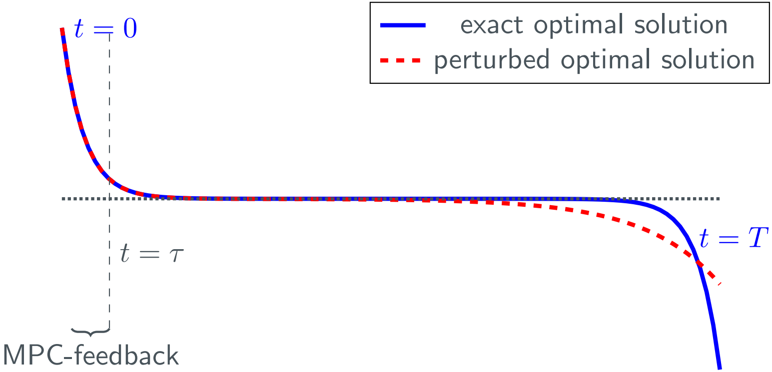
can be bounded independently of T.

Contents

A second look at the turnpike property



A second look at the turnpike property



$\hookrightarrow$ Can we show that perturbations close to $t = T$ do not really influence the MPC-feedback?

Influence of perturbations of the RHS

exact solution

$$M\begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix}$$

Influence of perturbations of the RHS

exact solution

$$M\begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix}$$

computed solution

$$M\begin{pmatrix} \tilde{y} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ 0 \\ \varepsilon_2 \\ 0 \end{pmatrix}$$

Influence of perturbations of the RHS

exact solution

$$M \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix}$$

computed solution

$$M \begin{pmatrix} \tilde{y} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ 0 \\ \varepsilon_2 \\ 0 \end{pmatrix}$$

$$M \begin{pmatrix} \tilde{y} - y \\ \tilde{\lambda} - \lambda \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ 0 \\ \varepsilon_2 \\ 0 \end{pmatrix}$$

Influence of perturbations of the RHS

exact solution

$$M\begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix}$$

computed solution

$$M\begin{pmatrix} \tilde{y} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ 0 \\ \varepsilon_2 \\ 0 \end{pmatrix}$$

$$M\begin{pmatrix} \tilde{y} - y \\ \tilde{\lambda} - \lambda \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ 0 \\ \varepsilon_2 \\ 0 \end{pmatrix}$$

How does $(\varepsilon_1(t), \varepsilon_2(t))$ influence $(\tilde{y}(t) - y(t), \tilde{\lambda}(t) - \lambda(t))$ for $t \in [0, T]$?

Influence of perturbations of the RHS

exact solution

$$M\begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix}$$

computed solution

$$M\begin{pmatrix} \tilde{y} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} C^*Cy_d \\ 0 \\ Bu_d + f \\ y_0 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ 0 \\ \varepsilon_2 \\ 0 \end{pmatrix}$$

$$M\begin{pmatrix} \tilde{y} - y \\ \tilde{\lambda} - \lambda \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ 0 \\ \varepsilon_2 \\ 0 \end{pmatrix}$$

How does $(\varepsilon_1(t), \varepsilon_2(t))$ influence $(\tilde{y}(t) - y(t), \tilde{\lambda}(t) - \lambda(t))$ for $t \in [0, T]$?

One can show, that perturbations act local in time.

Main message

Reminder: MPC controller solves on $[0, T]$, but implements $u\big|_{[0,\tau]}$ $\tau \ll T$.

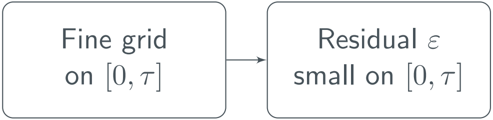
Main message

Reminder: MPC controller solves on $[0,T]$, but implements $u\big|_{[0,\tau]}$ $\tau \ll T$.

Fine grid
on $[0,\tau]$

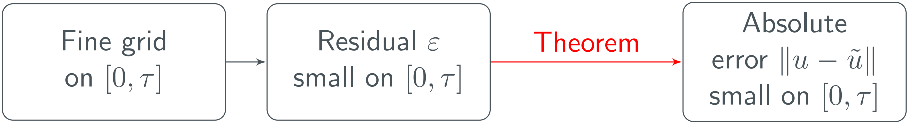
Main message

Reminder: MPC controller solves on $[0, T]$, but implements $u\big|_{[0,\tau]}$ $\tau \ll T$.



Main message

Reminder: MPC controller solves on $[0,T]$, but implements $u\big|_{[0,\tau]}$ $\tau \ll T$.



Recap

- If $\|M^{-1}\|$ independent of T (given if stabilizable+detectable), then
  - Turnpike property.
  - Influence of discretization errors decays exponentially in time
    $\hookrightarrow$ in MPC-context: coarsening of grids towards T.

Ongoing work:

- Goal oriented space-time error estimation techniques confirm these findings
- Numerical performance analysis

References

- Grüne, S., Schiela: Sensitivity analysis of optimal control for a class of parabolic PDEs motivated by model predictive control, SICON 2019
- Grüne, S., Schiela: Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations, Dec 2018, submitted
- Grüne, S., Schiela: Space-Time Adaptivity for Model Predicitve Control of parabolic PDEs, in preparation

References

- Grüne, S., Schiela: Sensitivity analysis of optimal control for a class of parabolic PDEs motivated by model predictive control, SICON 2019
- Grüne, S., Schiela: Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations, Dec 2018, submitted
- Grüne, S., Schiela: Space-Time Adaptivity for Model Predicitve Control of parabolic PDEs, in preparation

Thank you for your attention!

Perturbations stay local in time

Notation:

- $\|M^{-1}\|_{L_2,C} = \|M^{-1}\|_{(L_2(0,T;X)\times X)^2 \to C(0,T;X)^2}$
- $\|M^{-1}\|_{L_2,L_2} = \|M^{-1}\|_{(L_2(0,T;X)\times X)^2 \to L_2(0,T;X)^2}$

Theorem (Grüne, S., Schiela, 2018)

$(\tilde{y}, \tilde{u}, \tilde{\lambda})$ *computed solution,* (y, u, λ) *exact solution and*

- $(\delta y, \delta u, \delta\lambda) := (\tilde{y}, \tilde{u}, \tilde{\lambda}) - (y, u, \lambda)$
- $0 \le \mu < \frac{1}{\|M^{-1}\|_{(L_2,L_2)}}$
- $\|e^{-\mu\cdot}\varepsilon_1\|_{L_2(0,T;X)} + \|e^{-\mu\cdot}\varepsilon_2\|_{L_2(0,T;X)} \le \rho, \quad \rho \ge 0$

Then, there is a constant $C \ge 0$ *indep. of* T *s.t.*

$$\|e^{-\mu\cdot}\delta y\|_{C(0,T;X)} + \|e^{-\mu\cdot}\delta u\|_{L_2(0,T;U)} + \|e^{-\mu\cdot}\delta\lambda\|_{C(0,T;X)} \le C\rho\|M^{-1}\|_{L_2,C}$$

Adaptive grids in optimal control

Let (y, u) be exact solution, $(\tilde{y}, \tilde{u})$ numerical approximation.

Adaptive grids in optimal control

Let (y, u) be exact solution, $(\tilde{y}, \tilde{u})$ numerical approximation.

How to do adaptive grid refinement?

Classical Try to reduce $\int_0^T l(y,u) - l(\tilde{y},\tilde{u})\,dt$.

Adaptive grids in optimal control

Let (y, u) be exact solution, $(\tilde{y}, \tilde{u})$ numerical approximation.

How to do adaptive grid refinement?

Classical Try to reduce $\int_0^T l(y,u) - l(\tilde{y},\tilde{u})\,dt$.

MPC context Try to reduce $\int_0^\tau \tilde{l}(y,u) - \tilde{l}(\tilde{y},\tilde{u})\,dt$.

Let (y, u) be exact solution, $(\tilde{y}, \tilde{u})$ numerical approximation.

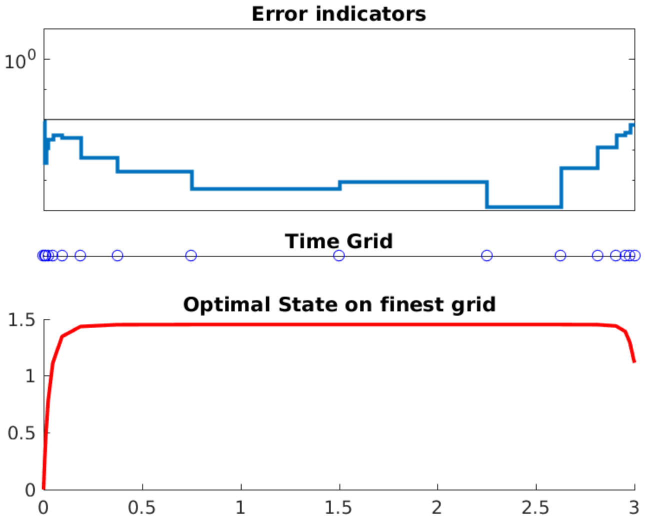
How to do adaptive grid refinement?

Classical Try to reduce $\int_0^T l(y,u) - l(\tilde{y},\tilde{u})\,dt$.

MPC context Try to reduce $\int_0^\tau \tilde{l}(y,u) - \tilde{l}(\tilde{y},\tilde{u})\,dt$.

How do goal oriented grids which minimize the MPC feedback error look like ?

Goal: Small error in cost functional ($[0, T]$)



Goal: Small error in MPC-feedback ($[0, \tau]$, here $\tau = 0.5$)